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*Lesław Gajek*, Marek Kałuszka**

# ON THE AVERAGE RETURN RATE FOR A GROUP OF INVESTMENT FUNDS[1]

**Abstract.** In the paper a new definition of the average return rate for a group of investment (or pension) funds is proposed. The definition is derived via integration of the financial results of the group of funds during a given period of time. It satisfies a set of postulates which every coherent definition is supposed to fulfil contrary to the definition which is used in the Polish law of August 1997 on Organisation and Operation of Pension Funds. A very simple formula for the average return rate is available provided that the fund's shares are stable in time.

## 1. INTRODUCTION

Consider a group of $n$ pension (or investment) funds which start their activity selling accounting (or participation) units at the same price. Denote by $k_i(t)$, $i = 1, 2, \ldots, n$, the number of all units possessed by the clients of the $i$-th fund at the moment $t$ and by $w_i(t)$ – the value of $i$-th fund unit at the moment $t$. The value $w_i(t)$ is established by dividing the total assets of the $i$-th fund, say $A_i(t)$, by the number of the units $k_i(t)$. The assets $A_i(t)$ can change due to the change of $k_i(t)$ or due to the change of the unit's value $w_i(t)$ according to the formula

$$A_i(t) = k_i(t)w_i(t).$$

For the individual investor the change of $w_i(t)$ is of main interest because it results in his own return rate. So define the return rate at the $i$-th fund during the time period $(t, t + \Delta t)$, by $[w_i(t + \Delta t) - w_i(t)]/w_i(t)$. Assume,

* Technical University of Łódź, Institute of Mathematics.

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for mathematical simplicity, that there exists a limit of this return rate divided by $\Delta t$, as $\Delta r \to 0$, and denote it by $\delta_i(t)$. Hence

$$\delta_i(t) = \frac{d}{dt} \log w_i(t) \tag{1}$$

Formula (1) may be derived also in another way. Assume that both $k_i(\cdot)$ and $w_i(\cdot)$ are differentiable functions. The infinitesimal relative change of the assets of the $i$-th fund during the time interval $(t, t+\Delta t)$ is

$$\frac{dA_i(t)}{A_i(t)} = \frac{dk_i(t)}{k_i(t)} + \frac{dw_i(t)}{w_i(t)}.$$

The first summand corresponds to the allocation of units as well as to appearing new clients or disappearing old ones and so on. The second summand describes a pure investment effect at the $i$-th fund, and is equal to $\delta_i(t)dt$. Hence $\delta_i(t)dt$ has two interpretations: it is the infinitesimal return rate for the accounting unit in the $i$-th fund and, simultaneously, it is the infinitesimal return rate for the assets of this fund, due to the pure investment effects (we shall use this duality in Section 2 to define the average return rate for the whole group).

Let $r_i$ denote the return rate of the $i$-th fund during a given time period $[T_1, T_2]$. Clearly

$$r_i = \exp\left[\int_{T_1}^{T_2} \delta_i(t)dt\right] - 1 \tag{2}$$

The rate $r_i$ informs the client what would be his return at time $T_2$ if he bought one accounting unit of the $i$-th fund at time $T_1$.

Now the problem arises how to define an average weighted return rate $\bar{r}(T_1, T_2)$ for the whole group of $n$ investment funds. The average return rate $\bar{r}$ should reflect the investment results of all the funds. In the Polish pension fund law it is also used in order to verify if a given pension fund achieves the so called minimum required return rate (compare: *Security through...*, 1997). If the return rate $r_i$ is smaller than the minimum required one, a deficiency arises which should be covered by the company managing the fund. Since the definition has severe financial consequences, it should be very carefully formulated taking into account the following "coherency postulates".

**Postulate 1.** In case the group consists of one fund $(n=1)$ $\bar{r}(T_1, T_2)$ should reduce to (2).

**Postulate 2.** If all funds have the same values of their accounting units all the time, i.e. $w_i(t) = \dots w_n(t)$ for all $t \in [\mathrm{T}_1, \mathrm{T}_2]$, then

$$\bar{r}(T_1, T_2) = \frac{w_1(T_2) - w_1(T_1)}{w_1(T_1)}.$$

It means that if the unit's value changes in time in the same way in all funds then it does not matter if the clients alocate from a fund to another one or where the newcomers place themselves; their individual return rates will always be the same.

