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FUTURE LIFE-TABLES BASED ON THE LEE-CARTER METHODOLOGY AND THEIR APPLICATION TO CALCULATING THE PENSION ANNUITIES[1]

Summary. In the paper a new recursive approach to the mortality forecasting is proposed based on the well-known Lee-Carter stochastic model. The standard Lee-Carter method and its modified version are presented and compared using mortality data for Poland in the time period 1990-2007. The results obtained indicate that the recursive approach gives more accurate forecasts in terms of the mean squared error.

Stochastic forecasts of age-specific death rates are also used to predict death probabilities and life expectancy being the main parameters of the life-tables. As an example, future life-tables for 2020 are calculated. Applications of Lee-Carter methodology in pension annuity calculations are presented.

Key words: mortality forecasting, Lee-Carter model, future life-tables, pension annuity.

1. Introduction

Long-standing observations of different characteristics of the survival distribution in human population show that the characteristics change in time. For instance, observations of annual probabilities of death giving survival to an age x, conducted in the developed countries, show that such probabilities have been decreasing in the recent decades, although the changes are irregular. Thus, the considered probabilities can be therefore viewed as processes that are characterised by certain stochastic variability, in addition to their general tendency. The other indicators have been showing a similar, stochastic character, for instance, life expectancy, mortality rates or rectangularization indices of the survival functions [Wilmoth, Horiuchi, 1999].

The trends and regularities observed in the developed countries in the second half of the 20th c. with respect to some of the aforementioned indicators can be summed up as follows:

- the mode of the death intensity curve moves towards very old ages,
- concentration of deaths around the mode is increasing,

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– the survival curve is undergoing rectangularization (which is a result from the tendencies outlined above),

– accidental deaths at young ages (especially male twenty-year-olds) have been growing (injuries, accidents, poisoning),

– both male and female life expectancy have been increasing.

The phenomena produce various socio-economic consequences. One of them is the growing numbers of people surviving to the retirement age, as well as an extending period of time in which the pension annuity providers need to pay out the benefits. Therefore, improving mortality rates have a direct effect on the present value of future liabilities and its related level of reserves held by institutions paying the benefits. A reliable estimation of mortality rates in the future periods becomes therefore a key approach.

This paper uses the Lee-Carter methodology to derive mortality forecasts for Poland up to 2020. The recursive version of the Lee-Carter stochastic approach is proposed. The standard and recursive Lee-Carter method are presented and compared using mortality data for Poland. The results obtained indicate that the new approach gives more accurate forecasts in terms of the mean squared error. Stochastic forecasts of age-specific death rates are also used to predict death probabilities and life expectancy being the main parameters of the life-tables.

The forecasts of probabilities of death will be used to calculate the pension annuities to be paid out under the so-called „second and third pillar" of the Polish pension system. Pension annuities are calculated from amounts accumulated on individual funds kept by the Open Pension Funds (OPF), the Employee Pension Schemes (EPS) or the Individual Pension Accounts (IPA).

2. The Lee-Carter methodology

The fist attempt at mathematical modelling of the intensity of deaths in dynamic terms was undertaken by Blaschke [1923] who considered so-called dynamic Makeham's law. The model assumes that the mortality intensity $\mu_x(t)$ is function of not only age x, but also of calendar time t. The effect of the time variable t was expressed in the model through some deterministic function.

The stochastic character of mortality-related processes justifies the need of reaching for the stochastic methods for modelling, forecasting and describing the phenomena. This is very important, especially with respect to the accuracy of the forecasts of their future development. Among the stochastic models for forecasting mortality that are popular today, there is the Lee-Carter model.

In the 1990s, Lee and Carter [Lee, Carter 1992] attempted to apply the theory of random walk with a drift to modelling and subsequently forecasting the crude age-specific death rates $m_x(t)$, cross-classified by age x, and by calendar year t. The model has the form

$$\ln m_x(t) = a_x+b_xk_t+\varepsilon_{x,t} \quad \text{for} \quad x = 0,1,...,\omega, \quad t = 1,2,..., T, \tag{1}$$

or equivalently

$$m_x(t) = \exp\{a_x+b_xk_t+\varepsilon_{x,t}\} \quad \text{for} \quad x = 0,1, ..., \omega, \; t = 1,2, ..., T, \tag{2}$$

where $\{a_x\}$ and $\{b_x\}$ are sets of some constants that are different for different age groups x, and k_t is an index viewed as a discrete-time stochastic process. Double-index terms $\varepsilon_{x,t}$ represent the random errors reflecting particular age-specific influences not captured by the model. It is assumed that $\varepsilon_{x,t}$ are independent random variables, normally distributed with the mean equal 0 and variance σ^2.

The model (1) or (2) is undetermined without additional constraints. Let us assume, for instance, that we have an empirical data matrix **M** of logarithms of specific mortality rates, i.e. a matrix with elements ln $m_x(t)$ in the body, where x=0,1,...,ω denotes the age group (matrix rows), whereas t=1,2,...,T are calendar years (matrix columns). Let for a set of parameters $\{a_x\}$, $\{b_x\}$, x=0,1,... ω and $\{k_t\}$, t=1,2,...,T the model (1) be valid. It is easy to verify that for any constant c and the set of parameters $\{a_x - cb_x\}$, $\{b_x\}$ $\{k_t+c\}$ or $\{a_x\}$, $\{cb_x\}$ $\{k_t/c\}$ the model (1) also holds. Hence, the parameters k_t are determined to the transformation k_t+c or k_t/c, parameters b_x are determined to the multiplicative constant, whereas the parameters a_x – to the additive constant.

To ensure the unique parameters of the model (1) it is necessary to define certain additional constraints. To this end, Lee and Carter assumed that the sum of the parameters b_x for all age groups (indexed by x) equals 1, whereas the sum of the parameters k_t (indexed by t) equals 0. Thus, the parameters b_x and k_t satisfy the following constrains

$$\sum_{x=0}^{\omega} b_x = 1, \quad \sum_{t=1}^{T} k_t = 0, \quad \text{so that} \quad a_x = \frac{1}{T}\sum_{t=1}^{T} \ln m_x(t).$$

It follows, that under these constrains the parameters a_x describe the age pattern of mortality averaged over time, whereas b_x describe deviations from the averaged pattern when k_t varies. The parameters k_t describe the effect of the calendar time t on a change in the mortality.

The model (1) is fitted to the crude death rates $m_x(t) = \dfrac{d_x(t)}{K_x(t)} \cdot 1000$, where $d_x(t)$ denotes the number of deaths observed at age x and time t, and $K_x(t)$

denotes the exposure to the risk of death at age x. In our approach, we will replace crude death rates by their estimates $\hat{m}_x(t)$ obtained from the expressions $\hat{m}_x(t) = \frac{2q_x(t)}{2-q_x(t)} \cdot 1000$, where $q_x(t)$ denotes the probability of death in year t, giving survival to age x. Probabilities $q_x(t)$ will be taken from the period life-tables published for years 1990–2007 by the Central Statistical Office.

