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THE ESTIMATION OF CES PRODUCTION FUNCTION PARAMETERS BY THE AXIAL DOUBLE ITERATION METHOD

1. Introduction

The earlier investigations [1], [2], [7] on the estimation methods for CES production function in the form

$$Y = \alpha\left[\delta K^{-\varrho} + (1-\delta)L^{-\varrho}\right]^{-\frac{\nu}{\varrho}} + \varepsilon, \tag{1}$$

where:

- Y - production output,
- K - fixed assets,
- L - employment,
- α - production scale parameter,
- δ - distribution parameter,
- ν - function homogeneity parameter,
- ϱ - substitution parameter,
- ε - random term,

proved that the estimation of a four-parameter function can be limited to the estimation of a two-parameter function, after having estimated the parameter ν by K m e n t a method ([3]) and determined the estimates of parameter α directly from the first equation of the system of normal equations. Hence, the

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above problem has been confined to the determination of such values of A, b, r for which the function

$$(2) \qquad Q(A,b,r) = \sum_{i=1}^{n} Y_i - A\left[bK_i^{-r} + (1-b)L_i^{-r}\right]^{-\frac{c}{r}}\Big\}^2,$$

would attain its minimum at an a priori given value of c and at the values of A determined according to

$$(3) \qquad A = \frac{\sum_{i=1}^{n} Y_i\left[bK_i^{-r} + (1-b)L_i^{-r}\right]^{-\frac{c}{r}}}{\sum_{i=1}^{n}\left[bK_i^{-r} + (1-b)L_i^{-r}\right]^{-\frac{2c}{r}}},$$

(The form of (3) was generated in [2] from the system of normal equations built for the function (2)).

To determine the values of b and r a so called double iteration method was suggested in [2]. The method consists in a step-by-step searching for extreme values of b and r. Then, similar calculations are made for the other parameter. The whole iteration process, constructing two interrelated blocks, is carried out so long that the two normal equations for the function (2) are solved with a predetermined accuracy.

The Monte-Carlo experiments confirmed the numerical efficiency of the double iteration method, i.e. regardless the choice of a starting point the predetermined parameters were always obtained. However, the iterative process - although converging with the predetermined values of parameters - was too slow and costly. Therefore investigations are carried out to make it more numerically efficient since the shortening of time needed for calculations is necessary also in the case when the properties of estimators of model parameters are studied by means of numerical experiments.

The paper presents a modification of the double iteration method, further called the axial double iteration method.

2. The Axial Double Iteration Method

It follows from the investigations [5] that the criterion function Q (A(b, r), b, r) in the form (2) defines the paraboloid-shaped area, and its level lines are the curves close to ellipses with common axes (from the earlier study [4] it followed that the "axis" of level lines symmetry was approximately a straight line). The point of intersection of these axes determined, with some numerical accuracy, the point in which function Q (A(b, r), b, r) attained its minimum. This was the basis for modifying the way of determination of the extreme values of b and r, i.e. for assuming another way of solution of the system of equations:

$$(4)\qquad \begin{cases} F_2(A(b,r),b,r) = \dfrac{\partial}{\partial b} Q(A(b,r),b,r) = 0, \\ F_3(A(b,r),b,r) = \dfrac{\partial}{\partial r} Q(A(b,r),b,r) = 0. \end{cases}$$

This way was different from the double iteration method proposed and discussed in [2]. Namely, at given r_o point b_o is looked for in such a way that the function F_2 differs from zero less than by a given accuracy WW, and similarly - for the given value of r a corresponding value of b_1 is searched for.

