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SOME CONSTRUCTION OF OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS

Abstract: The paper gives certain new construction method for optimum chemical balance weighing designs. It utilizes a relation between the incidence matrices of two group divisible designs with the same association scheme and the design matrix of a chemical balance weighing design.

Key words: chemical balance weighing design, group divisible design.

I. INTRODUCTION

Suppose we are given p objects to be weighed in n weighings on a chemical balance. The design matrix $\mathbf{X}$ has the elements –1, 1 or 0 if the object is placed on the left pan, right pan, or is not included in the particular weighing, respectively. If $\mathbf{X}$ is full column rank, the least squares estimates of the true weights are given by

$$\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} \tag{1}$$

with the variance matrix

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \sigma^2\mathbf{C} = \sigma^2\left(c_{jj'}\right) \tag{2}$$

where $\mathbf{w}$ is the column vector of unknown weights of p objects and $\mathbf{y}$ is the column vector of results in the n weighings. From Hotelling (1944) we have

Definition 1. Any nonsingular weighing design is said to be optimum if $c_{jj} = \frac{1}{n}$ and $c_{jj'} = 0,\ \ j, j' = 1,2,...,p,\ j \neq j'$.

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The problem is to choose the design matrix **X** in such a way that the variance factors are minimized. Several methods of construction of the design matrix **X** are available in the literature. Dey (1971), Saha (1975), Kageyama and Saha (1983), Ceranka and Katulska (1988) and others have shown how the optimum chemical balance weighing design can be constructed from the incidence matrices of the balanced incomplete block designs.

In the present paper we study a method of construction of the design matrix **X** for a chemical balance weighing design for a fixed number p of objects and a fixed number n of weighings. It is based on two incidence matrices of the group divisible designs with the same association scheme under the some condition.

II. CONSTRUCTION OF THE DESIGN MATRIX

An incomplete block design is said to be partially balanced with two associate classes if the experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block, there are $v\,(>k)$ treatments and each occurs in r blocks. There can be establish a relation of association between any two treatments satisfying the following requirements: two treatments are either first associates or second associates, each treatment has exactly n_i ith associates, $i=1,2$. For any two treatments which are ith associates the number of treatment common to the j th associate of the first and the kth associate of the second is p_{jk}^{i} and its independent of the pair of treatments we start with. Also $p_{jk}^{i}=p_{kj}^{i},\ i,j,k=1,2.$ Two treatments which are ith associate occur together in exactly λ_i blocks. The numbers $v, b, r, k, n_1, n_2, \lambda_1, \lambda_2$ are called the parameters of the first kind, whereas p_{jk}^{i} are called the parameters of the second kind. A group divisible design is a partially balanced incomplete block design with two associate classes for which the v treatments may be divided into m groups of q distinct treatments each such that treatments belonging to the same group are first associates and two treatments belonging to different groups are second associates. We have $v=mq,\quad n_1=q-1,\quad n_2=q(m-1)\ \ (q-1)\lambda_1+(m-1)\lambda_2=r(k-1).$

Consider two group divisible designs with the parameters $v, b_t, r_t, k_t, \lambda_{1t}, \lambda_{2t},\quad t=1,2$. We assume that association scheme is the same and additionally $\lambda_{11}+\lambda_{12}=\lambda_{21}+\lambda_{22}=\lambda$. Let $\mathbf{N}_t$ denote the binary matrix of order $v\times b_t,\quad t=1,2$. Then under our assumption

$$\mathbf{N}_1\mathbf{N}_1^{'} + \mathbf{N}_2\mathbf{N}_2^{'} = (r_1 + r_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v^{'}, \tag{3}$$

where $\mathbf{I}_v$ denotes the identity matrix of order v and $\mathbf{1}_v$ denotes the $v \times 1$ vector with the unit elements everywhere. Then

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} \\ 2\mathbf{N}_2^{'} - \mathbf{1}_{b_2}\mathbf{1}_v^{'} \end{bmatrix} \tag{4}$$

is the design matrix of a chemical balance weighing design. In this design $p = v$ and $n = b_1 + b_2$.

Theorem 1. For a nonsingular chemical balance weighing design with $\mathbf{X}$ given by (4) estimated weights are uncorrelated if and only if

$$b_1 + b_2 = 4(r_1 + r_2 - \lambda). \tag{5}$$

Proof. For the design matrix $\mathbf{X}$ given by (4) we have

$$\mathbf{X}^{'}\mathbf{X} = (b_1 + b_2 - a)\mathbf{I}_v + a\mathbf{1}_v\mathbf{1}_v^{'}, \tag{6}$$

where $a = b_1 + b_2 - 4(r_1 + r_2 - \lambda)$. From (6) we have

$$(\mathbf{X}^{'}\mathbf{X})^{-1} = \frac{1}{b_1 + b_2 - a}\left[\mathbf{I}_v - \frac{a}{b_1 + b_2 + a(v-1)}\mathbf{1}_v\mathbf{1}_v^{'}\right]. \tag{7}$$

Taking into consideration (7) we have $\mathrm{Cov}(\hat{w}_j, \hat{w}_{j'}) = 0$ for $j, j' = 1,2,...,v$, $j \neq j'$ if and only if $a = 0$. But, from its definition $a = 0$ if and only if $b_1 + b_2 = 4(r_1 + r_2 - \lambda)$. Hence the Theorem.