3. Forecasting from the Lee-Carter model

Let us assume that the parameters a_x and b_x in the model (1) are constant in time t, which means that estimates of the parameters, once derived, can be used in the future, and the mortality forecasts can be obtained by modelling k_t as a time series. The forecasts concerning the predicted values of k_t together with the estimates of the parameters a_x and b_x allow, based on the model (1), forecasting long-term mortality, and more specifically, forecasting the logarithms of deaths rates for future periods $t>T$.

As proposed by Lee and Carter, finding the values of k_t for $t=1,2,...,T$ provides a starting point for modelling a time series $\{k_t\}$ as a random walk with drift that can be described using the formula

$$k_t = c + k_{t-1} + e_t, \tag{3}$$

where c stands for a constant (a drift), while e_t is an error term with normal distribution with the mean 0 and a finite variance. This approach allows reducing accumulation of error terms resulting from the short-term variability in mortality rates, which largely influences the accuracy of the forecasts.

The estimator of the drift c has the form

$$\hat{c} = \frac{1}{T-1}(k_T - k_1), \tag{4}$$

whereas the variance estimator of $\hat{c}$ is given by the formula

$$\hat{\sigma}_c^2 = \frac{1}{T-1}\sum_{t=2}^{T}\left(k_t - k_{t-1} - \hat{c}\right)^2. \tag{5}$$

Estimation of the constant c allows making forecasts concerning k_t for $t>T$. Inserted in model (1), where the parameters a_x and b_x are replaced by their estimates, allows making the forecasts of future log-central mortality rates for $t>T$.

Then it is also possible to estimate other parameters for future periods in order to obtain the so-called future life-tables.

4. Model fitting

In the Lee-Carter methodology the Singular Value Decomposition (SVD) is applied to derive the parameters a_x, b_x, and k_t.

Let us consider the matrix

$$\mathbf{A} = \left[\ln \hat{m}_x(t) - a_x\right]_{(\omega+1)\times T} . \tag{6}$$

SVD allows representing each element of the matrix **A** as the following sum

$$\ln \hat{m}_x(t) - a_x = \sum_{i=1}^{r} \lambda_i \cdot u_i(x) \cdot v_i(t) , \tag{7}$$

where $r = \text{rank}\,\{\mathbf{A}\}$, $\lambda_i, i = 1,2,\ldots,r$ are the ordered (increasingly) singular values, and $\mathbf{u}_i, \mathbf{v}_i$ are the corresponding left and right singular vectors of **A**.

Let us consider the following representation

$$\ln \hat{m}_x(t) - a_x = \lambda_1 \cdot u_1(x) \cdot \nu_1(t) + \varepsilon_{x,t} ,$$

where $\varepsilon_{x,t} = \sum_{i=2}^{r} \lambda_i \cdot u_i(x) \cdot v_i(t).$

Such a representation leads directly to the Lee-Carter model

$$\ln \hat{m}_x(t) - a_x = b_x \cdot k_t + \varepsilon_{x,t} ,$$

with $b_x \cdot k_t = \lambda_1 \cdot u_1(x) \cdot v_1(t)$ and $\sum_{x=0}^{\omega} b_x = 1, \quad \sum_{t=1}^{T} k_t = 0.$

Thus the Lee-Carter model is simply the SVD approximation of the matrix $\mathbf{A} = \left[\ln \hat{m}_x(t) - a_x\right]$.

The percentage of the total variance explained by the Lee-Carter model is $\lambda_1^2 \Big/ \sum_{i=1}^{r} \lambda_i^2$. Since the first singular value λ_1 is usually much larger than all the others, therefore the percentage of the total variance explained by the model is usually very high. For instance, Tuljapurkar *et al*., (2000) found that for some countries (with low mortality) over 94% of the variance was explained by the model (1).

The Lee-Carter methodology and the subsequent modifications are broadly discussed in the literature [e.g. Carter, 1996; Lee, 2000; Booth et al., 2002; Brouhns et al., 2002a,b; Renshaw, Haberman, 2003a,b,c; Li et al., 2004; Lundström, Qvist, 2004; Brouhns et al., 2005; Koissi, Shapiro, 2006; Koissi et al., 2006; Denuit, Dhaene, 2007].

5. The recursive procedure of the Lee-Carter approach

In this section a modification of the Lee-Carter approach based on a recursive algorithm is proposed.

Let $\mathbf{A}^{(k)}$ be a matrix with elements obtained in the *k*-th step of the recursive procedure. If *k*=1 then $\mathbf{A}^{(k)}$ reduces to the matrix $\mathbf{A}$ with elements defined in (6). The procedure can be described as follows:

[1.] Apply the standard Lee-Carter approach using $\mathbf{A}^{(k)}$ and derive forecasts of $\ln \hat{m}_x(t)$, *x*=0,1,…,ω, i.e. for one year ahead.

[2.] Attach the values of $\ln \hat{m}_x(t)$ obtained in the step 1 to the matrix $\mathbf{A}^{(k)}$ as an additional column, giving the matrix $\mathbf{A}^{(k+1)}$.

[3.] Repeat the steps 1-2 taking *k*:=*k*+1, until *k* reaches the assumed horizon of forecast.

As it will be shown, the recursive Lee-Carter procedure (*RLC*) gives quite similar estimates of a_x, b_x, for *x*=0,1,…, ω as the standard Lee-Carter approach (*SLC*). However, the estimates of k_t for *t*=1,2,…,*T* differ substantially. What is more, the longer is the time horizon of forecasting the greater is the difference in the estimates of k_t.

Both approaches *RLC* and *SLC* will be used to forecast the crude death rates for Poland (male and female). The forecasts will be then used to derive the male's and female's future life-tables for the year 2020.

6. Estimation of the Lee-Carter parameters for Poland

To estimate the Lee-Carter's parameters for Poland we derived data from life-tables for time period 1990–2007 that are available at the website of Central Statistical Office (www.stat.gov.pl). The estimates of parameters a_x and b_x for x=0,1,…,100, obtained by means of the *SLC* and *RLC* approaches, are presented on Figures 1–2 (separately for men and women).

Based on the estimates of k_t the parameter was projected for future periods. To this end, the formula (3) was applied, where the constant c was estimated using (4). The estimates of k_t together with forecasts are presented in Figure 3.

