A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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A METHOD OF COMPUTING MOMENTS OF THE DURBIN-WATSON STATISTIC FOR LINEAR TREND

1. INTRODUCTION

To verify the hypothesis concerning the lack of autocorrelation in an econometric model

$$y = X\alpha + \varepsilon \tag{1}$$

the Durbin-Watson test is usually used because of simple calculations and quite a big power as well. The applications of the test, however, are limited because the so-called "nonconclusivity interval" does exist. It is connected with the fact that the distribution of the Durbin-Watson statistic depends on the matrix X, thus tabulating of the critical values is practically impossible. Commonly known tables contain only lower and upper limits of the quantiles corresponding with the most frequently applied significance levels.

There is a possibility, however, to build such tables when X matrix is fixed. The table for the linear trend case when

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \\ 1 & n \end{bmatrix} \tag{2}$$

is shown in the paper by T o m a s z e w i c z [4].

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2. THE PROBLEM

An other possibility of investigating the Durbin-Watson statistic distribution which seems to be worth taking into account is to find its moments or cumulants. This paper presents general formulae for computing moments of the statistic for linear trend model. The formulae for the mean and standard deviation are shown as well.

We consider econometric model (1) (where X has the form (2)) which fulfils classical assumptions i.e. ε is of n-dimensional normal distribution:

$$\varepsilon \sim N(o, \sigma^2 I).$$

The Durbin-Watson statistic is given by the formula

$$d = \frac{\sum_{t=2}^{n} (e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2} = \frac{e^T A e}{e^T e}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

and e is the vector of residuals e_t obtained by the ordinary least squares method:

$$e = y - Xa$$

where

$$a = (X^T X)^{-1} X^T y.$$

Vector e can be expressed in the following form

$$e = M\varepsilon$$

where

$$M = I - X(X^TX)^{-1}X^T \tag{3}$$

is very well known idempotent matrix. Further on we shall also use the symbols:

$$u = e^TAe, \quad v = e^Te,$$

so

$$d = \frac{u}{v}.$$

On the basis of von N e u m a n n' s result [3] D u r b i n and W a t s o n [1] proved that the variables $d = u/v$ and v are independent. Consequently, for given k we have

$$Eu^k = Ed^kv^k = Ed^kEv^k,$$

hence

$$Ed^k = \frac{Eu^k}{Ev^k}. \tag{4}$$

To find d statistic moments it is enough to calculate the moments Eu^k and Ev^k. D u r b i n and W a t s o n [1] found formulae for moments d on the basis of the sums of the form

$$\sum_{t=1}^{n-k_1} \nu_t^q = tr(MA)^q.$$

In particular

$$Ed = \frac{1}{n-k_1}\sum_{t=1}^{n-k_1} \nu_t, \tag{5}$$

$$Evd = \frac{2}{(n-k_1)(n-k_1+2)}\left(\sum \nu_t^2 - \frac{1}{n-k_1}\left(\sum \nu_t\right)^2\right) \tag{6}$$

where

$$\sum \nu_t = tr(MA) = trA - tr\ X^TAX(X^TX)^{-1},$$

$$\sum \nu_t^{\ 2} = tr(MA)^2 = trA^2 - 2tr(X^TA^2X(X^TX)^{-1}) +$$

$$+ tr(X^TAX(X^TX)^{-1})^2.$$

3. FORMULAE FOR LINEAR TREND

We present formulae (4), (5) and (6) for the case of the linear trend model in which the X matrix is given by (2).

Let us denote

$$M = I - N, \quad N = X(X^TX)^{-1}X^T.$$

Therefore

$$(MA)^k = (A - NA)^k = \sum_{h=0}^{k} (-1)^h \binom{k}{h} A^{k-h} (NA)^h.$$

As from the very well known properties of trace

$$tr\ A^{k-h} (NA)^h = tr\ A^{k-h} \underbrace{NANA \ldots NA}_{h \text{ times } NA} =$$

$$= tr\ A^{k-h} X(X^TX)^{-1} \underbrace{X^TAX(X^TX)^{-1}\ X^TA \ldots X(X^TX)^{-1}\ X^TA}_{h-1 \text{ terms } X^TAX(X^TX)^{-1}} =$$

$$= tr\ X^TA^{k-h+1}\ X(X^TX)^{-1}\ (X^TAX(X^TX)^{-1})^{h-1},$$

thus,

$$s_k = tr(MA)^k = tr\ A^k +$$

$$+ \sum_{h=1}^{k} (-1)^h \binom{k}{h} tr\ B_{k-h+1}\ (X^TX)^{-1}\ (B_1(X^TX)^{-1})^{h-1} \tag{7}$$