If estimated weights are uncorrelated then $\mathrm{Var}(\hat{w}_j) = \dfrac{\sigma^2}{b_1 + b_2}$, $j = 1,2,...,v$. Thus from Definition 1 and Theorem 1 we have.

Corollary 1. If the condition (5) holds then a chemical balance weighing design with $\mathbf{X}$ given by (4) is optimal.

III. GROUP DIVISIBLE DESIGNS LEADING TO OPTIMUM WEIGHING DESIGNS

We have seen in Corollary 1 that if the parameters of two group divisible designs satisfy the condition (5) then a chemical balance weighing design with $\mathbf{X}$ given by (4) is optimal. Under this condition the list of all parameters combinations of the existing group divisible designs with the same association scheme and for $\lambda_{11}+\lambda_{12}=\lambda_{21}+\lambda_{22}=\lambda$, in the range $v\leq 90$, $b_t\leq 100$, $r_t,k_t\leq 10$, $t=1,2$ (Clatworthy, 1973) is given in Table 1.

Table 1.

Weighing design		First group divisible design							Second group divisible design						
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
p	n	References number	v	b	r	k	λ_1	λ_2	References number	v	b	r	k	λ_1	λ_2
8	12	S18	8	4	3	6	3	2	R54	8	8	3	3	0	1
8	12	S18	8	4	3	6	3	2	R134	8	8	5	5	2	3
8	16	S19	8	8	6	6	6	4	SR36	8	8	4	4	0	2
8	24	S19	8	8	6	6	6	4	R55	8	16	6	3	0	2
8	24	S19	8	8	6	6	6	4	R136	8	16	10	5	4	6
8	24	S20	8	12	9	6	9	6	SR37	8	12	6	4	0	3
8	24	SR38	8	12	6	4	2	3	R164	8	12	9	6	7	6
8	36	S20	8	12	9	6	9	6	R57	8	24	9	3	0	3
8	36	R58	8	24	9	3	2	3	R164	8	12	9	6	7	6
9	12	S21	9	3	2	6	2	1	SR23	9	9	3	3	0	1
9	12	S21	9	3	2	6	2	1	SR65	9	9	6	6	3	4
9	24	S21	9	3	2	6	2	1	R62	9	21	7	3	1	2
9	24	S22	9	6	4	6	4	2	SR24	9	18	6	3	0	2
9	24	SR23	9	9	3	3	0	1	R59	9	15	5	3	2	1
9	24	SR23	9	9	3	3	0	1	R165	9	15	10	6	7	6
9	24	SR65	9	9	6	6	3	4	R59	9	15	5	3	2	1
9	24	SR65	9	9	6	6	3	4	R165	9	15	10	6	7	6
9	36	S23	9	9	6	6	6	3	SR25	9	27	9	3	0	3
9	36	SR23	9	9	3	3	0	1	R65	9	27	9	3	3	2
9	36	SR24	9	18	6	3	0	2	R60	9	18	6	3	3	1
9	36	SR65	9	9	6	6	3	4	R65	9	27	9	3	3	2
9	36	R59	9	15	5	3	2	1	R62	9	21	7	3	1	2
9	36	R62	9	21	7	3	1	2	R165	9	15	10	6	7	6
9	48	SR24	9	18	6	3	0	2	R67	9	30	10	3	4	2
9	48	R62	9	21	7	3	1	2	R65	9	27	9	3	3	2
9	48	SR25	9	27	9	3	0	3	R61	9	21	7	3	4	1
9	48	R60	9	18	6	3	3	1	R68	9	30	10	3	1	3
12	16	S82	12	4	3	9	3	2	R145	12	12	5	5	1	2
12	16	S82	12	4	3	9	3	2	R176	12	12	7	7	3	4

Table 1 (cont.)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
12	24	S27	12	6	3	6	3	1	SR42	12	18	6	4	0	2
12	32	SR66	12	8	4	6	0	2	R110	12	24	8	4	4	2
12	36	R110	12	24	8	4	4	2	R144	12	12	5	5	0	2
12	36	R110	12	24	8	4	4	2	R175	12	12	7	7	2	4
18	24	S37	18	12	4	6	4	1	SR99	18	12	6	9	0	3
18	36	SR72	18	18	6	6	0	2	R196	18	18	9	9	6	4
20	36	SR167	20	16	8	10	0	4	R180	20	20	7	7	6	2

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PEWNE KONSTRUKCJE OPTYMALNYCH CHEMICZNYCH UKŁADÓW WAGOWYCH

W pracy podano nową metodę konstrukcji optymalnych chemicznych układów wagowych wykorzystujące relacje pomiędzy macierzami incydencji częściowo zrównoważonych układów o blokach niekompletnych z podzielnymi grupami o takich samych schematach partnerstwa a macierzą chemicznego układu wagowego.