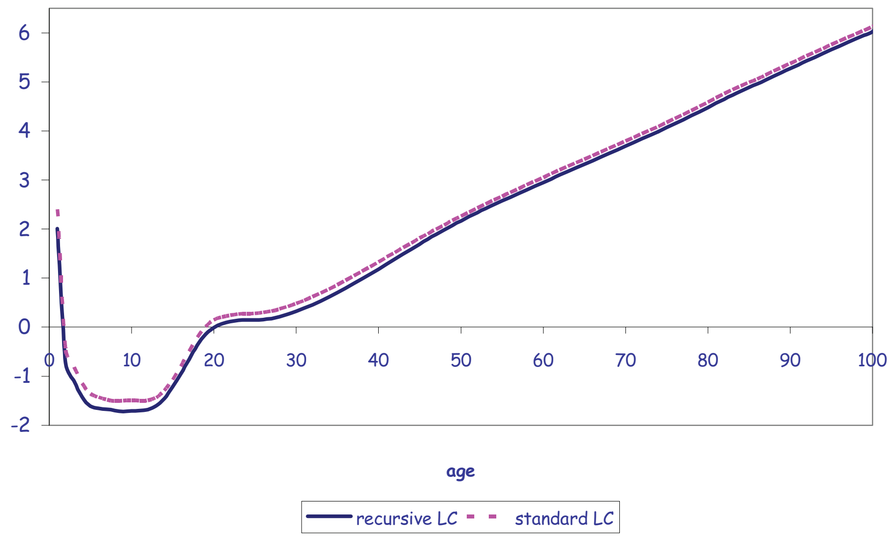

Fig. 1. Estimates of a_x for $x \in [0,100]$ from the recursive and standard Lee-Carter model (model fitting based on the period male's life-tables for 1990–2007)

Source: Developed by the author.

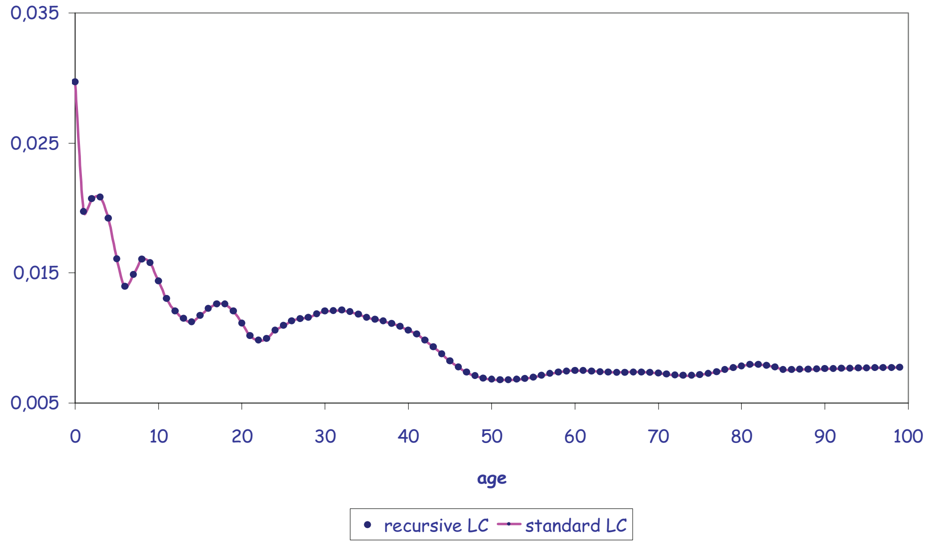

Fig. 2. Estimates of b_x for $x \in [0,100]$ from the recursive and standard Lee-Carter model (model fitting based on the period male's life-tables for 1990–2007)

Source: Developed by the author.

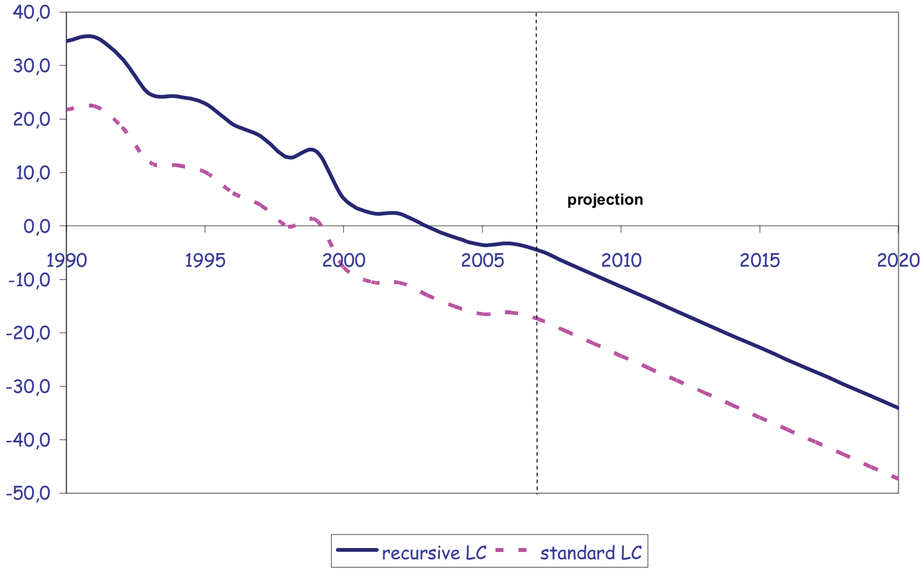

Fig. 3. Estimates of k_t from the recursive and standard Lee-Carter model for the years 1990–2007 and its 2008–2020 forecast (model fitting based on the period male's life-tables for 1990–2007)

Source: Developed by the author.

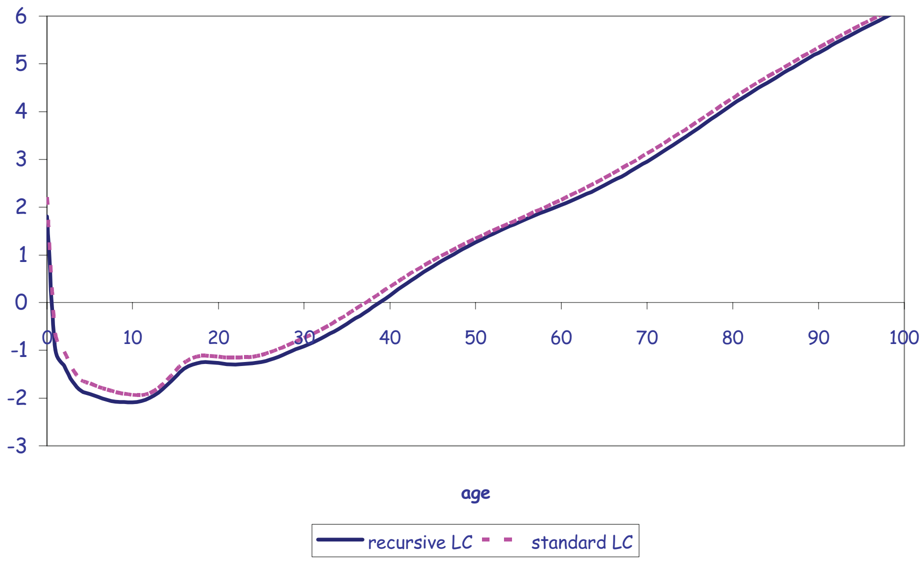

Fig. 4. Estimates of a_x for $x \in [0,100]$ from the recursive and standard Lee-Carter model (model fitting based on the period female's life-tables for 1990–2007)

Source: Developed by the author.

