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MULTIVALUED STOP-LOSS STOCHASTIC DOMINANCE TEST

Abstract. Stochastic Dominance tests can be employed to assist decision-makers in ordering uncertain alternatives. Thes tests require specification of alternatives probability distributions and the assumption of the utility function of the decision-maker. With these assumptions, decision alternatives can be partitioned into classes by stochastic dominance or inverse stochastic dominance (stop-loss dominance). This paper notices procedures to identify this class of alternatives in case of multivalued probability distributions.

1. INTRODUCTION

Stochastic Dominance tests can be employed to assist decision-makers in ordering uncertain alternatives. These tests require specification of alternative probability distributions and the assumption of the utility function of the decision-maker. With these assumptions decision alternatives can be partitioned into class by stochastic dominance or inverse stochastic dominance (stop-loss dominance) depending on describing model for gains or for looses. This paper notices procedures to identify this class of alternatives in case of multivalued probability distributions.

2. MULTIVALUED STOCHASTIC DOMINANCE

Let F and G be the cumulative distributions of two distinct uncertain alternatives X and Y. X dominates Y by first, second and third stochastic dominance (FSD, SSD, TSD) if and only if

$$H_1(x) = F(x) - G(x) \leqslant 0 \quad \text{for all} \quad x \in [a, b] \ (X \text{ FSD } Y) \tag{1}$$

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$$H_2(x) = \int_a^x H_1(y)dy \leqslant 0 \quad \text{for all} \quad x \in [a, b] \ (X \text{ SSD } Y) \tag{2}$$

$$H_3(x) = \int_z^x H_2(y)dy \leqslant 0 \quad \text{for all} \quad x \in [a, b], \quad \text{and } E(F(x)) \geqslant E(G(x))(X \text{ TSD } Y) \tag{3}$$

The relationship between the three stochastic dominance rules can be summarised by the following diagram: FSD ⇒ SSD ⇒ TSD, which means that dominance by FSD implies dominance by SSD and dominance by SSD in turn implies dominance by TSD. For proof of FSD and SSD see Hadar and Russell [1969], Hanoch and Levy [1969] and Rothschild and Stiglitz [1970]. The criterion for TSD was suggested by Whitmore [1970].

When we have ambiguities in probabilities and outcomes, we have no single-valued distribution, such a situation can be represented by a set of probability distributions. Each family has two extreme probability distributions the scalar outcome space X.

Definition 1. Lower probability distributions for all values $x_i \in X$, are denoted by

$$p^*(x_i) = \sum_{j: x_i = \min\{y: y \in A_j\}} p(A_j) \tag{4}$$

According to this definition we have: $\sum_i p^*(x_i) = 1$.

Definition 2. Upper probability distributions for all values $x_i \in X$, are denoted by

$$p^*(x_i) = \sum_{j: x_i = \max\{y: y \in A_j\}} p(A_j) \tag{5}$$

Now we also have: $\sum_i p^*(x_i) = 1$.

In case of the point values of random variables both distributions (lower and upper probability distributions) are exactly the same: $p_*(x_i) = p^*(x_i) = p(x_i)$ and we have probability distributions in classical sense.

Example 1. We determine lower and upper probability distributions for random variable **X**, whose outcomes are multivalued, included in some intervals:

A_j	[2, 4]	[3, 4]	[4, 5]	[5, 6]
$p(A_j)$	0.2	0.5	0.2	0.1

According to the Definitions 1 and 2 we have lower and upper probability distributions for random variable **X**:

x_j	2	3	4	5	6
$p_*(x_j)$	0.2	0.5	0.2	0.1	–
$p^*(x_j)$	–	–	0.7	0.2	0.1

When we have lower and upper probability distributions for random variables, whose outcomes are multivalued, we can rank such a stochastic alternative.

