ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 48, 1985

Czesław Domański*, Andrzej Tomaszewicz*

THE POWER OF TESTS BASED ON THE LENGTH OF RUNS

1. Introduction

The paper is a continuation of the research concerning the following tests based on the length of runs (see [2]):

- a test based on the maximum length of runs on one of the median (S_A),
- a test based on a smaller among the maximum lengths of runs above and below the median (S_D),
- a test based on a bigger among the maximum lengths of runs above and below the median (S_G).

These tests could be applied in verification of hypotheses on independence of the sequence of observations, in determination of the trend in the time series, in verification of the hypothesis on the linearity of the econometric model with one or more independent variables.

The aim of this paper is to formulate some conclusions concerning the power of tests based on the length of runs which are applied in verification of the hypothesis on independence of subsequent elements in a sample.

We shall confine our consideration to the case of stationary Markov chain with two states which are traditionally denoted as A and B and the transition matrix

$$\begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} = \begin{bmatrix} 1-q_0 & q_0 \\ q_1 & 1-q_1 \end{bmatrix}.$$

*Dr., Lecturer at the Institute of Econometrics and Statistics, University of Łódź.

Let $P_{n,\vartheta}$ be a distribution of this chain for each $\vartheta \in \Theta = \{(q_0, q_1): 0 < q_0 < 1, \ 0 < q_1 < 1\}$ and let $\Omega_n = \{A, B\}^n$ be a set of all-n-element sequences formed from elements A and B. Thus, we shall consider the probability spaces

(1) $$M_{n,\vartheta} = \left(\Omega_n, 2^{\Omega_n}, P_{n,\vartheta}\right) \text{ for } \vartheta \in \Theta.$$

The formulated conclusions are based on the numerically determined power of tests basing on the distribution of the length of runs for $n = 1, 2, \ldots, 100$ and on several dozen pairs chosen from the set Θ. The combinatorial formulae of probability connected with runs distribution presented in the literature (cf. [3], [4]) are inconvenient for numerical calculations. It is more efficient to use recursive formulae especially in the case when the calculations are made for subsequent values of n.

2. Recursive Formulae for Univariate Runs Distributions

We shall assign to each sequence

$$\omega = (x_1, x_2, \ldots, x_n) \in \Omega_n$$

the following numbers:

$N_A(\omega)$ - number of elements A in sequence ω,
$L_A(\omega)$ - number of runs consisting of elements A,
$L(\omega)$ - total number of runs.

Assume that sequences $\omega \in \Omega_n$ are the realizations of the stationary Markov chain with the transition matrix

$$\begin{bmatrix} P_{AA} & P_{BA} \\ P_{AB} & P_{BB} \end{bmatrix},$$

where $0 < P_{AB}, P_{BA} < 1$. Hence, the stationary probabilities are given by the formulae

(2) $$p_A = P(X_j = A) = \frac{P_{AB}}{P_{AB} + P_{BA}}, \quad p_B = P(X_j = B) = \frac{P_{BA}}{P_{AB} + P_{BA}}$$

for $j = 1, 2, \ldots, n$.

Under these assumptions the probability distribution on the set can be presented by the formula

$$(3)\quad P(\omega)=\frac{1}{p_{AB}+p_{BA}}\, p_{AA}^{(N_A-L_A)}\, p_{AB}^{L_A}\, p_{BA}^{(L-L_A)}\, p_{BB}^{(n-N_A-L+L_A)},$$

where n denotes sample size, $N_A(\omega)$ - the number of A-type elements in the sample, $L(\omega)$ - total number of runs, $L_A(\omega)$ - number of runs consisting of A-type elements.

Equation (3) results immediately from the identity

$$(4)\quad P(\omega)=P(X_1=x_1)\, P(X_2=x_2|X_1=x_1)\ \dots\ P(X_n=x_n|X_{n-1}=x_{n-1})$$

after taking into account equation (2).

Determine for given n the following random variables:

S_A - the maximum length of runs consisting of A-type elements,

S_B - the maximum length of runs consisting of B-type elements,

$S_D=\min\{S_A,S_B\}$,

$S_G=\max\{S_A,S_B\}$.

Having this notation we can formulate the following theorems whose proofs, as of little interest, are omitted.

T h e o r e m 1. The distribution of variable S_A determined on $M_{n,\vartheta}$ is expressed by the recursive formula

$$(5)\qquad P(S_A=s)=Q_0^A(n,s)+Q_1^A(n,s),$$

where

$$Q_0^A(n,s)=\sum_{v=1}^{n-s} Q_1^A(n-v,s)\, q_0(1-q_1)^{v-1},$$

$$Q_1^A(n,s)=\sum_{v=0}^{s-1} Q_0^A(n-s,v)\, q_1\,(1-q_0)^{s-1}+\sum_{w=1}^{s} Q_0^A(n-w,s)\, q_1(1-q_0)^{w-1}$$

under initial conditions

$$Q_0^A(0,0)=Q_1^A(0,0)=\frac{1}{q_0+q_1}.$$

In the same way the distribution of S_B statistic is obtained.