The obtained points (b_o, r_o) and (b_1, r_1) are the basis for determining the straight line corresponding to the "axis" of symmetry for level lines. This axis is determined as a straight line by the equation:

$$(5)\qquad b = D_1 r + D_2,$$

where:

$$D_1 = \frac{b_1 - b_o}{r_1 - r_o}, \quad D_2 = b_o - D_1 r_o.$$

Assuming that the straight line (5) determines the axis of ellipses, on this line we search the point (b(r), r) in which the function Q has a conditional minimum. This corresponds to the determination of a zero place of a respective derivative of

function $Q(A(b(r), r), b(r), r)$, i.e. to the solution of the equation

$$NF_3 = F_3(A,b,r) + D_1F_2(A,b,r) = 0, \tag{6}$$

where:

$$A = A(b,r), \quad b = D_1r + D_2.$$

For a numerical realization of the axial double iteration method the program # CES 7 has been developed (this program is available at the Library of Programs, Institute of Econometrics and Statistics, University of Łódź). The results of Monte-Carlo experiments obtained using the program # CES 7 allow us to evaluate properties of the presented method, and especially to answer the following questions:

1. Does the determined straight line (5) depend on the assumed values of starting points? If so, in what way?

2. What is the influence of the correlation level of variables K and L on the values taken by D_1 and D_2?

3. What is the relationship between the directions of the straight line (5) and the values of parameter ρ determining the substitution elasticity $\sigma = \frac{1}{1+\rho}$ in the function?

4. What is the behaviour of the estimates of function parameters (1) in relation to the number of observations (n) and random term ε?

3. A Numerical Model

To answer the above questions the following numerical model of experiments has been constructed:

a) the sequence of n values $L_1, L_2, \ldots, L_i, \ldots, L_n$, has been defined,

b) assuming s different levels of correlation between variables K and L (measured by the values of correlation coefficients $\rho(K, L)$ the s sequences of the values on variable K, i.e. $K_{1s}, K_{2s}\ldots, K_{is}, \ldots, K_{ns}$ were determined,

c) for the set of pairs $\{(L_i, K_{is}),\ i = 1, \ldots, n\}$ and for

the assumed values of parameters α, δ, ρ the values of $YT_{is} = \alpha [\delta K_{is}^{-\rho} + (1-\delta) L_{is}^{-\rho}]^{-1/\rho}$ were calculated.

Parameters α and δ, as shown in [6], can be determined at an arbitrary level by assuming measuring units for K, L and Y. In the experiments various levels were assumed for the parameter ρ.

In this way the s sets of points $D_s = \{(L_i, K_{is}, YT_{is}), i = 1, \ldots, n\}$ on respective functional surfaces were obtained.

In the experiments the following levels of parameters were assumed:

α: 2,

δ: 0.4,

ρ: -0.3, 0.2, 1.

$\rho(K, L)$: 0.324, 0.723, 0.936,

and starting values:

r_o : 0.3, 0.5,

r_1 : -0.5, -0.4, -0.3, -0.2, -0.1, 0.1, 0.15, 0.19, 0.20 0.21, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0,8, 0.9, 1.0, 1.2, 1.5.

The respective points $b_o = b(r_o)$ and $b_1 = b(r_1)$ were determined in such a way that

$$(7) \qquad \left| \frac{\partial}{\partial b} Q(A(b,r), b, r) \right| < 10^{-6}.$$

The sets D_s were treated as samples. They served for checking the numerical efficiency of the axial double iteration method (ADIM) at various choice of starting points r_o and r_1. They were also a basis for the analysis of the accuracy of ADIM estimates of function parameters (1) in the case of random term occurring in the variable Y.

For each set D_s the IP new sets of $D_s^1, \ldots, D_s^m, \ldots, D_s^{IP}$ were determined so that

$$D_s^m = \{(L_i, K_i, Y_{is}^m), \quad i = 1, \ldots, n\},$$

where Y_{is}^m is a realization of a random variable defined as

$$(8) \qquad Y_{is}^m = YT_{is} + \varepsilon_i^m, \qquad m = 1, \ldots, IP,$$

where ε_i^m is the i-th realization of the random variable ε^m

with the distribution $N(0, \sigma_\varepsilon)$ (ε was generated using the procedure NORGEN with alpha). The variance σ_ε^2 was chosen so as to be 0.1% and 1.0% of variability of Y. For each set D_s a sequence of sets $\{D_s^m, m = 1, \ldots, IP\}$ was obtained.