Let two distinct uncertain multivalued alternatives **X** and **Y** have lower cumulative distributions $F_*(x)$ and $G_*(x)$ respectively, upper cumulative distributions $F^*(x)$ and $G^*(x)$, for $x \in [a, b]$ respectively, then we have multivalued first, second and third stochastic dominance if and only if

$$H_1(x) = F_*(x) - G^*(x) \leqslant 0, \quad \text{for all} \quad x \in [a, b], \ (\mathbf{X}\,\mathrm{FSD}\,\mathbf{Y}) \tag{6}$$

$$H_2(x) = \int_a^x H_1(y)dy \leqslant 0, \quad \text{for all} \quad x \in [a, b], \ (\mathbf{X}\,\mathrm{SSD}\,\mathbf{Y}) \tag{7}$$

$$H_3(x) = \int_a^x H_2(y)dy \leqslant 0, \quad \text{for all} \quad x \in [a, b], \ (\mathbf{X}\,\mathrm{TSD}\,\mathbf{Y}) \\ \text{and} \quad E(F_*(x)) \geqslant E(G^*(x)) \tag{8}$$

For proof see Langewisch and Choobineh [1996].

Example 2. (Trzpiot, 1998) Let us take a random variable **C** and **D** whose outcomes are multivalued, included in some intervals as follows:

A_1	[0, 1]	[1, 2]	[2, 3]	[3, 4]
$p(\mathbf{C})$	–	0.2	0.4	0.4
$p(\mathbf{D})$	0.3	0.15	0.55	–

We determine lower and upper probability distributions for random variables **C** and **D** as follows:

x_j	0	1	2	3	4
C: $p_*(x_j)$	–	0.2	0.4	0.4	–
C: $p^*(x_j)$	–	–	0.2	0.4	0.4
D: $p_*(x_j)$	0.3	0.15	0.55	–	–
D: $p^*(x_j)$	–	0.3	0.15	0.55	–

Now we can receive the values of lower and upper cumulative distributions (see Fig. 1).

x_j	$(-\infty, 0]$	(0, 1]	(1, 2]	(2, 3]	(3, 4]	$(4, \infty)$
$C_*(x_j)$	0	0	0.2	0.6	1	1
$C^*(x_j)$	0	0	0	0.2	0.6	1
$D_*(x_j)$	0	0.3	0.45	1	1	1
$D^*(x_j)$	0	0	0.3	0.45	1	1