**Postulate 3.** If the number of units is constant at every fund during the time interval $[T_1, T_2]$, then

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} A_i(T_2) - \sum_{i=1}^{n} A_i(T_1)}{\sum_{i=1}^{n} A_i(T_1)} \tag{3}$$

Indeed, when none of the clients change the fund or come into or out of the business, then any change of the assets $A_i$ reflects only the investment results in the $i$-th fund. Treating all the unds as a solid one leads to the formula (3) then. Using the notation $k_i = k_i(t)$, we obtain from (3) that

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} r_i k_i w_i(T_1)}{\sum_{i=1}^{n} k_i w_i(T_1)} \tag{4}$$

where

$$r_i = \frac{w_i(T_2) - w_i(T_1)}{w_i(T_1)} \tag{5}$$

is the return rate of the $i$-th fund during the time period $[T_1, T_2]$. Clearly $r_i$ satisfies (1).

Postulate 3 implies

**Postulate 3'.** If $k_1(t) = \ldots = k_n(t) = k$ for every $t \in [T_1, T_2]$, then

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} r_i w_i(T_1)}{\sum_{i=1}^{n} w_i(T_1)} \tag{6}$$

Indeed, (4) implies (6).

Postulate 3 implies also

**Postulate 3''.** Assume that the number of units is constant at every fund during the time interval $[T_1, T_2]$, the initial assets (at $t = T_1$) of every fund have the same values and for some $k \leqslant n/2$, $r_1 = -r_{k+1}$, $r_2 = -r_{k+2}$, ... $r_k = -r_{2k}$, $r_{2k+1} = 0$, ..., $r_n = 0$. Then

$$\bar{r}(T_1, T_2) = 0.$$

Indeed, under the assumptions of Postulate 3'', the total assets of the group are constant and since the number of units does not change at any fund, the average return should be 0.

**Postulate 4 (Multiplication Rule).** For every $T \in [T_1, T_2]$ it should hold

$$1 + \bar{r}(T_1, T_2) = [1 + \bar{r}(T_1, T)][1 + \bar{r}(T, T_2)] \tag{7}$$

It means that the average return since $T_1$ until $T_2$ should equal the average return since $T$ until $T_2$ given the average return since $T_1$ to $T$. Clearly, the individual return rate $r_i$ defined by (5) satisfies (7).

**Postulate 5.** If there are numbers $n_1, n_2 \in \{1, 2, \ldots, n\}$ such that $\delta_{n_1}(t) \leqslant \delta_i(t) \leqslant \delta_{n_2}(t)$ for all $t \in [T_1, T_2]$ and every $i = 1, 2, \ldots, n$, then

$$\min_i r_i \leqslant \bar{r}(T_1, T_2) \leqslant \max_i r_i.$$

Clearly, $\min_i r_i = r_{n_1}$ and $\max_i r_i = r_{n_2}$.

Postulate 5 describes two extreme situations: all clients have chosen the best fund, or all have chosen the worst one. In both cases none of them alocate during the considered time period.

The next postulate takes into account that clients may change the fund when its return rate has changed comparing with other funds.

**Postulate 6.** It should hold

$$\exp\left[\int_{T_1}^{T_2}\min_i \delta_i(t)dt\right]-1\leqslant \bar{r}(T_1, T_2)\leqslant \exp\left[\int_{T_1}^{T_2}\max_i \delta_i(t)dt\right]-1.$$

Postulate 6 means that the average return rate $\bar{r}$ is not greater than the rate corresponding to the case all clients alocate at each $t\in[T_1, T_2]$ to the fund obtaining the highest return rate, and not smaller than the rate corresponding to the case all clients alocate to the fund obtaining the smallest return rate, respectively.

**Postulate 7.** Assume that $n\geqslant 2$ and $k_1(t)>0$, $k_i(t)=0$ for $i=2, ..., n$, $t\in[T_1, T_2-\Delta t]$, where $\Delta t>0$ is such that $T_2-\Delta t>T_1$. Then

$$\lim_{\Delta t\longrightarrow 0} \bar{r}(T_1, T_2)=r_1.$$

Similarly, if $k_i(t)=0$ for $i=2, ..., n$ and $t\in[T_1+\Delta t, T_2]$, then

$$\lim_{\Delta t\rightarrow 0} \bar{r}(T_1, T_2)=r_1.$$

It means that if all the clients were members of a one fund during almost all time then the average return rate would be approximately equal to the return rate of this fund.