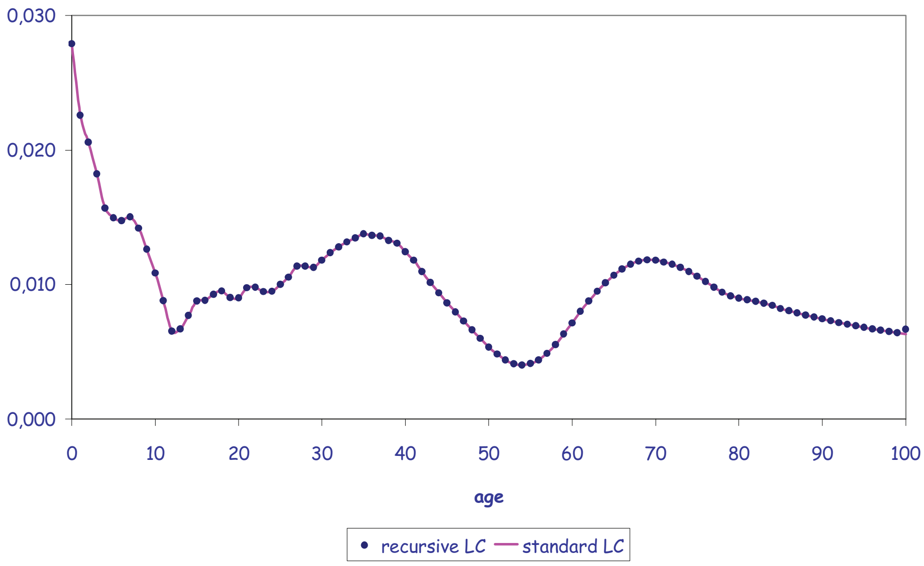

Fig. 5. Estimates of b_x for $x \in [0,100]$ from the recursive and standard Lee-Carter model (model fitting based on the period female's life-tables for 1990–2007)

Source: Developed by the author.

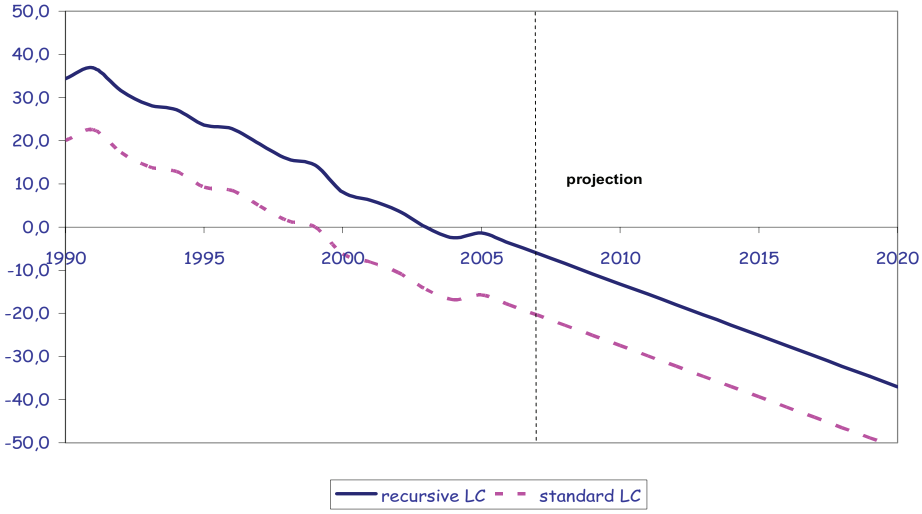

Fig. 6. Estimates of k_t from the recursive and standard Lee-Carter model for the years 1990–2007 and the 2008–2030 forecast (model fitting based on the period female's life-tables for 1990–2007)

Source: Developed by the author.

The projections of the parameters k_t for $t>T$ were then used to forecast some life-table parameters within the time horizon 2008–2020. Tables 1 and 2 present annual probabilities of death $q_x(t)$ giving survival to x and their estimates $q_x^{RLC}(t)$ and $q_x^{SLC}(t)$ for the year t=2009 (separately for men and women) obtained by means of both the recursive and standard Lee-Carter approach. Based on these estimates the mean squared errors for both types of estimators in relation to the real values of $q_x(t)$, for x=0,1,..., ω were calculated. Means squared errors were defined as follows:

$$MSE^{RLC}(t) = \sqrt{\frac{1}{\omega+1}\sum_{x=0}^{\omega}(q_x(t) - q_x^{RLC}(t))^2},$$

$$MSE^{SLC}(t) = \sqrt{\frac{1}{\omega+1}\sum_{x=0}^{\omega}(q_x(t) - q_x^{SLC}(t))^2} \quad ,$$

where $q_x^{RLC}(t)$, $q_x^{SLC}(t)$ represent estimates of $q_x(t)$ for the age group x=0,1,...,100 received by means of the recursive and standard Lee-Carter method for the year t=2009.

Tables 3 and 4 present the male's and female's life expectancies $e_x(2009)$ and their forecasts $e_x^{RLC}(t)$ for t=2009 and t=2020.