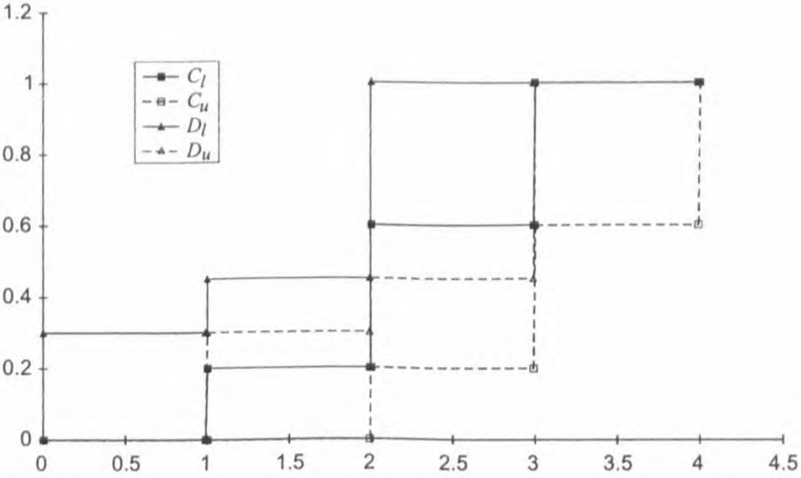

Fig. 1. Upper and lower distributions of C and D

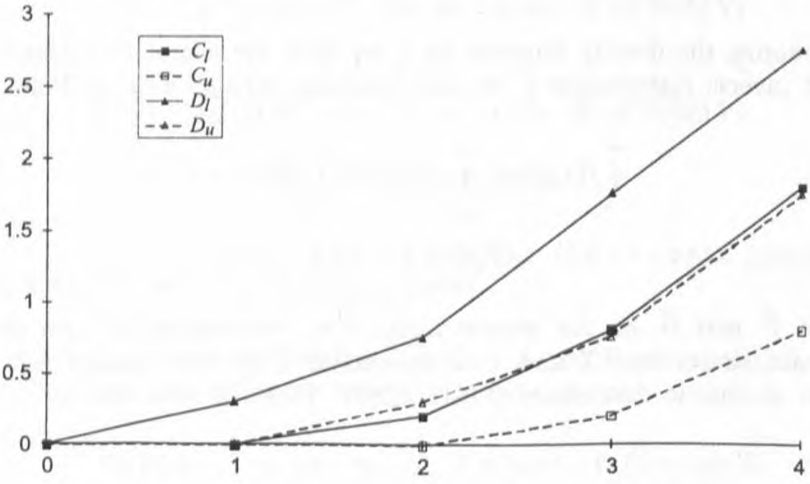

Fig. 2. Upper and lower integrals of cumulated distributions of C and D

It is easy to check, that the Definition 1 and the Definition 2 are not true (see Fig. 1 and. Fig. 2).

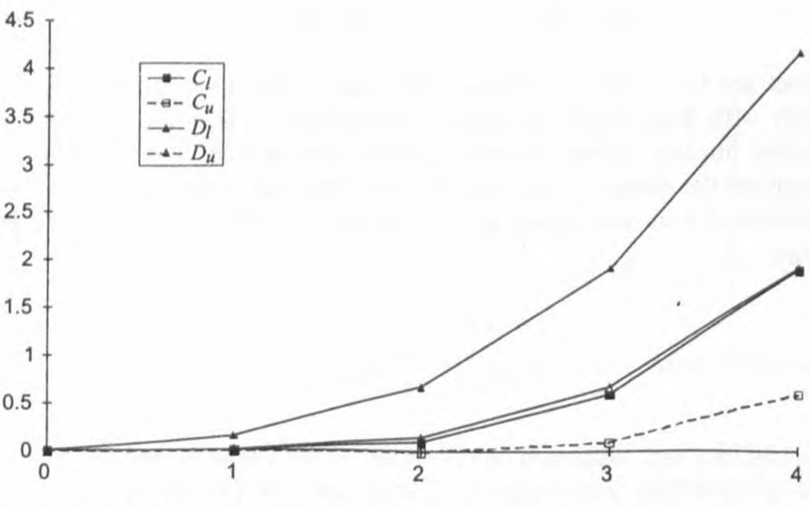

Fig. 3. Upper and lower double integrals of cumulated distributions of C and D

In this example we can establish a third degree multivalued stochastic dominance: **C** TSD **D** (see Fig. 3.).

3. MULTIVALUED INVERSE STOCHASTIC DOMINANCE

Denoting the density function by f, we have the cumulative distribution F and inverse distribution $\overline{F}$ by the following (Goovaerts, 1984).

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} dF(x) = \int_{-\infty}^{\infty} d\overline{F}(x) = 1 \tag{9}$$

$$\overline{F}(x) = 1 - F(x) \tag{10}$$

Let $\overline{F}$ and $\overline{G}$ be the inverse cumulative distributions of two distinct uncertain alternatives X and Y. X dominates Y by first, second and third inverse stochastic dominance (FISD, SISD, TISD) if and only if

$$\overline{H}_1(x) = \overline{F}(x) - \overline{G}(x) \geqslant 0 \quad \text{for all} \quad x \in [a, b] \ (X \, \text{FISD} \, Y) \tag{11}$$

$$\overline{H}_2(x) = \int_x^b \overline{H}_1(y)dy \geqslant 0 \quad \text{for all} \quad x \in [a, b] \ (X \, \text{SISD} \, Y) \tag{12}$$

$$\overline{H}_3(x) = \int_x^b \overline{H}_2(y)dy \geqslant 0 \quad \text{for all} \quad x \in [a, b]$$

$$\text{and} \quad E(F(x)) \geqslant E(G(x)) \ (\mathbf{X} \, \text{TISD} \, \mathbf{Y}) \tag{13}$$

For proof see Goovaerts [1984]. First degree stop-loss (inverse) dominance coincides with first degree stochastic dominance. The n-th degree stop-loss dominance implies higher degree stop-loss dominance.

