A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA OECONOMICA 235, 2010

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COHERENT RISK MEASURES IN MULTIPERIOD MODELS

Abstract. The framework of coherent risk measures has been introduced by Artzner et. al. (1999) in a single-period setting. Here we present a similar model in a multiperiod context. We add in axiom of dynamic consistency to the standard coherence axioms. We describe a set of property of multiperiod model. We present recursive formulas for the computation of price bounds and corresponding optimal hedges. We present a recursive formula for price bounds in terms of choosing risk measures.

Key words: coherent risk measures, multiperiod model.

I. INTRODUCTION

The seminal work of Artzner et al. (1999) on coherent risk measures is focused primarily on supervisory applications. The axioms for acceptability that were put forward by Artzner et al. differ from the ones that have been used traditionally in statistical decision theory. Yet it is clear that the regulatory issues can be formulated as decision problems, and moreover there are many other situations where both the classical axioms of decision theory and the coherence axioms adopted by Artzner et al. can be taken into consideration. One may for instance think of capital budgeting decisions, the determination of premium for insurance contracts, and the pricing of derivatives in incomplete markets.

The coherence axioms in Artzner et al. (1999) lead to risk measures of the form

$$\rho(X) = \sup_{\mathbf{P}\in P} E(X) \tag{1.1}$$

where X is a function from a (finite) set to R, and P is a class of probability measures on Ω. Actually, the paper Artzner et al. (1999) uses *risk measures* rather than *acceptability measures.* We changed the terminology here to facilitate comparison with common formulations in decision theory, and also to stress the applicability of the same notion in different contexts. The difference

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between the risk measures $\rho(X)$ of Artzner et al. (1999) and the acceptability measures $\phi(X)$ used here is just a change of sign ($\phi(X) = -\rho(X)$). For simplicity reasons we change discounting. Artzner et al. (2003) use the term "risk adjusted values" for what we call acceptability measures. We may compare this notation to the criterion obtained from a set of decision-theoretic axioms. This criterion corresponds to a utility specification of the form

$$U(X) = \sup_{\mathbf{P}\in P} E(u(X)) \qquad (1.2)$$

where X and P are defined as above, and where u(·) is a utility function. The interpretation given by Gilboa and Schmeidler to the class of probability measures P, and which their axiom system is designed to reflect, is that this class represents ambiguity in the sense of Ellsberg (1961). There is no explicit appeal to ambiguity in Artzner et al. (1999).

II. SINGLE-PERIOD SETTING

Here we briefly review the axiomatic setting for risk measures that was proposed by Artzner et al. (1999), with some (simple) modifications: acceptability measures can be used instead of risk measures, and we do not consider discounting. Let Ω be a finite set, say with n elements. The set of all functions from Ω to **R** will be denoted by **X**(Ω). An element X of **X**(Ω) is thought of as a representation of the position that generates outcome $X(\omega)$ when the state $\omega \in \Omega$ arises. An *acceptability measure* defined on is a mapping from **X**(Ω) to **R**.

The number $\rho(\boldsymbol{X})$ that is associated to the position $X \in \mathbf{X}(\Omega)$ by an acceptability measure ϕ is interpreted as the "degree of acceptability" of the position X. An acceptability measure ϕ is said to be coherent if it satisfies the four axioms, similar to coherent risk measures.

Definition 2.1.

An *risk measure* ρ: $L^2 \rightarrow (-\infty,\infty)$ is said to be coherent if it satisfies the following four axioms [1]

a) translation property:for all $c \in R$, $\rho(X + c\cdot r) = \rho(X) - c$
b) superadditivity:$\rho(X + Y) \leq \rho(X) + \rho(Y)$, for all $X, Y \in L^2$
c) monotonicity: $X, Y \in L^2$, if $X \leq Y$, then $\rho(Y) \leq \rho(X)$
d) positive homogeneity:$X \in L^2$ and $\lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$

