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ESTIMATION OF SOME DISEQUILIBRIUM MODELS BY MAXIMUM LIKELIHOOD METHOD

1. INTRODUCTION

The problem of market disequilibrium modelling has been appearing in econometric and statistic literature for many years (see: [1], [2], [3], [4], [5], [6], [7], [8], [9]). Different methods are proposed to estimate the econometric models parameters for markets where demand differs from supply. There are the following methods: the ordinary least squares method ([2], [3]), the two - stage least squares method ([1], [2], [3], [4]), the maximum likelihood method[1] ([1], [3], [4], [5], [7], [9]). Some attempts are also being made to apply the switching regression method in econometrics of disequilibrium (e.g. [6], p. 269-273).

This paper is devoted to the estimation of parameters of the disequilibrium models with quantitative indicator[2] by the maximum likelihood method. Market disequilibrium signifies the presence of the surplus in demand or supply in each period under the examination[3]. Theoretical works on this subject (see: [1], [2], [4], [6], [7], [9]) usually end with the presentation of

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[1] Wide studies on this subject are contained in [9].

[2] Quantitative disequilibrium indicator is a variable allowing us to determine the quantity of the market disequilibrium.

[3] There are some definitions of the market disequilibrium (see: [2], p. 9-11).

the likelihood function and they recommend iterative methods to determine its maximum.

The aim of this paper is to describe a noniterative algorithm for determining the values of the maximum likelihood (ML) estimator in the case of disequilibrium model for a single market. All the explanatory variables in the equations of demand and supply are exogenous. There are two versions of the algorithm presented according to the form of diseguilibrium indicator equation.

Section 2 introduces the discription of a disequilibrium model. Section 3 contains a presentation of noniterative algorithm for determining ML estimators of the model parameters. Section 4 describes a modification of the algorithm.

2. FORM OF THE DISEQUILIBRIUM MODEL

Econometric disequilibrium model represents market processes with the demand - supply imbalance. There are some types of the models. They are distinguished according to the form of the information concerning the market disequilibrium ratio ([2], p. 19-30).

In this paper we shall deal with the disequilibrium model with linear structural relations where in the equations of demand and supply as explanatory variables we have exogenous variables. This version has the following form ([2], p. 30-31):

$$d_t = x'_{Dt}\alpha + u_{Dt} \tag{2.1}$$

$$s_t = x'_{St}\beta + u_{St} \tag{2.2}$$

$$q_t = \min\{d_t, s_t\} \tag{2.3}$$

$$y_t = \gamma(d_t - s_t) \tag{2.4}$$

where;

$t = 1, 2, \ldots, T$;

d_t - effective demand on a given market;

s_t - supply on a given market;

x_{Dt}, x_{St} - non-random vectors of explanatory variables for demand and supply equation;

α, β, γ - structural parameters of the model (α, β - vectors, γ - scalar);

q_t - quantity transacted;

u_{Dt}, u_{St} - random components;

y_t - variable jointly interdependent with d_t and s_t, it represents the quantity of market disequilibrium (the so called "disequilibrium indicator");

T - the length of time series.

We assume that the random components u_{Dt} and u_{St} have normal distribution with zero mathematical expectation, constant variances δ_D^2 and δ_S^2 and they are not correlated with each other and in time.

Previous empirical researches of market disequilibrium employed variable y_t as the increase in price (e.g. [1], [3], [4], [5]). In disequilibrium models for the labour market it may represent, for example, the number of vacancies per one person looking for a job[4].

The variables y_t and q_t are the only observable endogenous variables in the model (2.1)-(2.4).

3. ML - ESTIMATORS FOR PARAMETERS OF DISEQUILIBRIUM MODEL

There are different methods known to estimate the disequilibrium model parameters. They are: the maximum likelihood method, the two - stage least squares method, the ordinary least squares method employed to transformated equations of the model[5].

The maximum likelihood method for the disequilibrium model is presented below. It consists in determining such values of the model parameters for which equivalent likelihood function achieves its maximum. For the model (2.1)-(2.4) it has the following form:

$$L(y_t, q_t; \alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2) =$$

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[4] Such a variable was used in [8] in the labour market model.