Theorem 2. The distribution of variable S_G determined on $M_{n,\vartheta}$ is expressed by the recursive formula

$$P(S_G=s) = Q_0^G(n,s) + Q_1^G(n,s), \tag{6}$$

where:

$$Q_h^G(n,s) = \sum_{v=0}^{s-1} Q_{1-h}^G(n-s,v)\, q_h (1-q_{1-h})^{s-1} + $$

$$+ \sum_{w=1}^{s} Q_{1-h}^G(n-w,s)\, q_h (1-q_{1-h})^{w-1}$$

for h = 0,1, at initial conditions

$$Q_0^G(0,0) = Q_1^G(0,0) = \frac{1}{q_0+q_1}.$$

The distribution of variable S_D can be determined on the basis of the following relation

$$P(S_D \le s) = P(S_A \le s) + P(S_B \le s) - P(S_G \le s). \tag{7}$$

3. Power Evaluation

On the basis of the recursive formulae presented in § 2 the power of randomized tests was determined numerically for n = 1, 2, ..., 100 and for some pairs (p, ϱ), where

$$p = p_A \frac{q_1}{q_0 + q_1}, \quad \varrho = 1 - q_0 - q_1. \tag{8}$$

The procedure was as follows:

- if $S^A < s_\alpha^A - 1$, then $H_0 : \varrho = 0$ is accepted,
- if $S^A \geq s_\alpha^A$, then H_0 is rejected in favour of $H_1 : \varrho > 0$,
- if $S^A = s_\alpha^A - 1$, then H_0 is accepted with probability r_α^A.

Let for the determined n, p, significance level α and $\varrho = 0$

$$F_A(s) = P(S_A \le s), \quad s = 0, 1, \ldots \tag{9}$$

The critical value of the test based on S_A statistic will be therefore

$$s_\alpha^A = \min\left\{ s : F_A(s) > 1 - \alpha \right\}. \tag{10}$$

To this value the randomizing probability corresponds

$$r_\alpha^A = \frac{F_A(s_\alpha^A) - (1 - \alpha)}{F(s_\alpha^A) - F_A(s_\alpha^A - 1)}, \tag{11}$$

which was presented in [2].

In the same way randomized tests based on statistics S_B, S_D and S_G were determined.

The results of power calculation for the tests S_A, S_B S_D (verifying the hypothesis $H_0 : \rho = 0$) for $\alpha = 0.05$, $p = 0.5$, 0.7, 0.9, and $\rho = 0.1$, 0.3, 0.5, 0.7, 0.9 and $n = 5, 10, \ldots, 100$, as well as for a simple alternative hypothesis $H_1 : \rho = \rho_1$ presented in Tables 1-4, allow to formulate the following conclusions.

1. The power of tests being considered is the highest for $p = 0.5$. (This is confirmed by already quoted result obtained by B a t e m a n [1]).

2. The test based on S_A statistic proved to be stronger than the test based on S_B statistic for $p > 0.5$, excluding the cases of very strong correlation ($\rho > 0.7$).

3. Among the tests of the runs length the most frequently used test S_G proved to be stronger than tests S_A and S_B, excluding the cases of big asymmetry ($\rho > 0.6$) and of very strong autocorrelation ($\rho > 0.7$).

4. The test S_G is stronger than the test S_D only for p close to 0.5 and for not very strong autocorrelation at a relatively small size of samples.

5. With the increase of p the difference between the power of tests S_A and S_G, and S_B and S_D decreases rapidly (excluding the cases of very small n ($n < 15$)). These differences remain significant only in the case of strong autocorrelation ($\rho > 0.5$).

Table 1

Power of tests of runs S_A for some pairs (p, ϱ) and $\alpha = 0.05$

n	p = 0.5					p = 0.7					p = 0.9				
	ϱ					ϱ					ϱ				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
5	68	118	190	289	420	59	81	109	148	184	52	57	62	67	73
10	74	145	243	360	467	69	124	214	353	562	55	67	81	98	118
15	78	165	292	436	524	71	134	236	386	586	58	79	107	143	190
20	82	184	336	505	583	73	146	265	432	625	62	93	140	207	306
25	84	197	371	560	634	75	157	294	485	668	65	110	183	302	492
30	87	212	406	613	682	77	168	320	532	712	68	123	221	391	683
40	90	232	455	686	756	80	183	363	605	785	69	127	229	404	694
50	93	250	500	746	814	82	197	400	664	839	70	133	243	428	714
60	95	268	542	795	859	84	208	431	711	879	71	139	260	458	741
80	98	290	593	853	916	87	227	482	781	932	73	152	295	526	807
100	101	313	644	898	950	89	242	522	830	961	75	163	327	588	872

Note: All probabilities have been multiplied by 1000.