4. Location of the Level Axis

On the level axis there are points with coordinates $(r_o, b(r_o))$. They were determined assuming r_o and defining $b(r_o)$ in such a way that relation (7) holds. Table 1 presents coordinates of the points on the level axes determined for three various functional areas differentiated by the correlation degree of K and L. The Table also presents the values of residuals $e(r_o)$ obtained as a difference between the values of $b(r_o)$ and $\hat{b}(r_o)$. The values of $\hat{b}(r_o)$ were calculated from the LSM regression line:

$$\hat{b}(r) = D_1^* r + D_2^* \tag{9}$$

determined for the obtained sets of points with coordinates $(r_o, b(r_o))$.

Three regression lines for three subsets Y_i, K_i, L_i $i = 1, \ldots, n$ corresponding to three levels $\varrho(K, L) = 0.324,\ 0.727,\ 0.936$ were determined. These are:

$$\hat{b} = 0.1198296r + 0.375885 \quad \text{for } \varrho(K,L) = 0.324, \tag{10}$$

$$\hat{b} = 0.1372636r + 0.3726333 \quad \text{for } \varrho(K,L) = 0.727, \tag{11}$$

$$\hat{b} = 0.1547703r + 0.3697843 \quad \text{for } \varrho(K,L) = 0.936. \tag{12}$$

From the results presented in Table 1 it follows that in each case being considered the values of $b(r_o)$ increase with the increase in r_o. At a fixed level of r_o the values of $b(r_o)$ increase with the increase in the correlation degree of variables K, L when $r_o > \varrho$, and they decrease when $r_o < \varrho$. Hence, the determined regression lines (10)-(12) are characterized by the increase of intercept value of D_1^* and the decrease of random term D_2^* for higher values of $\varrho(K, L)$.

An interesting tendency can be observed in the signs of re-

Table 1

Values of coordinates b (r_o) for selected values of r_o

r_o	$\varrho(K,L) = 0.324$		$\varrho(K,L) = 0.727$		$\varrho(K,L) = 0.936$	
	$b(r_o)$	$e(r_o)$	$b(r_o)$	$e(r_o)$	$b(r_o)$	$e(r_o)$
1.00	0.4950	-0.00071	0.5093	-0.00060	0.5252	+0.00065
0.90	0.4834	-0.00033	0.4961	-0.00007	0.5098	+0.00072
0.80	0.4717	-0.00005	0.4826	+0.00016	0.4943	+0.00070
0.70	0.4599	+0.00013	0.4691	+0.00038	0.4787	+0.00058
0.60	0.4481	+0.00032	0.4554	+0.00041	0.4630	+0.00035
0.50	0.4361	+0.00030	0.4416	+0.00033	0.4472	+0.00003
0.40	0.4241	+0.00028	0.4278	+0.00026	0.4314	-0.00029
0.30	0.4121	+0.00027	0.4139	+0.00009	0.4157	-0.00052
0.25	0.4060	+0.00018	0.4069	+0.00005	0.4078	-0.00068
0.21	0.4012	+0.00015	0.4014	-0.00006	0.4016	-0.00069
0.20	0.4000	+0.00015	0.4000	-0.00009	0.4000	-0.00074
0.19	0.3988	+0.00015	0.3986	-0.00011	0.3984	-0.00079
0.15	0.3940	+0.00014	0.3931	-0.00012	0.3922	-0.00080
0.10	0.3879	+0.00003	0.3861	-0.00026	0.3844	-0.00086
-0.10	0.3637	-0.00020	0.3585	-0.00041	0.3536	-0.00071
-0.20	0.3517	-0.00022	0.3448	-0.00038	0.3385	-0.00033
-0.30	0.3397	-0.00024	0.3313	-0.00015	0.3236	+0.00025
-0.40	0.3277	-0.00025	0.3178	+0.00007	0.3089	+0.00102
-0.50	0.3159	-0.00007	0.3046	+0.00060	0.2945	+0.00210

Note: In the column $e(r_o)$ the values of residuals i.e. $e(r_o) = b(r_o) - \hat{b}(r_o)$ are given.

siduals $e(r_o)$. Namely, the grouping of residuals of the same sign is an evidence that there is autocorrelation in the residuals series. From a formal point of view this autocorrelation can provide evidence that, first, there are systematic errors in numerical calculations, and secondly, there can be an erroneous hypothesis that the symmetry axis of level lines is a straight line. Changing absolute values of residuals back up the second variant rather, although they do not exclude the first one. It is worthwhile to note that with the increase of correlation degree

of variables the distribution of positive and negative residuals changes and their absolute value increases.