Tab. 1. Forecasts of male's death probabilities from the recursive and standard LC model (2009)

Age x	$q_x(2009)$	$q_x^{RLC}(2009)$	$q_x^{SLC}(2009)$	Age x	$q_x(2009)$	$q_x^{RLC}(2009)$	$q_x^{SLC}(2009)$
0	0.00604	0.00573	0.00573	**48**	0.00716	0.00747	0.00747
1	0.00035	0.00041	0.00041	**49**	0.00788	0.00817	0.00818
2	0.00024	0.00028	0.00028	**50**	0.00866	0.00892	0.00892
3	0.00017	0.00020	0.00020	**51**	0.00949	0.00970	0.00970
4	0.00015	0.00017	0.00017	**52**	0.01038	0.01052	0.01052
5	0.00014	0.00017	0.00017	**53**	0.01133	0.01137	0.01138
6	0.00015	0.00017	0.00017	**54**	0.01234	0.01227	0.01227
7	0.00016	0.00016	0.00016	**55**	0.01341	0.01322	0.01322
8	0.00015	0.00016	0.00016	**56**	0.01453	0.01422	0.01422
9	0.00015	0.00016	0.00016	**57**	0.01571	0.01530	0.01530
10	0.00014	0.00016	0.00016	**58**	0.01695	0.01647	0.01647
11	0.00014	0.00017	0.00017	**59**	0.01824	0.01774	0.01774
12	0.00016	0.00018	0.00018	**60**	0.01958	0.01910	0.01910
13	0.00020	0.00021	0.00021	**61**	0.02097	0.02058	0.02058
14	0.00027	0.00027	0.00027	**62**	0.02244	0.02216	0.02217
15	0.00038	0.00035	0.00035	**63**	0.02397	0.02387	0.02387
16	0.00053	0.00046	0.00046	**64**	0.02559	0.02569	0.02570
17	0.00071	0.00062	0.00062	**65**	0.02731	0.02765	0.02765
18	0.00089	0.00077	0.00077	**66**	0.02915	0.02974	0.02974
19	0.00103	0.00088	0.00089	**67**	0.03113	0.03197	0.03197
20	0.00112	0.00096	0.00096	**68**	0.03327	0.03437	0.03438
21	0.00114	0.00102	0.00102	**69**	0.03559	0.03698	0.03698
22	0.00113	0.00105	0.00105	**70**	0.03814	0.03980	0.03981
23	0.00111	0.00106	0.00106	**71**	0.04094	0.04287	0.04288
24	0.00109	0.00105	0.00105	**72**	0.04405	0.04620	0.04621
25	0.00108	0.00106	0.00106	**73**	0.04751	0.04982	0.04982
26	0.00109	0.00108	0.00108	**74**	0.05138	0.05371	0.05371
27	0.00112	0.00112	0.00112	**75**	0.05572	0.05793	0.05794
28	0.00117	0.00118	0.00118	**76**	0.06057	0.06254	0.06254
29	0.00125	0.00124	0.00124	**77**	0.06596	0.06758	0.06759
30	0.00133	0.00132	0.00132	**78**	0.07191	0.07315	0.07316
31	0.00143	0.00142	0.00142	**79**	0.07844	0.07926	0.07927
32	0.00155	0.00153	0.00153	**80**	0.08555	0.08591	0.08592
33	0.00169	0.00166	0.00166	**81**	0.09323	0.09306	0.09308
34	0.00185	0.00182	0.00182	**82**	0.10148	0.10050	0.10051
35	0.00203	0.00200	0.00200	**83**	0.11030	0.10826	0.10827
36	0.00223	0.00220	0.00220	**84**	0.11971	0.11642	0.11643
37	0.00245	0.00242	0.00242	**85**	0.12973	0.12470	0.12471
38	0.00270	0.00268	0.00268	**86**	0.14041	0.13422	0.13424
39	0.00297	0.00297	0.00297	**87**	0.15176	0.14436	0.14438
40	0.00327	0.00329	0.00329	**88**	0.16386	0.15514	0.15516
41	0.00360	0.00365	0.00365	**89**	0.17676	0.16661	0.16663
42	0.00397	0.00406	0.00406	**90**	0.19041	0.17876	0.17878
43	0.00438	0.00452	0.00452	**91**	0.20512	0.19169	0.19171
44	0.00484	0.00502	0.00503	**92**	0.22063	0.20535	0.20537
45	0.00534	0.00558	0.00558	**93**	0.23694	0.21976	0.21979
46	0.00589	0.00617	0.00617	**94**	0.25406	0.23493	0.23495
47	0.00650	0.00680	0.00680	**95**	0.27196	0.25085	0.25087
				MSE		**0,00685**	**0,00748**

Source: Developed by the author.

Tab. 2. Forecasts of female's death probabilities from the recursive and standard LC model (2009)

Age *x*	$q_x(2009)$	$q_x^{RLC}(2009)$	$q_x^{SLC}(2009)$	Age *x*	$q_x(2009)$	$q_x^{RLC}(2009)$	$q_x^{SLC}(2009)$
0	0.00507	0.00448	0.00449	**48**	0.00261	0.00271	0.00271
1	0.00030	0.00029	0.00029	**49**	0.00290	0.00301	0.00301
2	0.00022	0.00021	0.00021	**50**	0.00322	0.00333	0.00333
3	0.00017	0.00016	0.00016	**51**	0.00356	0.00366	0.00366
4	0.00014	0.00013	0.00013	**52**	0.00392	0.00401	0.00401
5	0.00013	0.00013	0.00013	**53**	0.00431	0.00437	0.00437
6	0.00013	0.00012	0.00012	**54**	0.00472	0.00474	0.00474
7	0.00013	0.00011	0.00011	**55**	0.00517	0.00512	0.00512
8	0.00013	0.00011	0.00011	**56**	0.00564	0.00551	0.00551
9	0.00012	0.00011	0.00011	**57**	0.00614	0.00590	0.00590
10	0.00012	0.00011	0.00011	**58**	0.00666	0.00630	0.00630
11	0.00013	0.00012	0.00012	**59**	0.00720	0.00671	0.00671
12	0.00015	0.00013	0.00013	**60**	0.00776	0.00714	0.00714
13	0.00018	0.00014	0.00014	**61**	0.00833	0.00762	0.00762
14	0.00021	0.00016	0.00016	**62**	0.00893	0.00816	0.00816
15	0.00023	0.00019	0.00019	**63**	0.00955	0.00878	0.00878
16	0.00024	0.00023	0.00023	**64**	0.01021	0.00949	0.00949
17	0.00025	0.00025	0.00025	**65**	0.01092	0.01030	0.01031
18	0.00026	0.00026	0.00026	**66**	0.01172	0.01124	0.01124
19	0.00026	0.00026	0.00026	**67**	0.01262	0.01232	0.01233
20	0.00026	0.00026	0.00026	**68**	0.01367	0.01358	0.01358
21	0.00025	0.00025	0.00025	**69**	0.01491	0.01504	0.01505
22	0.00025	0.00025	0.00025	**70**	0.01637	0.01674	0.01675
23	0.00026	0.00025	0.00025	**71**	0.01811	0.01872	0.01872
24	0.00026	0.00025	0.00025	**72**	0.02019	0.02098	0.02099
25	0.00026	0.00026	0.00026	**73**	0.02266	0.02359	0.02360
26	0.00028	0.00027	0.00027	**74**	0.02557	0.02660	0.02661
27	0.00030	0.00028	0.00028	**75**	0.02897	0.03007	0.03008
28	0.00033	0.00031	0.00031	**76**	0.03292	0.03407	0.03408
29	0.00037	0.00033	0.00033	**77**	0.03745	0.03866	0.03867
30	0.00039	0.00035	0.00035	**78**	0.04258	0.04388	0.04390
31	0.00043	0.00038	0.00038	**79**	0.04835	0.04973	0.04974
32	0.00046	0.00041	0.00041	**80**	0.05475	0.05615	0.05616
33	0.00050	0.00045	0.00045	**81**	0.06181	0.06314	0.06316
34	0.00055	0.00050	0.00050	**82**	0.06953	0.07058	0.07060
35	0.00061	0.00055	0.00055	**83**	0.07793	0.07857	0.07859
36	0.00068	0.00062	0.00062	**84**	0.08703	0.08723	0.08726
37	0.00075	0.00069	0.00070	**85**	0.09687	0.09641	0.09644
38	0.00084	0.00079	0.00079	**86**	0.10750	0.10705	0.10707
39	0.00094	0.00089	0.00089	**87**	0.11893	0.11857	0.11860
40	0.00105	0.00102	0.00102	**88**	0.13126	0.13105	0.13108
41	0.00118	0.00116	0.00116	**89**	0.14454	0.14453	0.14456
42	0.00132	0.00133	0.00133	**90**	0.15871	0.15901	0.15905
43	0.00148	0.00152	0.00152	**91**	0.17407	0.17465	0.17469
44	0.00167	0.00172	0.00172	**92**	0.19041	0.19136	0.19140
45	0.00187	0.00194	0.00194	**93**	0.20770	0.20916	0.20920
46	0.00210	0.00217	0.00217	**94**	0.22595	0.22804	0.22809
47	0.00234	0.00243	0.00243	**95**	0.24514	0.24801	0.24806
				MSE		**0,00143**	**0,00159**