[5] This method is proposed in [2], p. 86-93.

$$= \prod_{t=1}^{T} f(y_t, q_t; \alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2) \qquad (3.1)$$

where $f(y_t, q_t; \alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2)$ is the joint density function of the random variables possible to be observed (y_t, q_t). As the random disturbances u_{Dt}, u_{St} are not correlated and normally distributed, the likelihood function (3.1) may be defined as follows[6]:

$$L(y_t, q_t; \alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2) = |\gamma|^{-T} (2\pi\sigma_D\sigma_S)^{-T} \cdot$$

$$\cdot \exp\left\{-\frac{1}{2\sigma_D^2}\left[\sum_{t\in \mathcal{T}_1}\left(\frac{1}{\gamma}y_t + q_t - x'_{Dt}\alpha\right)^2 + \right.\right.$$

$$\left. + \sum_{t\in \mathcal{T}_2}(q_t - x'_{Dt}\alpha)^2\right] - \frac{1}{2\sigma_S^2}\left[\sum_{t\in \mathcal{T}_1}(q_t - x'_{St}\beta)^2 + \right.$$

$$\left.\left. + \sum_{t\in \mathcal{T}_2}\left(q_t - \frac{1}{\gamma}y_t - x'_{St}\beta\right)^2\right]\right\} \qquad (3.2)$$

where $\mathcal{T}_1 = \{t: 1 \leqslant t \leqslant T$ and $d_t > s_t\}$, $\mathcal{T}_2 = \{t: 1 \leqslant t \leqslant T$ and $d_t < s_t\}$.

The function L in (3.2) is a nonlinear function with respect to the parameters $\alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2$. Its maximum is difficult to be found. Former used iterative methods of determining this maximum are : Newton-Raphson method, Powell method, "quadratic hill climbing" method ([3], [4]) or noniterative methods of solving non-linear equations systems ([1], [5]).

Below we present a new analytic method of determining the maximum of the function (3.2).

To simplify the notation we shall leave arguments y_t and q_t and use the following symbols:

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[6] Analogous form of the likelihood function was obtained by A m e m i y a in [1].

$$\mathbf{X}_D = \begin{bmatrix} x'_{D1} \\ \cdot \\ \cdot \\ \cdot \\ x'_{DT} \end{bmatrix}, \quad \mathbf{X}_S = \begin{bmatrix} x'_{S1} \\ \cdot \\ \cdot \\ \cdot \\ x'_{ST} \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_T \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_T \end{bmatrix} \tag{3.3}$$

where

$$a_t = \begin{cases} q_t + \frac{1}{\gamma} y_t & \text{for} \quad t \in \mathcal{T}_1 \\ q_t & \text{for} \quad t \in \mathcal{T}_2 \end{cases}$$

$$b_t = \begin{cases} q_t & \text{for} \quad t \in \mathcal{T}_1 \\ q_t - \frac{1}{\gamma} y_t & \text{for} \quad t \in \mathcal{T}_2 \end{cases}$$

The vectors **a** and **b** can be presented in the following form:

$$\mathbf{a} = \mathbf{q} + \frac{1}{\gamma}\, \mathbf{g}, \qquad \mathbf{b} = \mathbf{q} - \frac{1}{\gamma}\, \mathbf{h} \tag{3.4}$$

where:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_T \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_T \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_T \end{bmatrix}$$

$$g_t = \begin{cases} y_t & \text{for} \quad t \in \mathcal{T}_1 \\ 0 & \text{for} \quad t \in \mathcal{T}_2 \end{cases}$$

$$h_t = \begin{cases} 0 & \text{for} \quad t \in \mathcal{T}_1 \\ y_t & \text{for} \quad t \in \mathcal{T}_2 \end{cases}$$

Then we have

$$L\,(\alpha,\, \beta,\, \gamma,\, \sigma_D^2,\, \sigma_S^2) = |\gamma|^{-T} (2\pi \sigma_D \sigma_S)^{-T} \cdot \tag{3.5}$$

$$\cdot \exp \left\{ - \frac{1}{2\sigma_D^2} (a - X_D\alpha)'(a - X_D\alpha) - \frac{1}{2\sigma_S^2}(b - X_S\beta)'(b - X_S\beta) \right\}$$

so the logarithm of the likelihood function has the form:

$$\ln L\,(\alpha, \beta, \gamma, \sigma_D^2, \sigma_S^2) = - T\ln|\gamma| - T\ln 2\pi - \frac{T}{2}\ln\sigma_D^2 + \tag{3.6}$$

$$- \frac{T}{2}\ln\sigma_S^2 - \frac{1}{2\sigma_D^2}(a - X_D\alpha)'(a - X_D\alpha) - \frac{1}{2\sigma_S^2}(b - X_S\beta)'(b - X_s\beta)$$