The above postulates describe partly a kind of economical intuition and partly mathematical self consistency of any good definition of a weighted average return rate of a group of investment funds. In the Polish law regulations (*The Law on Organisation and Operation of Pension Funds*, "Dziennik Ustaw" nr 139 poz. 934, Art. 173; for the English translation, see: *Polish Pension...*, 1997) the following definition of the average return rate appears

$$\bar{r}_0(T_1, T_2)=\sum_{i=1}^{n}\frac{1}{2}r_i\left(\frac{A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_i(T_2)}{\sum_{i=1}^{n}A_i(T_2)}\right) \tag{8}$$

Unfortunately, $\bar{r}(T_1, T_2)$ defined by (6) does not satisfy Postulates 3, 3', 3", 4 and 7. In Section 2 we derive a definition of the average return rate basing on the integration of the financial results of the whole group of funds. The definition satisfies all the Postulates 1–7. In Section 3 we derive a simple formula

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} r_i \alpha_i w_i(T_i)}{\sum_{i=1}^{n} \alpha_i w_i(T_1)}$$

for the average return rate which is valid when the relative shares $k_i(t)/\sum_{i=1}^{n} k_i(t) \equiv \alpha_i$ are constant in time for $i = 1, ..., n$. We show that (8) always overestimates $\bar{r}(T_1, T_2)$ in that case.

## 2. DEFINITION OF THE AVERAGE RETURN RATE

Let $A(t)$ denote the total assets of the group at the moment $t \in [T_1, T_2]$, i.e.

$$A(t) = \sum_{i=1}^{n} k_i(t) w_i(t).$$

Assume that both $k_i(\cdot)$ and $w_i(\cdot)$ are differentiable functions. Then

$$\frac{d}{dt} A(t) = \sum_{i=1}^{n} \left( \frac{d}{dt} k_i(t) \right) w_i(t) + \sum_{i=1}^{n} k_i(t) \frac{d}{dt} w_i(t) \tag{9}$$

After rescaling (9) by the total assets, we get

$$\frac{\frac{d}{dt} A(t)}{A(t)} = \frac{\sum_{i=1}^{n} \left( \frac{d}{dt} k_i(t) \right) w_i(t)}{\sum_{i=1}^{n} k_i(t) w_i(t)} + \frac{\sum_{i=1}^{n} k_i(t) \frac{d}{dt} w_i(t)}{\sum_{i=1}^{n} k_i(t) w_i(t)} \tag{10}$$

The first sum on the right side of (10) corresponds to the influence on the total assets value of fluctuations of the number of units at each fund; the second sum corresponds to the influence of fluctuations of the unit's values. The second sum is corresponding only to the effects of investing the assets, not to alocating the clients between the funds or so. This is exactly what we are interested in when defining the average return rate. The second sum on the right side of (10) may be written as

$$\sum_{i=1}^{n} \frac{k_i(t)w_i(t)}{\sum_{i=1}^{n} k_i(t)w_i(t)} \delta_i(t),$$

where $\delta_i(t) = \frac{d}{dt}[\log w_i(t)]$, for $i = 1, ..., n$. Similarly as in Section 1, the infinitesimal return rate for the group of funds during the time $(t, t+dt)$ is equal to

$$\sum_{i=1}^{n} \frac{k_i(t)w_i(t)}{\sum_{i=1}^{n} k_i(t)w_i(t)} \delta_i(t)dt$$

and the weighted average return rate of the group, during a given time period $[T_1, T_2]$, is

$$\bar{r}(T_1, T_2) = \exp\left[\int_{T_1}^{T_2} \sum_{i=1}^{n} \frac{k_i(t)w_i(t)}{\sum_{i=1}^{n} k_i(t)w_i(t)} \delta_i(t)dt\right] - 1 \tag{11}$$

(compare (1)). From the economical point of view this is the main candidate to be used as the average return rate of the group.

### 3. BASIC PROPERTIES

**Proposition 1.** The average return rate $\bar{r}(T_1, T_2)$ defined by (11) satisfies all Postulates 1–7. Additionally, if $k_1(t)w_1(t) = ... = k_n(t)w_n(t)$, then:

$$1 + \bar{r}(T_1, T_2) = (1 + r_1)...(1 + r_n),$$

where $r_i$ are defined by (2).