Source: Developed by the author.

As we can see from Tables 1 and 2, the death probabilities predicted by means of the *RLC* and *SLC* methods are similar for small *x*, but the differences increase for old ages.

Due to these differences the mean squared errors (attached in the last rows of Tables 1 and 2) satisfy the inequality $MSE^{RLC}(2009) < MSE^{SLC}(2009)$.

Tab. 3. The male's life expectancy and its forecasts from the recursive LC model (for 2009 and 2020)

Age *x*	$e_x(2009)$	$e_x^{RLC}(2009)$	$e_x^{RLC}(2020)$	**Age *x***	$e_x(2009)$	$e_x^{RLC}(2009)$	$e_x^{RLC}(2020)$
a	b	c	d	e	f	g	h
0	71.5	71.5	74.2	**48**	26.7	26.7	28.6
1	71.0	70.9	73.4	**49**	25.9	25.9	27.8
2	70.0	69.9	72.4	**50**	25.1	25.1	27.0
3	69.0	69.0	71.4	**51**	24.4	24.3	26.2
4	68.0	68.0	70.4	**52**	23.6	23.5	25.4
5	67.0	67.0	69.4	**53**	22.8	22.8	24.6
6	66.0	66.0	68.4	**54**	22.1	22.0	23.9
7	65.1	65.0	67.4	**55**	21.4	21.3	23.1
8	64.1	64.0	66.4	**56**	20.6	20.6	22.4
9	63.1	63.0	65.4	**57**	19.9	19.9	21.6
10	62.1	62.0	64.5	**58**	19.2	19.2	20.9
11	61.1	61.0	63.5	**59**	18.6	18.5	20.2
12	60.1	60.1	62.5	**60**	17.9	17.8	19.5
13	59.1	59.1	61.5	**61**	17.2	17.2	18.8
14	58.1	58.1	60.5	**62**	16.6	16.5	18.1
15	57.1	57.1	59.5	**63**	16.0	15.9	17.4
16	56.2	56.1	58.5	**64**	15.4	15.2	16.8
17	55.2	55.1	57.5	**65**	14.7	14.6	16.1
18	54.2	54.2	56.6	**66**	14.1	14.0	15.5
19	53.3	53.2	55.6	**67**	13.6	13.5	14.9
20	52.3	52.3	54.6	**68**	13.0	12.9	14.3
21	51.4	51.3	53.7	**69**	12.4	12.3	13.7
22	50.4	50.4	52.7	**70**	11.8	11.8	13.1
23	49.5	49.4	51.8	**71**	11.3	11.2	12.5
24	48.6	48.5	50.8	**72**	10.8	10.7	12.0
25	47.6	47.5	49.8	**73**	10.2	10.2	11.4
26	46.7	46.6	48.9	**74**	9.7	9.7	10.9
27	45.7	45.6	47.9	**75**	9.2	9.3	10.4
28	44.8	44.7	47.0	**76**	8.7	8.8	9.9
29	43.8	43.7	46.0	**77**	8.3	8.3	9.4
30	42.9	42.8	45.0	**78**	7.8	7.9	9.0
31	41.9	41.8	44.1	**79**	7.4	7.5	8.5
32	41.0	40.9	43.1	**80**	6.9	7.1	8.1
33	40.0	40.0	42.2	**81**	6.6	6.7	7.7
34	39.1	39.0	41.2	**82**	6.2	6.4	7.3
35	38.2	38.1	40.3	**83**	5.8	6.0	6.9
36	37.3	37.2	39.3	**84**	5.5	5.7	6.5
37	36.3	36.2	38.4	**85**	5.1	5.4	6.2
38	35.4	35.3	37.5	**86**	4.8	5.1	5.8
39	34.5	34.4	36.6	**87**	4.5	4.8	5.5
40	33.6	33.5	35.6	**88**	4.3	4.5	5.2
41	32.7	32.6	34.7	**89**	4.0	4.2	4.9

Tab. 3. (cont.)

a	b	c	d	e	f	g	h
42	31.8	31.8	33.8	**90**	3.8	4.0	4.6
43	31.0	30.9	32.9	**91**	3.5	3.7	4.3
44	30.1	30.0	32.0	**92**	3.3	3.5	4.1
45	29.3	29.2	31.2	**93**	3.1	3.2	3.8
46	28.4	28.3	30.3	**94**	2.9	3.0	3.6
47	27.6	27.5	29.5	**95**	2.7	2.8	3.4

Source: Developed by the author.