It is not difficult to prove that equations $\partial \ln L / \partial\sigma_D^2 = 0$ and $\partial \ln L / \partial\sigma_s^2 = 0$ are true for:

$$\hat{\sigma}_D^2 = \frac{1}{T}(a - X_D\alpha)'(a - X_D\alpha)$$

$$\hat{\sigma}_s^2 = \frac{1}{T}(b - X_s\beta)'(b - X_s\beta) \tag{3.7}$$

Having substituted σ_D^2 and σ_s^2 in (3.6) by (3.7) and leaving aside the constant $-T\ln 2\pi$ we have the concentrated likelihood function:

$$L^*(\alpha, \beta, \gamma) = -T\ln|\gamma| - \frac{T}{2}\ln(a - X_D\alpha)'(a - X_D\alpha) +$$

$$- \frac{T}{2}\ln(b - X_S\beta)'(b - X_S\beta) - T\ln\frac{1}{T} - T \tag{3.8}$$

Differentiating L^* with respect to α and β and equating the derivatives to zero leads to equations:

$$\hat{\alpha} = (X_D' X_D)^{-1} X_D' a$$

$$\hat{\beta} = (X_S' X_S)^{-1} X_S' b \tag{3.9}$$

It can be observed that for $y_t = 0$ formulas (3.7) and (3.9) show the form of estimators of parameters α, β, σ_D^2, σ_S^2 obtained by the least squares method. These formulas correspond with the model of equilibrium market.

After replacing α and β in (3.6) by (3.7) we have the following forms of the concentrated likelihood function:

$$L^{**}(\gamma) = -\ln|\gamma| - \frac{1}{2}\ln a'M_D\, a - \frac{1}{2}\ln b'M_S\, b \qquad (3.10)$$

where[7]

$$M_D = I - X_D\,(X_D'X_D)^{-1}\,X_D' \qquad (3.11)$$
$$M_S = I - X_S\,(X_S'\,X_S)^{-1}\,X_S'$$

Taking into account formulas (3.4), function (3.10) can be written as:

$$L^{**}(\gamma) = -\ln|\gamma| - \frac{1}{2}\ln\,(q + \frac{1}{\gamma}g)'\,M_D\,(q + \frac{1}{\gamma}g) + \qquad (3.12)$$
$$-\frac{1}{2}\ln\,(q - \frac{1}{\gamma}h)'\,M_S\,(q - \frac{1}{\gamma}h).$$

Equating its derivative with respect to γ to zero leads to the following equation:

$$a\gamma^4 + b\gamma^3 + c\gamma^2 + d\gamma + e = 0 \qquad (3.13)$$

where:

$$a = q'\,M_S q q'\,M_D q,$$

$$b = q'\,M_D g q' M_S q - q' M_s h q' M_D q$$

$$c = 0$$

$$d = q' M_S h g' M_D g - q' M_D g h' M_S h$$

$$e = -g' M_D g h' M_S h$$

Roots of the equation (3.13) meet the necessary condition for the existence of the function (3.12) extreme[8]. These roots should be put into the formulas (3.7) and (3.9) to find stationary points of the likelihood function. The estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\sigma}_D^2$, $\hat{\sigma}_S^2$ are the points in which partial derivatives of the first order of the function (3.6) are equal to zero.

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[7] I is an identity matrix.

[8] Similar version to the method presented was used in [8].

We should notice that the function from (3.6) can be rewritten as the following function:

$$\ln L(\gamma) = -T \ln|\gamma| + K_1 + \frac{1}{\gamma} K_2 + \frac{1}{\gamma^2} K_3, \qquad (3.14)$$

where

$$K_1 = -T\ln(2\pi\sigma_D\sigma_S) - \frac{1}{2\sigma_D^2} \sum_{\mathcal{T}_1 \cup \mathcal{T}_2} (q_t - x'_{Dt}\alpha)^2 +$$

$$- \frac{1}{2\sigma_S^2} \sum_{\mathcal{T}_1 \cup \mathcal{T}_2} (q_t - x'_{St}\beta)^2,$$

$$K_2 = - \frac{1}{\sigma_D^2} \sum_{t \in \mathcal{T}_1} y_t (q_t - x'_{Dt}\alpha) + \frac{1}{\sigma_S^2} \sum_{t \in \mathcal{T}_2} (q_t - x'_{St}\beta) y_t,$$

$$K_3 = - \frac{1}{2}\left[\frac{1}{\sigma_D^2} \sum_{t \in \mathcal{T}_1} y_t^2 + \frac{1}{\sigma_S^2} \sum_{t \in \mathcal{T}_2} y_t^2\right].$$

Equation $d\ln L/d\gamma = 0$ leads to the equation $2K_3 + K_2\gamma + T\gamma^2 = 0$, which has the following roots:

$$\gamma' = \left(-K_2 - \sqrt{K_2^2 - 8TK_3}\right)/2T < 0,$$

$$\gamma'' = \left(-K_2 + \sqrt{K_2^2 - 8TK_3}\right)/2T > 0.$$

(The expression under the square root sign is positive because $K_3 < 0$).