[1] $L^2 = L^2(\Omega, A, P)$ for probability space (Ω, A, P)

Theorem 2.1 (Artzner et al. (1999)

An risk measure ρ defined on a finite set is coherent if and only if there exists a family P of probability measures on such that, for all $\boldsymbol{X} \in \mathbf{X}(\Omega)$

$$\rho(\boldsymbol{X}) = \sup_{\mathbf{P} \in P} E(X) \tag{2.1}$$

If an acceptability measure ϕ satisfies (2.1), it is said to be represented by the family P of probability measures, and the probability measures in the collection P are sometimes referred to as "test measures" for ϕ. The above theorem can be generalized to the case of infinite sample spaces Ω if either the representation by probability measures is replaced by a representation in terms of finitely additive measures, or a continuity property is added to the coherence axioms (see Delbaen (2002).

If ϕ is a coherent acceptability measure, then $\phi(c\mathbf{1}) = c$ for all $c \in R$; this follows from the axiom of positive homogeneity (which implies $\phi(0) = 0$) and from the translation property. For convenience we introduce a separate term for this property.

Definition 2.2 An acceptability measure ϕ on a sample space Ω is said to be normalized if $\phi(c\mathbf{1}) = c$ for all $c \in R$.

In particular, if the set Ω consists of only one element ω, then the only normalized acceptability measure is $\phi(X) = X(\omega)$. The case in which all uncertainty has been resolved will be used below in the multiperiod context as a starting point for backward recursions.

III. MULTIPERIOD SETTING

We now pass to a multiperiod setting. To keep the context as simple as possible we still work with a finite sample space (following Artzner et al. (1999) and Carr et al. (2001)), but we consider each sample now as a discrete-time trajectory. We begin with introducing some notation and terminology that will be needed below.

Let T be a positive integer indicating the number of time periods over which we consider our economy. Let A be a finite set which we shall refer to as the "event set." This terminology is appropriate in particular for tree models; for instance in binomial models the event set consists of two elements ("up" and "down"). The framework that we use below applies equally well however to models obtained from discretization of a continuous state space, where A would rather be thought of as a representation of a grid in the state space.

Define Ω as the set of all sequences $(\alpha_1, \ldots, \alpha_T)$ with $\alpha_i \in A$; we refer to such sequences as "full histories." The collection of sequences $(\alpha_1, \ldots, \alpha_T)$ of length t $(1 \le t \le T)$ will be denoted by Ω_t .

For $\omega = (\alpha_1, \ldots, \alpha_T) \in \Omega$ and $1 \le t \le T$, define the *t - restriction* $\omega\big|_t$ as $(\alpha_1, \ldots, \alpha_t)$. If a sequence $\omega' = (\alpha_1, \ldots, \alpha_t)$ is a *prefix* of $\omega \in \Omega$ we write $\omega' \prec \omega$. The collection of all sequences beginning with a given sequence $F(\omega')$ is denoted by $F(\omega') := \{ \omega \in \Omega \mid \omega' \prec \omega \}$. We denote by $\mathbb{F}_t$ the algebra generated by the sets $F(\omega')$ with ω' in the set Ω_t of sequences of length exactly t ; in the present setting in which we have a finite sample space, this is of course the same as the σ-algebra generated by these sets. We write Ω' the sum of the set of all sequence of length at most t: $\Omega' = \bigcup_{1 \le i \le t} \Omega_t$ and we write $\Omega'' := \Omega'_{T-1}$. In the context of a non-recombining tree model, there is a one-one relation between the elements of Ω' and the nodes of the tree. The elements of Ω correspond to final nodes, and those of Ω'' to pre-final nodes.

Multiperiod acceptability measures

Consider a sequence space Ω as defined in the previous subsection. In the multiperiod setting, the acceptability of a given position should be considered not only as a function of the position itself, but also as a function of available information.

