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RELATIONS BETWEEN CHEMICAL BALANCE WEIGHING DESIGNS FOR $p = v$ AND $p = v + 1$ OBJECTS

ABSTRACT. The incidence matrices of ternary balanced block designs for v treatments have been used to construct chemical balance weighing designs for $p = v$ and $p = v+1$ objects with uncorrelated estimators of weights. Conditions under which the existence of a chemical balance weighing designs with uncorrelated estimators of weights for v objects implies the existence of the design with the same restrictions for $v+1$ objects are given. The existence of a chemical balance weighing design with uncorrelated estimators of weights for $v+1$ objects implies the existence of the design with the same restrictions for $p < v+1$ objects.

Key words: chemical balance weighing design, ternary balanced block design.

I. INTRODUCTION

In the presented paper we study the problem of constructing the design matrix **X** for the chemical balance weighing designs for $p = v$ and $p = v+1$ objects and relations between these designs, when matrix **X** is based on the incidence matrices of ternary balanced block designs. The problem is how to choose the matrix **X** in such a manner that the estimators of weights are uncorrelated. Several methods of constructing matrix **X** are available in the literature. Ceranka, Katulska and Mizera (1998), Ambroży and Ceranka (1999), Ceranka and Graczyk (2000) have shown how chemical balance weighing design with uncorrelated estimators of weights can be constructed from the incidence matrices of ternary balanced block

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constructed chemical balance weighing designs with uncorrelated estimators of weights for $p = v + 1$ objects from incidence matrices of ternary balanced block designs for v treatments. C e r a n k a and K a t u l s k a (1991) and (1999) have shown relations between parameters of chemical balance weighing designs in situation, when matrix $\mathbf{X}$ of chemical balance weighing design was based on the incidence matrices of balanced incomplete block designs and on balanced bipartite block designs, respectively.

The results of n weighing operations to determine the individual weights of p objects with a balance that is corrected for bias will fit into the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where $\mathbf{y}$ is an $n \times 1$ random observed vector of the recorded results of the weights, $\mathbf{X} = (x_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$, is an $n \times p$ matrix of known elements with $x_{ij} = -1, 1, 0$ if the j-th object is kept on the right pan, left pan, or is not included in the i-th weighing operation, respectively, $\mathbf{w}$ is the $p \times 1$ vector representing the unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors such that $\mathsf{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathsf{E}(\mathbf{e}\mathbf{e}') = \sigma^2 \mathbf{I}_n$, where $\mathbf{0}_n$ is the $n \times 1$ vector with zero elements everywhere, $\mathbf{I}_n$ is the $n \times n$ identity matrix, "E" stands for the expectation and $\mathbf{e}'$ is used for transpose of $\mathbf{e}$. The matrix $\mathbf{X}$ is the design matrix and we refer to the chemical balance weighing design $\mathbf{X}$ with the covariance matrix $\sigma^2 \mathbf{I}_n$.

If the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, the least squares estimates of the true weights are given by

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

and the variance-covariance matrix of $\hat{\mathbf{w}}$ is

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}.$$

A ternary balanced block design to be a design consisting of b blocks, each of size k, chosen from a set of size v in such a way that each of the v elements occurs r times altogether and 0, 1 or 2 times in each block, and each of the $\binom{v}{2}$ distinct pairs of elements occurs λ times. Any ternary balanced block design is regular, that is, each element occurs singly in ρ_1 blocks and is repeated in

ρ_2 designs for $p = v$ objects. C e r a n k a and G r a c z y k (2002) have blocks, where ρ_1 and ρ_2 are constant for the design. Accordingly we write the parameters of ternary balanced block design in the form $v,\ b,\ r,\ k,\ \lambda,\ \rho_1,\ \rho_2$.

Let $\mathbf{N}$ be the incidence matrix of ternary balanced block design. It is straightforward to check that:

$$vr = bk,$$

$$r = \rho_1 + 2\rho_2,$$

$$\lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2) = r(k-1) - 2\rho_2,$$

$$\mathbf{NN}' = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v' = (r + 2\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'.$$

II. MATRIX X BASED ON ONE INCIDENCE MATRIX OF TERNARY BALANCED BLOCK DESIGN

Definition 2.1. In a chemical balance weighing design the estimators of weights are uncorrelated if matrix $\mathbf{X}'\mathbf{X}$ is diagonal.

Let $\mathbf{N}$ denote the incidence matrix of order $v \times b$ of ternary balanced block design. From this matrix we define matrix $\mathbf{X}$ of chemical balance weighing design in the form:

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}' - \mathbf{1}_b\mathbf{1}_v' \\ \mathbf{1}_b\mathbf{1}_v' - \mathbf{N}' \end{bmatrix}, \tag{1}$$

where $\mathbf{1}_a$ is the $a \times 1$ vector of ones. In this design in $n = 2b$ weighings we check weights of $p = v$ objects.

We can see, that matrix $\mathbf{X}$ is the nonsingular matrix if and only if $v \neq k$.

Lemma 2.1. In the chemical balance weighing design with matrix $\mathbf{X}$ given by (1) the estimated weights are uncorrelated if and only if

$$b + \lambda - 2r = 0. \tag{2}$$

Proof is given by A m b r o ż y and C e r a n k a (1999).

Now we consider the design matrix $\mathbf{X}$ in the following form:

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}' - \mathbf{1}_b\mathbf{1}_v' & \mathbf{1}_b \\ \mathbf{1}_b\mathbf{1}_v' - \mathbf{N}' & \mathbf{1}_b \end{bmatrix}. \quad (3)$$

In this design in $n = 2b$ weighing operations we check weights of $p = v+1$ objects.