Values of the second derivatives of the function (3.14) in points γ' and γ'' can be calculated according to the formulas:

$$\left.\frac{d^2 \ln L}{d\gamma^2}\right|_{\gamma = \gamma'} = - \frac{8T^3 \sqrt{K_2^2 - 8TK_3}}{\left(\sqrt{K_2^2 - 8TK_3} + K_2\right)^3} < 0,$$

$$\left.\frac{d^2 \ln L}{d_\gamma^2}\right|_{\gamma = \gamma''} = - \frac{8T^3 \sqrt{K_2^2 - 8TK_3}}{\left(\sqrt{K_2^2 - 8TK_3} - K_2\right)^3} < 0.$$

It results from our reasoning[9] that for established values of α, β, σ_D^2, σ_S^2 function (3.6) reaches its maxima in points γ' and γ'' (it is so, because of the indeterminancy of the function L in the point $\gamma = 0$). Moreover for the fixed value of γ the function (3.6) reaches its maximum for α, β, σ_D^2, σ_S^2 determined by the formulae (3.7) and (3.9). To find the global maximum of the likelihood function one should consider the highest from among the local maxima. The $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$; $\hat{\sigma}_D^2$, $\hat{\sigma}_S^2$ values corresponding with it are the estimates of the (2.1)-(2.4) model parameters obtained by the maximum likelihood method.

Premises, based e.g. on the theory of economy, allow us to limit the range of γ variation to the interval with only one local maximum. If, for example, variable y_t in the model (2.1)-(2.4) signifies the price increase, then γ is positive and positive root of the equation (3.13) stands for its estimate.

4. ML - ESTIMATORS OF DISEQUILIBRIUM MODEL'S PARAMETERS WITH MODIFIED INDICATOR EQUATION

In the model considered above we assumed that the market disequilibrium is described by the equation $d_t - s_t = y_t / \gamma$. The parameter γ does not have to be of the same value during the demand surplus as during the supply surplus.

It can be assumed that $d_t - s_t = y_t/\gamma_1$ when $d_t > s_t$ and $d_t - s_t = y_t/\gamma_2$ when $d_t < s_t$. It causes some modifications of the above presentation.

The likelihood function has the following form:

$$L(\alpha, \beta, \gamma_1, \gamma_2, \sigma_D^2, \sigma_S^2) = |\gamma_1|^{-\tau_1} |\gamma_2|^{-\tau_2} (2\pi\, \sigma_D\, \sigma_S)^{-T} .$$

$$\cdot \prod_{t \in T_1} \exp\left\{-\frac{1}{2}\left[\frac{(y_t/\gamma_1 + q_t - x_{Dt}'\alpha)^2}{\sigma_D^2} + \frac{(q_t - x_{St}'\beta)^2}{\sigma_D^2}\right]\right\}. \qquad (4.1)$$

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[9] Analogous studies are included in [7], where stationary points of the likelihood function are established by iterative methods of solving system of non-linear equations, and y_t stands for the price increase.

$$\cdot \prod_{t \in \mathcal{T}_2} \exp\left\{-\frac{1}{2}\left[\frac{(q_t - x'_{Dt}\alpha)^2}{\sigma_D^2} + \frac{(q_t - y_t/\gamma_2 - x'_{St}\beta)^2}{\sigma_S^2}\right]\right\}$$

where $\tau_1 = \text{card}\mathcal{T}_1$, $\tau_2 = \text{card}\mathcal{T}_2$ ("card" signifies the cardinal number of a set). Introducing the notation:

$$c_t = \begin{cases} y_t/\gamma_1 + q_t & \text{when } t \in \mathcal{T}_1 \\ q_t & \text{when } t \in \mathcal{T}_2 \end{cases}$$