We still define a "position" as a mapping X from Ω to R. Such a mapping may be restricted to the set $F(\omega')$ consisting of all sequences beginning with ω'. The restricted mapping $X|_{F(\omega')}$ defines a position on $F(\omega')$. We extend slightly the definition by acceptability measures that was given before by allowing that the degree of acceptability of a given position can be 1. So, an acceptability measure on $F(\omega')$ is a mapping from $F(\omega')$ to the extended real line $\mathbf{R} \cup \{\infty\}$.

Definition 3.1

A multiperiod acceptability measure on the sequence space Ω is a mapping that assigns to each partial history $\omega' \in \Omega'$ an acceptability measure on $F(\omega')$.

The acceptability measure on $F(\omega')$ that is provided by a multiperiod acceptability measure will be denoted by $\phi(\cdot \mid \omega')$; the element of the extended real line associated by this mapping to a position X on $F(\omega')$ is denoted by $\phi(X \mid \omega')$. When X is a position on Ω, we also write $\phi(X \mid \omega')$ instead of $\phi(X|_{F(\omega')} \mid \omega')$. The situation at the initial time is represented by the sequence of zero length;

instead of ρ (X | 0), we write ρ (X). Under the normalization condition (Def. 2.2), we have ϕ(X | ω) = X(ω) for all ω ∈ Ω .

We say that a multiperiod risk measure is *coherent* if all partial-information acceptability measures ρ (· | ω') are coherent on *F*(ω').

This implies in particular that, for all positions X and Y and for all partial histories ω', the following holds:

$$\text{if } \rho(X \mid \omega') \le \rho(Y \mid \omega') \text{ for all } \omega' \prec \omega, \text{ then } \rho(X \mid \omega) \le \rho(Y \mid \omega). \quad (3.1)$$

We shall say that a multiperiod risk measure satisfies the *stepwise monotonicity condition* if the following condition holds for all positions X and Y and for all partial histories
ω' ∈ Ω''

$$\text{if } \rho(X \mid \omega'\alpha) \le \rho(Y \mid \omega'\alpha) \text{ for all } \alpha \in A, \text{ then } \rho(X \mid \omega') \le \rho(Y \mid \omega') \quad (3.2)$$

The example below shows that there exists situations in which the monotonicity property (3.1) is satisfied but the stepwise monotonicity property (3.2) does not hold.

Example 3.1

Consider a two-period binomial tree; that is, let A = {*u, d*} and Ω = {*uu, ud, du, dd*}. Specify an acceptability measure for products on by

$$\phi(X \mid \omega') = \inf_{i=1,2} E_{p_i}(X \mid \omega')$$

where P_1 is the probability measure that is obtained by assigning probability 0.6 to a u event and 0.4 to a d event, and P_2 is obtained by reversing these probabilities. Clearly, ϕ is a coherent multiperiod acceptability measure. Consider a position X that pays 100 if *ud* or *du* occurs, and that pays nothing otherwise (a "butterfly"). As is easily computed, we have ϕ (X) = 48 whereas ϕ (X | u) = ϕ (X | d) = 40. Comparing the position X to the position Y that pays 44 in all states of nature, we see that ϕ is not stepwise monotonic.

Some of authors (see Wang (2003)) use the term "dynamic consistency" for essentially the property that we refer to here as stepwise monotonicity. To define the notion of dynamic consistency on the basis of a stronger premise, which may this property easier to establish in some cases. The notions of stepwise monotonicity and dynamic consistency are actually the same under the coherence assumption.

Definition 3.2

A multiperiod risk measure ρ defined on a sequence space Ω is said to be *dynamically consistent* if for all partial histories $\omega' \in \Omega''$ and all positions X and Y we have

$$\text{if } \rho(X \mid \omega'\alpha) = \rho(Y \mid \omega'\alpha) \text{ for all } \alpha \in A, \text{ then } \rho(X \mid \omega') = \rho(Y \mid \omega')$$

Proposition 3.1.

A coherent multiperiod acceptability measure is dynamically consistent if and only if it is stepwise monotonic.