Denoting the density function by f, we have the lower inverse probability distributions $\overline{F}_*(x)$ and upper inverse probability distributions $\overline{F}^*(x)$ by the following:

$$\overline{F}_*(x) = 1 - F_*(x) \tag{14}$$

$$\overline{F}^*(x) = 1 - F^*(x) \tag{15}$$

Let two distinct uncertain multivalued alternatives **X** and **Y** have lower inverse probability distributions $\overline{F}_*(x)$ and $\overline{G}_*(x)$ respectively, upper inverse probability distributions $F^*(x)$ and $\overline{G}^*(x)$, for $x \in [a, b]$ respectively, then we have multivalued first, second and third inverse stochastic dominance if and only if

$$\overline{H}_1(x) = \overline{F}_*(x) - \overline{G}^*(x) \geqslant 0 \quad \text{for all} \quad x \in [a, b] \ (\mathbf{X} \, \text{FISD} \, \mathbf{Y}) \tag{16}$$

$$\overline{H}_2(x) = \int_x^b \overline{H}_1(y)dy \geqslant 0 \quad \text{for all} \quad x \in [a,\ b]\ (\mathbf{X}\ \mathbf{SISD}\ \mathbf{Y}) \tag{17}$$

$$\overline{H}_3(x) = \int_x^b \overline{H}_2(y)dy \leqslant 0 \quad \text{for all} \quad x \in [a,\ b]\ (\mathbf{X}\ \mathbf{TISD}\ \mathbf{Y})$$

$$\text{and } E(F(x)) \geqslant E(G(x)) \tag{18}$$

The proof proceeds parallel to provided by Goovaerts [1984] and Langewisch and Choobineh [1996].

Example 3. Let us take random variables **A** and **B** whose outcomes are multivalued, included in some intervals as follows:

A_j	[−6, −5]	[−5, −2]	[−5, −1]	[−4, −3]	[−1, 0]
$p(\mathbf{A})$	–	–	–	0.5	0.5
$p(\mathbf{B})$	0.25	0.6	0.15	–	–

We determine lower and upper probability distributions for random variables **A** and **B** as follows:

x_j	−6	−5	−4	−3	−2	−1	0
A: $p_*(x_j)$	–	–	0.5	–	–	0.5	–
A: $p^*(x_j)$	–	–	–	0.5	–		0.5
B: $p_*(x_j)$	0.25	0.75	–	–	–	–	–
B: $p^*(x_j)$	–	0.25	–	–	0.6	0.15	–

Now we can receive the values of lower and upper cumulative distributions.

x_j	$(-\infty, -6]$	(−6, −5]	(−5, −4]	(−4, −3]	(−3, −2]	(−2, −1]	(−1, 0]	$(0, \infty)$
$A_*(x_j)$	0	0	0	0.5	0.5	0.5	1	1
$A^*(x_j)$	0	0	0	0	0.5	0.5	0.5	1
$B_*(x_j)$	0	0.25	1	1	1	1	1	1
$B^*(x_j)$	0	0	0.25	0.25	0.25	0.85	1	1

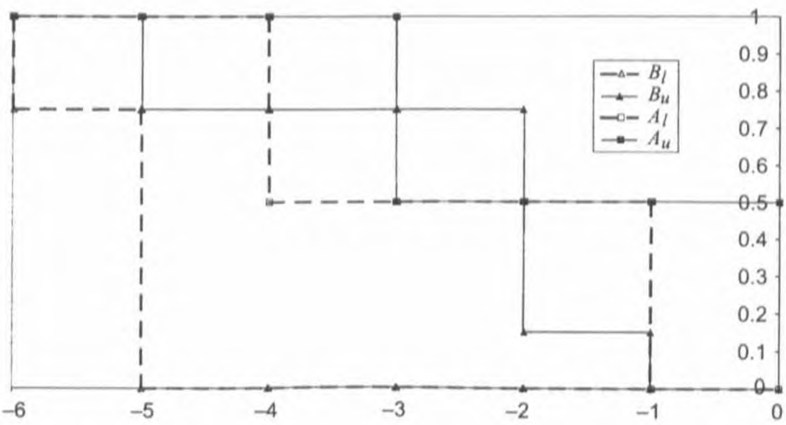

Fig. 4. Upper and lower inverse cumulated distributions of A and B

It is easy to check that the Definition 1 is not true. We determine lower and upper inverse cumulative distributions for random variables **A** and **B** as follows (Fig. 4):

x_j	(–∞, –6]	(–6, –5]	(–5, –4]	(–4, –3]	(–3, –2]	(–2, –1]	(–1, 0]	(0, ∞)
$\overline{A}_*(x_j)$	1	1	1	0.5	0.5	0.5	0	0
$\overline{A}^*(x_j)$	1	1	1	1	0.5	0.5	0.5	0
$\overline{B}_*(x_j)$	1	0.75	0	0	0	0	0	0
$\overline{B}^*(x_j)$	1	1	0.75	0.75	0.75	0.15	0	0