Proof. Omitted.

From Postulate 3 we get a very simple and useful formula for the average return rate $\bar{r}(T_1, T_2)$ if the number of units $k_i(t)$ of the $i$-th fund does not change in time (i.e. $k_i(t) = k_i$):

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} r_i k_i w_i(T_1)}{\sum_{i=1}^{n} k_i w_i(T_1)}.$$

The following proposition shows the relationship between $\bar{r}(T_1, T_2)$ and $\bar{r}_0(T_1, T_2)$, defined by (8), in the case $k_i(t) = const.$

**Proposition 2.** Assume that $k_i(t) = k_i$, $i = 1, \ldots, n$. Then

$$\bar{r}_0(T_1, T_2) = \bar{r}(T_1, T_2) + \frac{D^2}{2(1+\bar{r})},$$

where

$$D^2 = \sum_{i=1}^{n} (r_i - \bar{r}(T_1, T_2))^2 \frac{k_i w_i(T_1)}{\sum_{i=1}^{n} k_i w_i(T_1)}$$

is the variance of return rates, corresponding to $\bar{r}(T_1, T_2)$.

**Proof.** Denote $\bar{r}(T_1, T_2)$ by $\bar{r}$ and $w_i(T_i)$ by $w_i$. Then

$$\bar{r}_0 = \frac{1}{2} \sum_{i=1}^{n} r_i \left[ \frac{k_i w_i}{\sum_{i=1}^{n} k_i w_i} + \frac{k_i w_i(1+r_i)}{\sum_{i=1}^{n} k_i w_i(1+r_i)} \right] =$$

$$= \bar{r} + \frac{1}{2} \sum_{i=1}^{n} r_i \frac{k_i w_i}{\sum_{i=1}^{n} k_i w_i} \left[ (1+r_i) \frac{\sum_{i=1}^{n} k_i w_i}{\sum_{i=1}^{n} k_i w_i + \sum_{i=1}^{n} r_i k_i w_i} - 1 \right] =$$

$$= \bar{r} + \frac{1}{2} \sum_{i=1}^{n} r_i \frac{k_i w_i}{\sum_{i=1}^{n} k_i w_i} \left[ (1+r_i) \frac{1}{1-\bar{r}1} - 1 \right] =$$

$$= \bar{r} + \frac{1}{2} \sum_{i=1}^{n} r_i \frac{k_i w_i(r_i - \bar{r})}{\sum_{i=1}^{n} k_i w_i(1+\bar{r})} =$$

$$= \bar{r} + \frac{1}{2(1+\bar{r})}\left[\sum_{i=1}^{n} r_i^2 \frac{k_i w_i}{\sum_{i=1}^{n} r_i k_i w_i} - (\bar{r})^2\right] = \bar{r} + \frac{D^2}{2(1+\bar{r})}.$$

**Corollary 1.** Under the assumptions of Proposition 2,

$$\bar{r}_0(T_1, T_2) > \bar{r}(T_1, T_2)$$

unless $r_1 = \ldots = r_n$. Hence the formula (8), used in the Polish pension fund law, overestimates the real average return rate.

**Example 1.** Assume that the group consists of $n = 10$ funds for which the return rates and the initial unit's values are as follows

| $i$ | $w_i$ | $r_i$ |
|---|---|---|
| 1 | 10 | 30% |
| 2 | 10 | 25% |
| 3 | 10 | 17% |
| 4 | 10 | 23% |
| 5 | 10 | 28% |
| 6 | 10 | 11% |
| 7 | 10 | 24% |
| 8 | 10 | 26% |
| 9 | 10 | 27% |
| 10 | 10 | 24% |