Table 2

Power of tests of runs S_B for some pairs (p, ϱ) and $\alpha = 0.05$

n	p = 0.5					p = 0.7					p = 0.9				
	ϱ					ϱ					ϱ				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
5	68	118	190	289	420	73	127	185	241	285	76	114	133	134	116
10	74	145	243	360	467	82	167	263	338	344	90	154	191	193	146
15	78	165	292	436	524	88	194	319	415	401	96	180	238	246	178
20	82	184	336	505	583	91	214	365	482	455	102	207	284	299	209

Table 2 (contd.)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
25	84	197	371	560	634	95	236	411	545	507	108	234	328	349	240
30	87	212	406	613	682	100	259	456	602	556	115	260	370	395	270
40	90	232	455	686	756	104	286	512	678	633	128	311	447	479	327
50	93	250	500	746	814	107	303	553	734	696	142	360	515	552	380
60	95	268	542	795	859	110	322	593	782	749	155	401	571	612	427
80	98	290	593	853	916	116	361	667	856	830	158	421	616	682	504
100	101	313	644	898	950	122	401	731	906	885	161	442	659	741	572

N o t e: All probabilities have been multiplied by 1000.

T a b l e 3

Power of tests of runs S_D
for some pairs (p, ϱ) and $\alpha = 0.05$

n	$p = 0.5$					$p = 0.7$					$p = 0.9$				
	ϱ					ϱ					ϱ				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
5	54	59	56	43	18	58	67	66	51	21	71	98	102	81	34
10	70	120	175	202	124	75	135	192	211	122	90	153	186	176	90
15	78	159	261	335	234	87	189	304	366	239	96	180	237	241	140
20	85	197	350	468	350	91	211	353	448	325	102	207	284	298	185
25	89	217	397	543	486	94	231	402	523	406	108	234	328	348	255
30	91	235	443	614	517	99	256	452	590	479	115	260	370	395	260
40	101	289	550	741	654	106	296	527	685	593	128	311	447	479	323
50	103	304	588	796	741	108	309	561	737	672	142	360	515	552	378
60	107	333	646	853	812	111	326	598	759	735	155	401	571	612	427
80	116	385	730	917	899	116	364	669	820	825	158	421	616	682	504
100	119	409	773	948	944	123	403	732	861	884	161	442	659	741	572

N o t e: All probabilities have been multiplied by 1000.

Table 4

Power of tests of runs S_G
for some pairs (p, ϱ) and $\alpha = 0.05$

n	p = 0.5					p = 0.7					p = 0.9				
	ϱ					ϱ					ϱ				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
5	73	143	253	418	652	60	86	123	175	248	52	57	63	70	79
10	80	179	344	579	862	69	126	226	406	739	55	67	81	99	124
15	85	209	419	701	954	71	135	247	443	778	58	79	107	143	196
20	89	233	480	780	980	73	147	276	497	838	62	93	140	208	313
25	92	249	519	827	991	75	158	304	551	900	65	110	183	302	500
30	95	267	562	869	996	77	167	329	594	926	68	123	221	391	691
40	98	292	618	914	999	80	183	371	663	963	69	127	229	404	702
50	101	313	665	944	1000	82	197	407	717	980	70	133	243	428	722
60	104	337	711	966	1000	84	208	437	759	989	71	139	260	458	749
80	108	360	756	981	1000	87	227	487	820	997	73	152	295	526	815
100	111	387	802	991	1000	89	243	527	861	999	75	163	327	588	879

Note: All probabilities have been multiplied by 1000.

BIBLIOGRAPHY

[1] Bateman G. (1948); On the Power Function of the Longest Run as a Test for Randomness in a Sequence of Alternatives "Biometrika" 35, p. 97-112.

[2] Domański C., Tomaszewicz A. (1980), Variants of Tests Based on the Length of Runs. Paper presented at the conference "Problems of Building and Estimation of Large Econometric Models", Polanica Zdrój.

[3] Mood A. M. (1940), The Distribution Theory of Runs, Ann. of Math. Statist. 11, p. 367-392.

[4] Omsted P. S. (1958); Runs Determined in a Sample by an Arbitrary Cut, Bell System Techn. Jour. 37, p. 55-58.

Czesław Domański, Andrzej Tomaszewicz

MOC TESTÓW OPARTYCH NA DŁUGOŚCI SERII

Artykuł dotyczy analizy mocy testów opartych na maksymalnej długości serii z jednej strony mediany (S_A), mniejszej z maksymalnych długości serii z każdej strony mediany (S_D), większej z maksymalnych długości serii z każdej strony mediany (S_G) weryfikujących hipotezę o niezależności kolejnych elementów w próbie. W świetle przeprowadzonych badań uzyskano między innymi następujące wnioski: test S_G okazał się mocniejszy od testów S_A i S_B, wyjąwszy przypadki dużej asymetrii i autokorelacji; test S_G jest mocniejszy od testu S_D tylko w przypadku p bliskich 0,5 i małej autokorelacji; moc rozważanych testów jest największa dla $p = 0,5$.

U.Ł.