Tab. 4. The female's life expectancy and its forecasts from the recursive LC model (for 2009 and 2020)

Age x	$e_x(2009)$	$e_x^{RLC}(2009)$	$e_x^{RLC}(2020)$	**Age x**	$e_x(2009)$	$e_x^{RLC}(2009)$	$e_x^{RLC}(2020)$
a	b	c	d	e	f	g	h
0	80.1	80.1	82.6	**48**	33.6	33.6	35.6
1	79.5	79.5	81.7	**49**	32.7	32.7	34.7
2	78.5	78.5	80.7	**50**	31.8	31.8	33.8
3	77.5	77.5	79.8	**51**	30.9	30.9	32.9
4	76.5	76.5	78.8	**52**	30.0	30.0	32.0
5	75.5	75.6	77.8	**53**	29.1	29.1	31.1
6	74.5	74.6	76.8	**54**	28.2	28.3	30.2
7	73.5	73.6	75.8	**55**	27.4	27.4	29.3
8	72.6	72.6	74.8	**56**	26.5	26.5	28.5
9	71.6	71.6	73.8	**57**	25.7	25.7	27.6
10	70.6	70.6	72.8	**58**	24.8	24.8	26.8
11	69.6	69.6	71.8	**59**	24.0	24.0	25.9
12	68.6	68.6	70.8	**60**	23.2	23.1	25.0
13	67.6	67.6	69.8	**61**	22.3	22.3	24.2
14	66.6	66.6	68.8	**62**	21.5	21.5	23.3
15	65.6	65.6	67.8	**63**	20.7	20.6	22.5
16	64.6	64.7	66.8	**64**	19.9	19.8	21.6
17	63.7	63.7	65.9	**65**	19.1	19.0	20.8
18	62.7	62.7	64.9	**66**	18.3	18.2	20.0
19	61.7	61.7	63.9	**67**	17.5	17.4	19.1
20	60.7	60.7	62.9	**68**	16.7	16.6	18.3
21	59.7	59.7	61.9	**69**	16.0	15.8	17.5
22	58.7	58.7	60.9	**70**	15.2	15.1	16.7
23	57.7	57.8	59.9	**71**	14.4	14.3	15.9
24	56.8	56.8	58.9	**72**	13.7	13.6	15.1
25	55.8	55.8	58.0	**73**	13.0	12.9	14.3
26	54.8	54.8	57.0	**74**	12.3	12.2	13.6
27	53.8	53.8	56.0	**75**	11.6	11.5	12.8
28	52.8	52.8	55.0	**76**	10.9	10.8	12.1
29	51.8	51.9	54.0	**77**	10.2	10.2	11.4
30	50.9	50.9	53.0	**78**	9.6	9.6	10.8
31	49.9	49.9	52.0	**79**	9.0	9.0	10.1
32	48.9	48.9	51.0	**80**	8.5	8.4	9.5
33	47.9	47.9	50.1	**81**	7.9	7.9	8.9
34	46.9	46.9	49.1	**82**	7.4	7.4	8.4
35	46.0	46.0	48.1	**83**	6.9	6.9	7.9
36	45.0	45.0	47.1	**84**	6.5	6.5	7.4
37	44.0	44.0	46.1	**85**	6.0	6.0	6.9

Tab. 4. (cont.)

a	b	c	d	e	f	g	h
38	43.1	43.1	45.2	**86**	5.6	5.6	6.4
39	42.1	42.1	44.2	**87**	5.3	5.2	6.0
40	41.1	41.1	43.2	**88**	4.9	4.9	5.6
41	40.2	40.2	42.2	**89**	4.6	4.5	5.2
42	39.2	39.2	41.3	**90**	4.2	4.2	4.8
43	38.3	38.3	40.3	**91**	3.9	3.9	4.5
44	37.3	37.3	39.4	**92**	3.7	3.6	4.2
45	36.4	36.4	38.4	**93**	3.4	3.4	3.9
46	35.5	35.5	37.5	**94**	3.2	3.1	3.6
47	34.5	34.5	36.5	**95**	3.0	2.9	3.4

Source: Developed by the author.

7. Application of the future life-tables to calculation of pension annuities

Let us consider the amount of a retirement income to be paid out by a pension provider to a person aged x years (we assume that age x is rounded to an integer) in the form of a life annuity. To determine the amount of monthly payments, we shall use in this investigation an actuarial formula employed to calculate the present value of the life annuity payable at the beginning of each month [see Skałba, 2002].

Let us assume then that life annuity is paid out m times within a year (for the monthly payments we have m=12), at the beginning of each subperiods the length of which is given by $1/m$ of a year, with the instalment amount being $1/m$ zlotys (so that the total annual amount due will be 1 zl; this is a so-called normalized case). Let us assume further that the last payment is effected at the beginning of the subperiod in which the recipient dies.

Let $T(x)$ be a non-negative random variable. representing the remaining lifetime of the x-year-old person, while $T^*(x) = [T(x)]$. Besides, let $S^{(m)}$ denote a rounded-up portion of the last year of recipient's life with accuracy to a subperiod lasting $1/m$.

Note that $S^{(m)}$ is a discrete random variable taking its values from the set

$$\left\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\right\}. \tag{8}$$

The present value $Y^{(m)}$ of the payments is a random variable, which can be written using the following formula

$$Y^{(m)} = \frac{1}{m}\left(1 + v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^{T^*(x)+S^{(m)}-\frac{1}{m}}\right).$$

where $v = 1/(1+i)$ is the so-called discount rate and i is an average rate of return calculated for several periods [see Skałba, 2002, p. 13]).

Using the formula for the sum of a geometric series

$$aq^0 + aq^1 + aq^2 + \ldots + aq^{m-1} = a\frac{1-q^m}{1-q},$$

the variable $Y^{(m)}$ can be expressed as follows

$$Y^{(m)} = \frac{1}{m}\left((v^{\frac{1}{m}})^0 + (v^{\frac{1}{m}})^1 + (v^{\frac{1}{m}})^2 + \ldots + (v^{\frac{1}{m}})^{mT^*(x)+mS^{(m)}-1}\right) =$$

$$= \frac{1}{m}\cdot\frac{1-(v^{\frac{1}{m}})^{mT^*(x)+mS^{(m)}}}{1-v^{\frac{1}{m}}} = \frac{1}{m}\cdot\frac{1-v^{T^*(x)+S^{(m)}}}{1-v^{\frac{1}{m}}}.$$

We shall calculate the expected value of $Y^{(m)}$ which is denoted by $\ddot{a}_x^{(m)}$.