where

$$B_l = X^T A^l X$$

for $l = 1, 2, \ldots, k$. Let

$$X = [j \quad x].$$

Of course $Aj = 0$, hence each matrix B_k for $k \geqslant 1$ is of the form

$$B_k = \begin{bmatrix} 0 & 0 \\ 0 & x^T A^k x \end{bmatrix}.$$

Therefore the first row of each of $B_k (X^T X)^{-1}$ matrices consists of zeroes, thus, the trace of the product

$$B_k (X^T X)^{-1} (B_1 (X^T X)^{-1})^{h-1}$$

is equal to the product of diagonal elements of the second row of each element:

$$\operatorname{tr} B_k (X^T X)^{-1} (B_1 (X^T X)^{-1})^{h-1} = b_k b_1^{h-1},$$

where

$$b_k = \frac{12}{n(n-1)(n+1)} x^T A^k x.$$

However, for $k \geqslant 2$

$$x^T A^k x = \frac{2}{k-1} \binom{2(k-2)}{k-2},$$

thus, finally

$$b_k = \frac{24}{n(n-1)(n+1)(k-1)} \binom{2(k-2)}{k-2} \tag{8}$$

for $k \geqslant 2$ and

$$b_1 = \frac{12}{n(n+1)} \tag{9}$$

because

$$x^T A x = n - 1.$$

Taking into account (8), (9) and the formula

$$\mathrm{tr} A^k = n \frac{(2k - 1)!!}{k!} 2^k - 2^{2k-1},$$

(7) can be rewritten in the form

$$s_k = \mathrm{tr}(MA)^k = n \frac{(2k - 1)!!}{k} 2^k - 2^{k-1} +$$

$$+ \sum_{h=1}^{k} (-1)^h \binom{k}{h} \frac{1}{(n - 1)(k - 1)} \qquad (10)$$

$$+ \frac{1}{n-1} \sum_{h=1}^{k} (-1)^h \binom{k}{h} \frac{1}{k - h} \left(\frac{2(k - h - 1)}{k - h - 1}\right) \left(\frac{24}{n (n + 1)}\right)^h.$$

In particular

$$s_1 = \mathrm{tr}\, MA = \mathrm{tr}\, A - b_1$$

$$+ 2(n - 1) - 2(n - 1) \frac{12}{n (n - 1)(n + 1)}$$

$$= 2(n - 1) \left(1 - \frac{12}{n (n - 1)(n + 1)}\right), \qquad (11)$$

$$s_2 = \mathrm{tr}\, A^2 - 2b_2 + b_1^2$$

$$= 2 (3n - 4) - \frac{48}{n (n + 1)(n - 1)} + \frac{576}{n^2 (n + 1)^2}.$$

Values s_k (10) put in the formulae for the moments m_k of the numerator u distribution can be expressed as functions of its cumulants $\varkappa_k(u)$ (see: D u r b i n and W a t s o n [1], p. 99-106 also K e n d a l l and S t u a r t [2], p. 511-512):

$$\varkappa_k(u) = 2^{k-1}\,(k-1)!\,s_k.$$

Particularly

$$\varkappa_1(u) = s_1, \quad \varkappa_2(u) = 2s_2,$$

so

$$m_1 = Eu = \varkappa_1(u) = s_1, \tag{12}$$

$$m_2 = Eu^2 = \varkappa_2(u) - \varkappa_1(u)^2 = 2s_2 + s_1^2.$$

The moments of the denominator v which is of χ^2 distribution with $n-2$ degress of freedom are given by the formula

$$Ev^k = \frac{(n-4+2k)!!}{(n-4)!!},$$

Especially

$$Ev = n - 2, \tag{13}$$

$$Ev^2 = (n-2)n.$$

It is easy to obtain from (4) and (11)-(13)

$$Ed = \frac{s_1}{n-2} = 2\,\frac{n-1}{n-2}\left(1 - \frac{12}{n\,(n-1)(n+1)}\right) = \tag{14}$$

$$= 2 - \frac{20}{n\,(n+1)(n-2)}$$

and

$$D^2d = \frac{Eu^2}{Ev^2} - (Ed)^2$$

$$= \frac{2s_1 + s_2}{n\,(n-2)} - \frac{s_1^2}{(n-2)^2} \tag{15}$$

$$= \frac{2s_2}{n\,(n-2)} - \frac{2s_1}{n\,(n-2)^2}.$$

Table 1

Moments of the Durbin-Watson statistics:
lower DL, upper DU and for linear trend DT