To each $\omega' \in \Omega''$, one can associate a single-period economy in which the events that may occur (equivalently, the states of nature that may arise after one time step) are parametrized by the event set A. A single-period position is a mapping from A to R. A single-period acceptability measure assigns real numbers to single-period positions.

Let a multiperiod acceptability measure ϕ be given. For any partial history $\omega' \in \Omega''$, one can generate a position X_Y on $F(\omega')$ from a given single-period position $Y : A \to R$ by defining

$$X_Y(\omega) = Y(\omega'\alpha) \text{ if } \omega'\alpha \prec \omega. \tag{3.3}$$

In this way we can introduce for each $\omega' \in \Omega''$ a single-period acceptability measure denoted

by $\phi_{\omega'}$:

$$\phi_{\omega'} : Y \to \phi_{\omega}(X_Y \mid \omega'). \tag{3.4}$$

The following lemma is easily verified directly from the coherence axioms.

Lemma 3.1.

If ϕ is a coherent multiperiod acceptability measure, then all single-period acceptability measures $\phi_{\omega'}$ derived from ϕ are coherent as well.

Given a product X on the sequence space and a multiperiod acceptability measure ϕ, we can define for each partial history a single-period position $\phi(X \mid \omega')$ in the following

way:

$$\phi_{\omega}(X_Y \mid \omega') : \alpha \to \phi(X \mid \omega'\alpha) \tag{3.5}$$

Since this is a single-period position, its acceptability may be evaluated by means of the $\rho_{\omega'}$. If ϕ is dynamically consistent, we have

$$\phi\,(X \mid \omega') = \phi_{\omega'}\,(\phi\,(X \mid \omega'). \tag{3.6}$$

This implies that the multiperiod acceptability measure ϕ can be described in terms of the conditional single-period acceptability measures $\phi_{\omega'}$.

IV. PRICING IN INCOMPLETE MARKETS

The main purpose of this section is to obtain price bounds for derivative products in incomplete markets. For this purpose we suppose that an acceptability measure (i.e. a collection of test measures) has been selected in a sufficiently conservative way so that, in a straightforward extension of the standard arbitrage argument, any opportunity that is acceptable according this measure would be quickly eliminated in the market. After this adjustment has taken place, the price of any asset must be such that it is not possible to create acceptable opportunities by means of any admissible portfolio strategy. We aim to compute the resulting price bounds explicitly.

We assume that n basic assets are present in the market, whose prices are described by a function S: $\Omega' \rightarrow R^n$. For each $0 \leq t \leq T$, an F_t-measurable function $S_t : \Omega \rightarrow R^n$ is defined by $S_t(\omega) = S(\omega\,|_t)$. A trading strategy is a function from Ω' to R^n, interpreted as a rule that assigns to each partial history $\omega' \in \Omega'$ a position in the basic instruments.

If g: $\Omega' \rightarrow R^n$ is a strategy, we write $g_t(\omega) = g(\omega\,|_t)$. Each trading strategy defines a position, namely the total result of the strategy which is given by

$$H^g := \sum_{t=0}^{T-1} g_t^T (S_{t+1} - S_t) \tag{4.1}$$

for a self-financing strategy with zero initial investment. Given a basic acceptability measure ϕ, we define the acceptability measure of a position X subject to a strategy g by

$$\phi^g(X) := \phi\,(X + H^g) \tag{4.2}$$

Let us assume that a nonempty set $\mathbb{G}$ of allowed hedging strategies has been fixed. We can then define, for any position X, the optimal degree of acceptability taking hedging into account:

$$\phi^*(X) := \sup_{g \in \mathbb{G}} \phi^g(X). \tag{4.3}$$

Consider first, as in Carr et al. (2001), a single-period economy with traded assets $S^0, \ldots, S^n$ and with a collection *P* of probability measures on the finite set Ω of states of nature. The price of asset i at time t (t = 0, 1) is given by S^i_t ; S^0 is the numeraire which always has price 1.