Tables 2-4 present the values of coefficients D_1 and D_2 of straight lines determined in (6). They are obtained in such a way that all of them have one common point $r_o = 0.5$ and $b(r_o)$ and the point $(r_1, b(r_1))$ changes. In most of the investigated

Table 2

Coefficients D_1 and D_2 of straight line (5) determined for $(r_o, b_o) = (0.5, 0.4361)$ at $\varrho(K,L) = 0.324$

r_1	D_1	D_2	$d^*(1, (0.2, 0.4))$
1.00	0.117630	0.377326	0.000846 (p)
0.90	0.118138	0.377072	0.000695 (p)
0.80	0.118747	0.376727	0.000473 (p)
0.70	0.119035	0.376624	0.000428 (p)
0.60	0.119415	0.376434	0.000314 (p)
0.40	0.120039	0.376121	0.000129 (p)
0.30	0.120282	0.376000	0.000056 (p)
0.25	0.120381	0.375951	0.000027 (p)
0.21	0.120455	0.375916	0.000005 (p)
0.20	0.120471	0.375906	0.000000
0.19	0.120486	0.375898	0.000005 (n)
0.15	0.120547	0.375868	0.000023 (n)
0.10	0.120608	0.375838	0.000041 (n)
-0.10	0.120716	0.375783	0.000073 (n)
-0.20	0.120687	0.375798	0.000064 (n)
-0.30	0.120599	0.375841	0.000039 (n)
-0.40	0.120453	0.375915	0.000007 (p)
-0.50	0.120248	0.376018	0.000008 (p)

*Symbol d (1,(0.2, 0.4)) denotes the distance between the real point (0.2, 0.4) and the straight line:l : $b = D_1 r + D_2$ symbols (p) and (n) in the fourth column denote that the point (0.2, 0.4) is below the straight line l and above the straight line l, respectively.

cases the real point (ϱ, δ) lies below these straight lines which back up the statement that the determined estimates of parameter δ can be overestimated to a great extent. However, when r_1 is

chosen near the actual value of parameter δ the straight lines being determined are quite near to the point (ϱ, δ).

T a b l e 3

Coefficients D_1 and D_2 of straight line (5) determined for $(r_o, b_o) = (0.5, 0.4416)$ at $\varrho(K,L) = 0.727$

r_1	D_1	D_2	d(1,(0.2, 0.4))
1.00	0.135495	0.373847	0.000937 (p)
0.90	0.136142	0.373524	0.000746 (p)
0.80	0.136724	0.373233	0.000572 (p)
0.70	0.137238	0.372976	0.000420 (p)
0.60	0.137676	0.372757	0.000289 (p)
0.40	0.138326	0.372432	0.000096 (p)
0.30	0.138500	0.372350	0.000050 (p)
0.25	0.138599	0.372295	0.000015 (p)
0.21	0.138641	0.372274	0.000002 (p)
0.20	0.138650	0.372270	0.000000
0.19	0.138658	0.372266	0.000002 (n)
0.15	0.138678	0.372256	0.000008 (n)
0.10	0.138683	0.372254	0.000009 (n)
-0.10	0.138483	0.372353	0.000049 (p)
-0.20	0.138248	0.372471	0.000119 (p)
-0.30	0.137922	0.372634	0.000216 (p)
-0.40	0.137507	0.372842	0.000340 (p)
-0.50	0.137002	0.373094	0.000490 (p)

N o t e: Cf. Table 2.