Mean				Variance		Std. deviation	
N	DL	DU	DT	DL,DU	DT	DL,DU	DT
5	1.4607	2.5393	2.5333	0.3346	0.3342	0.5784	0.5781
6	1.5670	2.4330	2.4286	0.3542	0.3551	0.5951	0.5959
7	1.6396	2.3604	2.3571	0.3488	0.3499	0.5906	0.5915
8	1.6920	2.3080	2.3056	0.3340	0.3350	0.5780	0.5788
9	1.7315	2.2685	2.2667	0.3163	0.3170	0.5624	0.5631
10	1.7622	2.2378	2.2364	0.2982	0.2988	0.5461	0.5466
11	1.7868	2.2132	2.2121	0.2810	0.2814	0.5301	0.5305
12	1.8068	2.1932	2.1923	0.2649	0.2653	0.5147	0.5150
13	1.8235	2.1765	2.1758	0.2502	0.2504	0.5002	0.5004
14	1.8375	2.1625	2.1619	0.2367	0.2369	0.4865	0.4867
15	1.8495	2.1505	2.1500	0.2244	0.2246	0.4737	0.4739
16	1.8599	2.1401	2.1397	0.2132	0.2133	0.4617	0.4619
17	1.8689	2.1311	2.1307	0.2030	0.2031	0.4505	0.4506
18	1.8769	2.1231	2.1228	0.1936	0.1937	0.4400	0.4401
19	1.8840	2.1160	2.1158	0.1850	0.1851	0.4301	0.4302
20	1.8903	2.1097	2.1095	0.1771	0.1772	0.4209	0.4209
21	1.8959	2.1041	2.1039	0.1698	0.1699	0.4121	0.4122
22	1.9010	2.0990	2.0988	0.1631	0.1632	0.4039	0.4039
23	1.9056	2.0944	2.0942	0.1569	0.1569	0.3961	0.3961

24	1.9099	2.0901	2.0900	0.1511	0.1511	0.3887	0.3888
25	1.9137	2.0863	2.0862	0.1457	0.1457	0.3817	0.3818
26	1.9173	2.0827	2.0826	0.1407	0.1407	0.3751	0.3751
27	1.9205	2.0795	2.0794	0.1360	0.1360	0.3688	0.3688
28	1.9236	2.0764	2.0764	0.1316	0.1316	0.3628	0.3628
29	1.9264	2.0736	2.0736	0.1275	0.1275	0.3570	0.3570
30	1.9290	2.0710	2.0710	0.1236	0.1236	0.3515	0.3516
31	1.9314	2.0686	2.0685	0.1199	0.1199	0.3463	0.3463
32	1.9337	2.0663	2.0663	0.1165	0.1165	0.3413	0.3413
33	1.9358	2.0642	2.0642	0.1132	0.1132	0.3365	0.3365
34	1.9378	2.0622	2.0622	0.1101	0.1101	0.3319	0.3319
35	1.9396	2.0604	2.0603	0.1072	0.1072	0.3274	0.3274
40	1.9475	2.0525	2.0524	0.0946	0.0946	0.3076	0.3076
45	1.9536	2.0464	2.0464	0.0847	0.0847	0.2910	0.2910
50	1.9584	2.0416	2.0416	0.0766	0.0766	0.2768	0.2768
55	1.9623	2.0377	2.0377	0.0699	0.0699	0.2645	0.2645
60	1.9656	2.0344	2.0344	0.0643	0.0643	0.2536	0.2536
65	1.9683	2.0317	2.0317	0.0596	0.0596	0.2440	0.2440
70	1.9706	2.0294	2.0294	0.0554	0.0554	0.2355	0.2355
75	1.9726	2.0274	2.0274	0.0519	0.0519	0.2277	0.2277
80	1.9744	2.0256	2.0256	0.0487	0.0487	0.2207	0.2207
85	1.9759	2.0241	2.0241	0.0459	0.0459	0.2143	0.2143
90	1.9773	2.0227	2.0227	0.0434	0.0434	0.2084	0.2084
95	1.9785	2.0215	2.0215	0.0412	0.0412	0.2030	0.2030
100	1.9796	2.0204	2.0204	0.0392	0.0392	0.1979	0.1979

As it can be seen, the Durbin-Watson statistic moments even in the easiest case of the linear trend model are expressed by quite complicated formulae, which are difficult to be applied in further analytic research. Nevertheless, using those formulae largely facilitates the numerical analysis in comparison with direct calculations $tr(MA)^k$. That procedure was applied to find the first two moments of d statistics for some sample sizes. The results are shown in Table 1. Of course, the same procedure can be applied to compute higher-order moments.

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O PEWNEJ METODZIE OBLICZANIA MOMENTÓW STATYSTYKI DURBINA-WATSONA ZWIĄZANEJ Z LINIOWYM TRENDEM

Celem artykułu jest opis pewnej numerycznie atrakcyjnej metody obliczania momentów statystyki testu Durbina-Watsona dla hipotezy o braku autokorelacji w liniowym modelu trendu.

D u r b i n i W a t s o n [1] podali sposób obliczania momentów tej statystyki w zależności od sum wartości własnych potęg macierzy MA.

Autorowi udało się znaleźć wyrażenie, za pomocą którego można te sumy wyznaczyć.

Metoda została zastosowana do obliczenia dwóch pierwszych momentów. Wyniki są zawarte w tab. 1.