The economy is said to allow *strictly acceptable opportunities* if it is possible to form a strictly acceptable portfolio at zero cost; that is, if there exist portfolio weights $a_0, \ldots, a_n$ such that

$$\sum_{i=0}^{n} a_i S_i = 0$$

$$E_P \sum_{i=0}^{n} a_i S_i \geq 0, \text{ for all } P \in P$$

$$E_P \sum_{i=0}^{n} a_i S_i > 0 \text{ ,for some } P \in P$$

Carr et al. (2001) have argued that if a collection of test measures is chosen sufficiently large so as to reflect a widely held market view, it can be assumed that there will be no strictly acceptable opportunities in the economy. The NSAO condition ("no strictly acceptable opportunities") is a stronger requirement than absence of arbitrage, and in incomplete markets it therefore leads in general to tighter bounds on prices of contingent claims than would be obtained by the no-arbitrage condition alone.

In a multiperiod setting, we interpret the NSAO condition as the requirement that no self-financing investment strategy with zero initial cost should produce a strictly acceptable result. It is easily verified that a necessary condition for the NSAO condition to hold for a given multiperiod economy is that each of the associated single-period economies should be free of strictly acceptable opportunities. As can be seen from simple examples, however, this condition is not sufficient.

Example 4.1

Consider the two-period binomial tree of Example 3.1 again, with the same collection of two test measures. Suppose there are two assets S and B. The value of B is always 100, whereas for S we have

S(0) = 100, S(u) = 110, S(d) = 90,

S(uu) = 120, S(ud) = S(du) = 100, S(dd) = 80.

It is easily verified that none of the single-period economies derived from this model allows strictly acceptable opportunities. Now consider the dynamic strategy that is defined as follows. Take no position at the initial time; at time 1, take a position 1 in the asset S (and −1.1 in B) if an "up" movement occurs, and take the opposite position (and 0.9 in B) if a "down" step takes place. The expected result of this strategy under test measure P_1 is

$$0.6 \cdot (0.6 \cdot 10 + 0.4 \cdot (-10)) + 0.4 \cdot (0.6 \cdot (-10) + 0.4 \cdot 10) = 0.4$$

while under P_2 we find

$$0.4 \cdot (0.4 \cdot 10 + 0.6 \cdot (-10)) + 0.6 \cdot (0.4 \cdot (-10) + 0.6 \cdot 10) = 0.4.$$

So the expected result is positive in both cases; the "momentum" strategy creates a strictly acceptable opportunity.

CONCLUSIONS

We have considered the multiperiod extension of the notion of coherent risk measures as introduced by Artzner et al. (1999). Our main conclusions are that the addition of a dynamic consistency axiom leads to a simple and attractive characterization of coherent dynamic acceptability measures, and that the framework obtained in this way allows characterizations of price bounds for derivative assets in incomplete markets that are natural counterparts of the well-known results for the complete market case.

We have worked within the simplest possible setting that allows consideration of partially revealed information and dynamic hedging. Interpreting our results from a continuous-time perspective, one might say that the product property would favor approaches that measure discrepancy between models in terms of an entropy rate. In the absence of axiomatic underpinnings of continuous-time utility specifications, however, it remains difficult to make any definitive statements. In this paper we have worked with the axiom of positive homogeneity, following Artzner et al. (1999).

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KOHERENTNE MIARY RYZYKA W MODELU DYNAMICZNYM

Podstawy teorii koherentnych miar ryzyka były omówione w pracy Artzner i in. (1999) w ujęciu statycznym. Przedstawimy analogiczny model w podejściu dynamicznym z czasem dysktretnym. Zapiszemy standardowe aksjomaty definiujące miary koherentne dynamicznie. Omówimy własności przedstawionego modelu oraz pokażemy możliwość wykorzystania modelu do zabezpieczenia pozycji finansowej wykorzystując wybraną klasę miar ryzyka.