T a b l e 4

Coefficients D_1 and D_2 of straight line (5) determined for $(r_o, b_o) = (0.5, 0.4472)$ at $\varrho(K,L) = 0.936$

r_1	D_1	D_2	d (1,(0.2, 0.4))
1	2	3	4
1.00	0.155913	0.369247	0.000424 (p)
0.90	0.156469	0.368969	0.000260 (p)
0.80	0.156923	0.368742	0.000125 (p)

Table 4 (contd.)

1	2	3	4
0.70	0.157271	0.368568	0.000022 (p)
0.60	0.157516	0.368445	0.000051 (n)
0.40	0.157655	0.368376	0.000092 (n)
0.30	0.157559	0.368424	0.000063 (n)
0.25	0.157465	0.368471	0.000036 (n)
0.21	0.157370	0.368518	0.000008 (n)
0.20	0.157333	0.368533	0.000000
0.19	0.157316	0.368545	0.000008 (p)
0.15	0.157195	0.368606	0.000044 (p)
0.10	0.157015	0.368696	0.000098 (p)
-0.10	0.156008	0.369199	0.000396 (p)
-0.20	0.155334	0.369536	0.000596 (p)
-0.30	0.154551	0.369928	0.000785 (p)
-0.40	0.153660	0.370374	0.001093 (p)
-0.50	0.152665	0.370871	0.001388 (p)

N o t e: Cf. Table 2.

5. Results of the Monte-Carlo Experiments

The Monte-Carlo experiments concerned a chosen sample $\{(YT_i, K_i, L_i),\ i = 1, \ldots, n\}$ for which the correlation coefficient $\rho(K, L)$ was about 0.723. The sample contained: n = 20, 30, 40 elements, respectively. In each sample there were randomly distributed the values of variable Y assuming Y_i defined by (8) instead of YT_i.

Thus, three sets of samples in the form

$$\left\{\{(Y_i^m, K_i, L_i)\ ,\ i = 1, \ldots, n\}\ m = 1, \ldots IP\right\},$$

were obtained (according to the number n). They were used to determine the realization of sample estimators of expected values of parameter estimates in model (1) and their selected characteristics such as variance estimates, variability coefficients, bias values, RSME (variances of evaluations around the actual values of parameters).

The above measures were determined for IP = 5, 10, 15, 20, 25, ..., 95, 100 in order to observe their changes when the number of samples increased and then to formulate corresponding hypotheses on the properties of the estimators of model parameters (1) obtained using the axial double iteration method.

It follows from the experiments carried out by the authors that with the increase in the number of IP the fluctuations of mean estimates corresponding to A, b, r decrease but the estimates of parameter α are not biased numerically, while the estimates of parameter δ are generally overestimated. Mean values of bias are lower than the standard deviation. The estimates of parameter ρ are characterized by higher variability. The value of r is estimated to be the worst. The increase in the sample size (n) affects positively the results of estimation, i.e. with the increase of n the values of bias and respective variances decrease.

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ESTYMACJA PARAMETRÓW FUNKCJI PRODUKCJI TYPU CES OSIOWĄ METODĄ PODWÓJNEJ ITERACJI

W artykule omówiono metodę estymacji parametrów funkcji produkcji typu CES nazwaną osiową metodą podwójnej iteracji oraz wyniki eksperymentu Monte-Carlo przeprowadzonego dla tej metody. Eksperymenty te miały na celu zbadanie numerycznych własności otrzymanych ocen parametrów funkcji CES. Przeprowadzono je dla wybranych prób dwudziesto-, trzydziesto-, czterdziestoelementowych, w których współczynnik korelacji między zmiennymi objaśniającymi wynosi 0,723. Z badań tych wynika, że wraz ze wzrostem ilości iteracji maleją wahania ocen średnich odpowiednich realizacji estymatorów parametrów: skali produkcji (α), podziału (δ), substytucji(ρ). Oceny parametru α nie wykazują numerycznego obciążenia, natomiast oceny parametru δ są z reguły przeszacowane. Średnie wielkości obciążeń są jednak mniejsze niż jedno odchylenie standardowe z próby. Większą zmiennością charakteryzują się oceny parametru ρ.