Assuming that $T^*(x)$ and $S^{(m)}$ are independent we have

$$\begin{aligned}\ddot{a}_x^{(m)} = \mathrm{E}\left(Y^{(m)}\right) &= \frac{1}{m}\cdot\frac{1}{1-v^{\frac{1}{m}}}\left(1-\mathrm{E}\left(v^{T^*(x)+S^{(m)}}\right)\right) = \\ &= \frac{1}{m}\cdot\frac{1}{1-v^{\frac{1}{m}}}\left(1-\mathrm{E}\left(v^{T^*(x)+1}\right)\cdot\mathrm{E}\left(v^{S^{(m)}-1}\right)\right).\end{aligned} \quad (9)$$

The expected value $\mathrm{E}\left(v^{T^*(x)+1}\right)$ represents the so-called present value of a benefit (equal 1 zloty), payable at the end of year of death. In other words, it is an actuarial value of whole life insurance on being 1 zloty, payable at the end of the year of death. In the actuarial notation, it is usually denoted by A_x. The value is

$$A_x = \mathrm{E}\left(v^{T^*(x)+1}\right) = \sum_{k=0}^{\infty} v^{k+1}\cdot {}_kp_x\cdot q_{x+k},. \quad (10)$$

where probabilities

$$ {}_k p_x = P\big(T(x) > k\big), \quad q_{x+k} = P\big(T(x+k) \le 1\big) \tag{11} $$

denote, respectively, the probability of surviving k consecutive years giving survival to x, and the probability of dying within a year giving survival to $x+k$.

The other expected value $\mathbf{E}\left(v^{S^{(m)}-1}\right)$ can be determined assuming that the variable $S^{(m)}$ takes values from (8) with identical probabilities equal $1/m$. Then we have

$$ \mathbf{E}\left(v^{S^{(m)}-1}\right) = \sum_{k=1}^{m} v^{\frac{k}{m}-1} \cdot \frac{1}{m} = \frac{1}{m} \cdot \frac{1}{v} \cdot v^{\frac{1}{m}} \sum_{k=0}^{m-1} (v^{\frac{1}{m}})^k = \frac{1}{m} \cdot \frac{1}{v} \cdot v^{\frac{1}{m}} \cdot \frac{1-v}{1-v^{\frac{1}{m}}} $$

or after straightforward transformation

$$ \mathbf{E}\left(v^{S^{(m)}-1}\right) = \frac{1}{m} \cdot \frac{1-v}{v} \cdot \frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}. \tag{12} $$

Formulae (9), together with (10), (12), and using probabilities (11) allow calculating the expected value $\ddot{a}_x^{(m)}$ (also $\ddot{a}_x^{(12)}$).

Let K denote the amount of funds accumulated at the OPF by a person retiring at the age x years. Le B denote the monthly pension annuity (benefit) that a pensioner receives from the annuity provider. We will assume that the pension amount B is derived from the following equation, related to the present value of the life annuity paid monthly in advance [see e.g. Szumlicz T. ed., 2007]

$$ K = 12 \cdot B \cdot (1-\gamma) \cdot \ddot{a}_x^{(12)}. \tag{13} $$

where γ is a share of charges for the annuity provider.

In order to derive a benefit B from (13) it is necessary to take certain assumptions about the value of the accumulated capital K and the share of charges γ, etc. Besides, it is necessary to find the actuarial value of life annuity $\ddot{a}_x^{(12)}$. which is calculated using the formulae (9), (10), (12), and by using estimates of probabilities (11). The probabilities will be derived for future periods using the Lee-Carter methodology. The results will be compared with analogous calculations that were made using probabilities derived from life-tables published for 2007 year by the Central Statistical Office [www.stat.gov.pl].

8. Scenarios of pension annuity calculations

To calculate illustrative amounts of future old-age pension annuity using (13) the following assumptions were made:

– the minimum retirement age *x*: 60, 65 or 70 years;
– the calendar year at retirement – 2008;
– the pension capital *K* equal 100000 or 400000 zlotys;
– the share of charge γ=7%;
– the rate of return *i* equal 3% or 5%.

The probabilities (11) were used in two ways: case I – based on the future probabilities (future life-tables) derived using the Lee-Carter methodology and case II – applying a period life-table for the year 2007. Besides, to analyse the impact of gender on the level of the benefit, the calculations were made separately for men and women. Result obtained are presented in tables 5-8.

Tab. 5. The effect of gender and minimum retirement age on the benefit level (K=100000 zlotys, γ=7%, i=3%)

The minimum retirement age *x*		Monthly pension annuity (zlotys)	
		women	men
Case I	*x*=60 years	466.75	600.84
	x=65 years	570.51	724.36
	x=70 years	695.12	872.43
Case II	*x*=60 years	536.99	674.99
	x=65 years	635.87	794.90
	x=70 years	760.84	947.45

Source: Developed by the author.

Tab. 6. The effect of gender and minimum retirement age on the benefit level (K=400000 zlotys, γ=7%, i=3%)

The minimum retirement age *x*		Monthly pension annuity (zlotys)	
		women	men
Case I	*x*=60 years	1867.02	2403.36
	x=65 years	2282.03	2897.44
	x=70 years	2780.49	3489.72
Case II	*x*=60 years	2147.97	2699.96
	x=65 years	2543.50	3179.62
	x=70 years	3043.38	3789.80

Source: Developed by the author.

Tab. 7. The effect of gender and minimum retirement age on the amount of benefit (K=100000 zlotys, γ=7%, i=5%)

The minimum retirement age x		Monthly pension annuity (zlotys)	
		women	men
Case I	x=60 years	621.11	759.33
	x=65 years	712.73	875.29
	x=70 years	837.38	1025.12
Case II	x=60 years	674.20	813.55
	x=65 years	763.94	928.73
	x=70 years	889.18	1082.53

Source: Developed by the author.

Tab. 8. The effect of gender and minimum retirement age on the benefit level (K=400000 zlotys, γ=7%, i=5%)

The minimum retirement age x		Monthly pension annuity (zlotys)	
		women	men
Case I	x=60 years	2484.43	3037.00
	x=65 years	2850.90	3501.16
	x=70 years	3349.51	4100.46
Case II	x=60 years	2696.77	3254.20
	x=65 years	3055.78	3714.96
	x=70 years	3573.92	4341.06

Source: Developed by the author.

9. Conclusions

The results obtained reveal substantial variations in the benefits calculated using the future and current life tables (see cases I and II) and indicate that the pension annuities grow when the minimum retirement age is moved upwards. Lower values of monthly payments are provided from using future life tables in the calculations (case I) instead of period life-tables for the year 2007 (case II).

The illustrative results presented in tables 5–8 show that calculating the annuities using the period life-tables may expose the annuity provider to a risk of the considerable overestimation of benefits and thus may cause troubles with covering future liabilities.

We can also see that the female's pension annuities are lower both in the case I and II. This can be explained through the fact that expected lifetime in the female population is longer than in the male population, which directly contributes to lower amounts of benefits. It is also worth noting that even though gender is a distinct determinant of different benefits, in practice they are calculated using the common life-tables. This leads to the overestimation of benefits for females and underestimation of benefits for males.

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