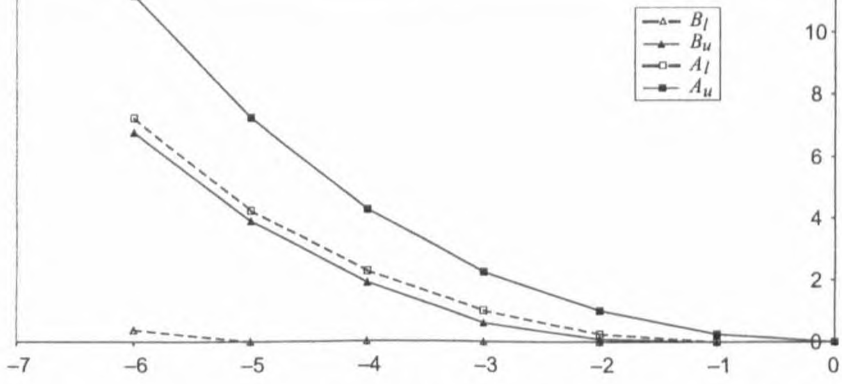

Fig. 5. Upper and lower double integrals of inverse cumulated distributions of A and B

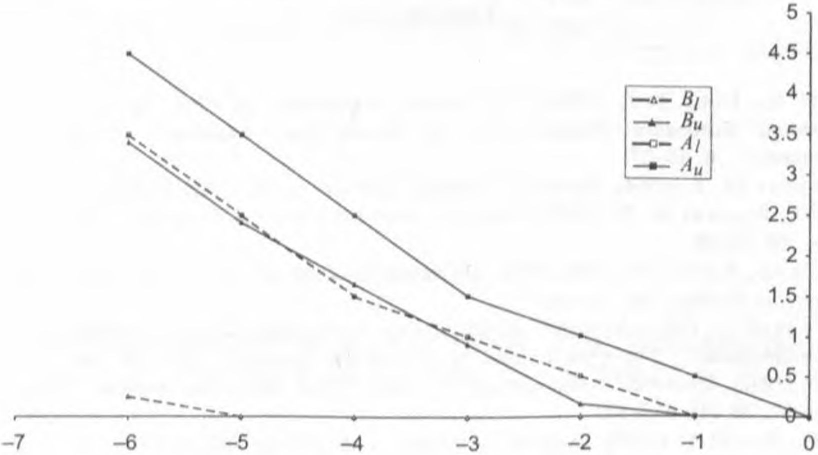

Fig. 6. Upper and lower integrals of inverse cumulated distributions of A and B

We can observe that the formulas (16) and (17) are not true (Fig. 4, 5). In this example we can establish a third degree multivalued stochastic dominance: **A** TISD **B** (see Fig. 6).

4. CONCLUSIONS

Stop-loss dominance ranking is an extension of stop-loss ordering in analogy to the stochastic dominance. We can apply two different ranking by stochastic dominance or stop-loss dominance which depends on our goals, whether we look for a good criterion for maximising gains or for minimising losses. Sometimes we need both and this is a case of operation research problem. The current use of stochastic dominance assumes that probability distributions are known and unique. In empirical study specifying unique probability may be unjustifiable, so we need multivalued view of the problem. Simple example is the continuous observation of assets, each day we have price from min to max price of the day. When we want to rank these assets we should apply multivalued stochastic dominance or multivalued inverse (stop-loss) stochastic dominance depending on our problem. The extended multivalued stochastic dominance to multivalued inverse (stop-loss) stochastic dominance, established here, provide a valuable technique to pare down the number of alternatives.

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