We can see, that matrix $\mathbf{X}$ is the nonsingular matrix if and only if $v \neq k$.

Lemma 2.2. In the chemical balance weighing design with matrix $\mathbf{X}$ given by (3) the estimated weights are uncorrelated if and only if (2) holds.

Proof of this Lemma was given by Ceranka and Graczyk (2000).

From Lemma 2.1 and Lemma 2.2 we have the following Theorem.

Theorem 2.1. In a chemical balance weighing design with matrix $\mathbf{X}$ given by (1) the estimated weights are uncorrelated if and only if in a chemical balance weighing design with matrix $\mathbf{X}$ given by (3) the estimated weights are uncorrelated.

Theorem 2.2. If in a chemical balance weighing design with matrix $\mathbf{X}$ given by (3) the estimated weights are uncorrelated, then any $p < v+1$ columns of this matrix constitute a chemical balance weighing design for p objects in $2b$ weighings, in that the estimated weights are uncorrelated.

Proof. In a chemical balance weighing design with matrix $\mathbf{X}$ given by (3) the estimated weights are uncorrelated if and only if matrix $\mathbf{X}'\mathbf{X}$ is diagonal matrix. This means that in a chemical balance weighing design with matrix $\mathbf{X}$ the estimated weights are uncorrelated if and only matrix $\mathbf{X}$ is a $(2b)\times(v+1)$ matrix of such elements -1,1 and 0, columns of this matrix are orthogonal, which yields the assertion of the theorem.

III. MATRIX X BASED ON TWO INCIDENCE MATRICES OF TERNARY BALANCED BLOCK DESIGNS

Let $\mathbf{N}_i$ denote the incidence matrices of order $v \times b_i$ of ternary balanced block designs with the parameters: v, b_i, r_i, k_i, λ_i, ρ_{1i}, ρ_{2i}, for $i = 1, 2$. From these matrices we define matrix $\mathbf{X}$ of chemical balance weighing design in the form:

$$\mathbf{X}=\begin{bmatrix}\mathbf{N}_1^{'}-\mathbf{1}_{b_1}\mathbf{1}_v^{'}\\ \mathbf{N}_2^{'}-\mathbf{1}_{b_2}\mathbf{1}_v^{'}\end{bmatrix}. \qquad (4)$$

In this design in $n=b_1+b_2$ weighing operations we check weights of $p=v$ objects. We can see, that matrix **X** is the nonsingular matrix if and only if $v \neq k_1$ or $v \neq k_2$.

Lemma 3.1. In the chemical balance weighing design with matrix **X** given by (4) the estimated weights are uncorrelated if and only if

$$b_1+b_2+\lambda_1+\lambda_2-2(r_1+r_2)=0. \qquad (5)$$

Proof of this Lemma was given by C e r a n k a and G r a c z y k (2000).

Now we consider the design matrix **X** in the following form:

$$\mathbf{X}=\begin{bmatrix}\mathbf{N}_1^{'}-\mathbf{1}_{b_1}\mathbf{1}_v^{'} & \mathbf{1}_{b_1}\\ \mathbf{N}_2^{'}-\mathbf{1}_{b_2}\mathbf{1}_v^{'} & \mathbf{0}_{b_2}\end{bmatrix}. \qquad (6)$$

In this design in $n=b_1+b_2$ weighing operations we check weights of $p=v+1$ objects.

We can see, that matrix **X** is the nonsingular matrix if and only if $v \neq k_2$.

Lemma 3.2. In the chemical balance weighing design with matrix **X** given by (6) the estimated weights are uncorrelated if and only if (5) holds and

$$b_1=r_1. \qquad (7)$$

Proof of this Lemma was given by C e r a n k a and G r a c z y k (2002).

From Lemma 3.1 and Lemma 3.2 we have the following Theorem:

Theorem 3.1. If in a chemical balance weighing design with matrix **X** given by (4) the estimated weights are uncorrelated and condition (7) is satisfied, then in a chemical balance weighing design with matrix **X** given by (6) the estimated weights are uncorrelated.

Theorem 3.2. If in a chemical balance weighing design with matrix **X** given by (6) the estimated weights are uncorrelated, then any $p < v+1$ columns of this matrix constitute a chemical balance weighing design for p objects in $b_1 + b_2$ weighings, in that the estimated weights are uncorrelated.

Proof. In a chemical balance weighing design with matrix **X** given by (6) the estimated weights are uncorrelated if and only if matrix **X'X** is the diagonal matrix of order $v+1$. When conditions (5) and (7) hold we obtain theorem.

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RELACJE POMIĘDZY CHEMICZNYMI UKŁADAMI WAGOWYMI DLA $p = v$ ORAZ $p = v + 1$ OBIEKTÓW

W pracy zajmujemy się chemicznymi układami wagowymi o macierzy układu skonstruowanej z macierzy incyndencji trójkowych układów zrównoważonych o blokach niekompletnych. Rozważamy zależności pomiędzy parametrami tych układów dla $p = v$ i $p = v+1$ obiektów. Przedstawiamy warunki, przy których istnienie chemicznego układu wagowego dla $p = v$ obiektów o nieskorelowanych estymatorach wag obiektów implikuje istnienie chemicznego układu wagowego dla $p = v+1$ obiektów, w którym estymatory wag obiektów są nieskorelowane. Z kolei istnienie chemicznego układu wagowego dla $p = v+1$ obiektów o nieskorelowanych estymatorach wag obiektów implikuje istnienie chemicznego układu wagowego dla dowolnego $p < v+1$ o nieskorelowanych estymatorach wag obiektów.