Assume that the number of units is constant during the considered period of time and $k_1 = \ldots = k_n$. Then the average return rate $\bar{r} = 23.500\%$ while the return rate defined in law is $\bar{r}_0 = 23.614\%$. Though the difference seems to be relatively small, it would result in a large amount of deficiency. After 5–6 years the assets in a typical pension fund in Poland will be larger than 4 bln PLN. Then a fund where the return for the last 24 months is lower than the minimum required return (11.0807%) is obliged to cover the resulting deficiency. Due to overestimating the financial results of the funds, the fund $N^0$ 6 would have to cover an additional $2.28 \cdot 10^6$ PLN of deficiency. All that concerns a very typical situation but what would happen if some of the funds had very bad financial results. Suppose for instance that $r_3 = -50\%$ $r_4 = -70\%$. Then $\overline{r} = 7.5000\%$ while $\overline{r}_0 = 12.99977\%$, hence the overestimating the financial results of the group is $\overline{r}_0 - \overline{r} = 5.49977\%$. In that case the additional deficiency to be covered by a fund, due to a wrong definition, would be $110.00 \cdot 10^6$ PLN. The largest differences between $r_i$, the more strange values $\overline{r}_0$ produces. If, for instance, five of the funds have the return rates equal to 50% and the rest five equal to $-50\%$, then the real average return rate is 0%, because the total assets after two years are the same. However, the definition used in law gives $\overline{r}_0 = 12.5\%$.

The next proposition provides a simple formula for the average return rate in the case $k_i(t)$ are not constant in time but the relative share of each fund is constant.

**Proposition 3.** Assume that there are a function $\varphi : [T_1, T_2] \to \mathbf{R}_+$ and reals $\alpha_i \geqslant 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and

$$k_i(t) = \alpha_i \varphi(t)$$

for (almost) all $\in [T_1, T_2]$, $i = 1, \ldots, n$. Then

(*i*)

$$\overline{r}(T_1, T_2) = \frac{\sum_{i=1}^{n} r_i \alpha_i w_i(T_1)}{\sum_{i=1}^{n} \alpha_1 w_1(T_1)} \tag{12}$$

(*ii*)

$$\overline{r}_0(T_1, T_2) = \overline{r}(T_1, T_2) + \frac{D^2}{2(1+\overline{r})},$$

where

$$D^2 = \sum_{i=1}^{n} (r_i - \overline{r}(T_1, T_2))^2 \frac{\alpha_i w_i(T_1)}{\sum_{i=1}^{n} \alpha_i w_i(T_1)}.$$

**Proof.** (i) Observe that

$$\int_{T_1}^{T_2} \sum_{i=1}^{n} \frac{k_i(t) w_i(t)}{\sum_{i=1}^{n} k_i(t) w_i(t)} \delta_i(t) dt = \int_{T_1}^{T_2} \sum_{i=1}^{n} \frac{\alpha_i w_i'(T_1)}{\sum_{i=1}^{n} \alpha_i w_i(T_1)} dt =$$

$$= \int_{T_1}^{T_2} \frac{d}{dt} \log\left(\sum_{i=1}^{n} \alpha_i w_i(T_1)\right) dt = \log\left(\sum_{i=1}^{n} \alpha_i w_i(T_2)\right) - \log\left(\sum_{i=1}^{n} \alpha_i w_i(T_1)\right).$$

Using (11) and the equality $r_i = w_i(T_2)/w_i(T_1) - 1$, we get (i). To prove (ii), observe that

$$\bar{r}_0 = \frac{1}{2}\sum_{i=1}^{n} r_i \left[\frac{k_i(T_1)w_i(T_1)}{\sum_{i=1}^{n} k_i(T_1)w_1(T_1)} + \frac{k_i(T_2)w_i(T_1)(1+r_i)}{\sum_{i=1}^{n} k_i(T_2)w_i(T_1)(1+r_i)}\right] =$$

$$= \frac{1}{2}\sum_{i=1}^{n} r_i \left[\frac{k_i(T_1)\alpha_i}{\sum_{i=1}^{n} k_i(T_1)\alpha_1} + \frac{\alpha_i w_i(T_1)(1+r_i)}{\sum_{i=1}^{n} \alpha_i w_i(T_1)(1+r_i)}\right].$$

The rest of the proof is similar to the proof of Proposition 2. □

Observe that $\alpha_i$ from Proposition 3 satisfy the equations

$$\alpha_i = k_i(t) / \sum_{i=1}^{n} k_i(t), \quad i = 1, \ldots, n.$$

Hence a very simple formula (12) may be used as a definition of the average return rate even when $k_i(t)$ are not constant in time, provided the relative share of each fund in the total number of units is constant.

## REFERENCES

*Security through Diversity* (1997), Office of the Government Plenipotentiary for Social Security Reform.

*Polish Pension Reform Package* (1997), Office of the Government Plenipotentiary for Social Security Reform.