$$d_t = \begin{cases} q_t & \text{when } t \in \mathcal{T}_1 \\ q_t - y_t/\gamma_2 & \text{when } t \in \mathcal{T}_2 \end{cases}$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_T \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ \cdot \\ \cdot \\ \cdot \\ d_T \end{bmatrix}$$

we obtain:

$$\ln L(\alpha, \beta, \gamma_1, \gamma_2, \sigma_D^2, \sigma_S^2) = -\tau_1 \ln|\gamma_1| - \tau_2 \ln|\gamma_2| - T\ln 2\pi +$$
$$-\frac{T}{2}\ln\sigma_D^2 - \frac{T}{2}\ln\sigma_S^2 - \frac{1}{2\sigma_D^2}(\mathbf{c} - X_D\alpha)'(\mathbf{c} - X_D\alpha) + \tag{4.2}$$
$$-\frac{1}{2\sigma_S^2}(\mathbf{d} - X_S\beta)'(\mathbf{d} - X_S\beta)$$

Analogously as for $\gamma_1 = \gamma_2 = \gamma$, we can formulate the formulae:

$$\hat{\sigma}_D^2 = (\mathbf{c} - X_D\alpha)'(\mathbf{c} - X_d\alpha)/T$$

$$\hat{\sigma}_S^2 = (\mathbf{d} - X_s\beta)'(\mathbf{d} - X_s\beta)/T \tag{4.3}$$

$$\hat{\alpha} = (X_D'X_D)^{-1}X_D'\mathbf{c}$$

$$\hat{\beta} = (X_S' X_S)^{-1} X_S' d$$

The concentrated likelihood function has then the form:

$$\ln L(\gamma_1, \gamma_2) = (T - \tau_1) \ln|\gamma_1| + (T - \tau_2) \ln|\gamma_2| +$$

$$- \frac{T}{2} \ln (\gamma_1 q + g)' M_D (\gamma_1 q + g) + \tag{4.4}$$

$$- \frac{T}{2} \ln (\gamma_2 q - h)' M_S (\gamma_2 q - h)$$

Equating its partial derivatives to zero leads to the following two quadratic equations:

$$a_1 \gamma_1^2 + b_1 \gamma_1 + c_1 = 0 \tag{4.5}$$

where

$$a_1 = \tau_1 q' M_D q,$$

$$b_1 = (\tau_1 - T) q' M_D g,$$

$$c_1 = (\tau_1 - T) g' M_D g$$

and

$$a_2 \gamma_2^2 + b_2 \gamma_2 + c_2 = 0 \tag{4.6}$$

where

$$a_2 = \tau_2 q' M_S q,$$

$$b_2 = (T - \tau_2) q' M_S h,$$

$$c_2 = (\tau_2 - T) h' M_S h.$$

Let γ_1' and γ_1'' be the roots of (4.5) and γ_2' and γ_2'' be the roots of (4.6). The likelihood function has four stationary points: (γ_1', γ_2'), (γ_1', γ_2''), (γ_1'', γ_2'), (γ_1'', γ_2''). As it was in the case when $\gamma_1 = \gamma_2 = \gamma$ we can prove that maximum exists in these points and find the global maximum of the likelihood function (4.4).

4. FINAL REMARKS

There are two versions of the algorithm for obtaining ML - estimators of values of one market disequilibrium model's parameters presented in this paper. The algorithm consists in the analitical determining of the maximum for the established likelihood function. It can be defined as the solution of the fourth degree equation or of two quadratic equations with the use of formulae analogous to those of the estimatores obtained by the least squares method (in the case of linear regression). Thanks to the algorithm we are able to find such values of the model parameters for which the likelihood function reaches its global maximum. Iterative procedures recommended in the literature, to solve similar problems, may prove to be divergent or convergent to the local maximum which is not the global one. The wrong choice of the initial values in iterative methods may make the estimation of the model parameters by the maximum likelihood method very difficult and increases the time of computation. This problem does not exist in the algorithm presented above.

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ESTYMACJA KILKU MODELI NIERÓWNOWAGI
ZA POMOCĄ METODY NAJWIĘKSZEJ WIARYGODNOŚCI

Przedmiotem artykułu jest opis nieiteracyjnego algorytmu, określającego wartości estymatora największej wiarygodności w przypadku modelu nierównowagi dla pojedynczego rynku. Wszystkie zmienne objaśniające w równaniach popytu i podaży są egzogeniczne. Prezentowane są dwie wersje algorytmu, odpowiadające zadanej postaci równania indykatora nierównowagi.