Mieczysław Król*, Tomasz Pisula**

ECONOMETRIC EVALUATION OF EFFECTIVE CONTROL IN A SYSTEM OF "TRANSPORT–RESERVES" TYPE

Abstract. The paper presents an econometric evaluation of possibilities of effective control in a system of 'transport–reserves' type with an emergency supply. The system consists of one sender of resources and one receiver of resources, and has a stock reservoir, which secures continuous operation of the receiver even if there is a supply failure.

The effectiveness of the system operation (in economical terms) was assessed on the basis of the so-called function of losses. The function takes into account the total system losses, which are connected with: resources supply costs, storage costs and emergency supply costs from external suppliers. The function of losses also accounts for additional profits resulting from the sale of the excess resources to external buyers.

The detailed econometric analysis of the function of losses for different variants of the system operation provided a lot of practical conclusions, which make it possible (by means of appropriate choice of the parameters) to control the system in such a way that its operation is the most effective.

Key words: inventory systems, inventory management, optimal control, simulation and forecasting.

1. INTRODUCTION

Our investigations are closely related with the theory of storage, which since the beginning of the 30s of the XX century (i.e. the time of publication a classic Wilson's Economic Order Quantity model) has begun its own very fast progress. The theory of storage began another progress in the 50s of the last century (i.e. first Moran's works related to the water dams systems), and this time because of the applications of probabilistic methods in that theory.

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In the theory of storage very different models are examined. Generally, these models can be classified according to the following criteria:
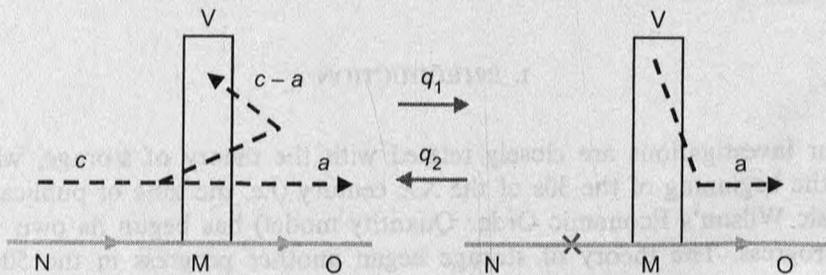
- First criterion refers to the manner of interpretation of time in the models — in scientific literature the systems with continuous and discreet time are considered,

- Second criterion refers to the manner of change of the stock level — in scientific literature the systems with continuous and jump stock level changes are considered.

The models with jump stock level changes embrace: the so-called "inventory models" (which were analyzed by Arrow, Karlin, Scarf) and "insurance models" (which were analyzed by Lundberg, Cramer, Buhlman). The models with continuous stock level changes embrace: "dams models" (for example water dams models) which were analyzed by Moran, Gani, Prabhu and "transport–reserves models" (which were analyzed by Gładysz, Galanc, Król). The systems of "transport–reserves" type are the subject of our analysis.

2. MATHEMATICAL MODEL OF THE SYSTEM OPERATION

Mathematical model of the system operation can be briefly characterized by the following scheme:

The system consists of one sender of resources (marked "N" on the scheme) and one receiver of resources (marked "O"). The receiver requires that resources must be delivered to him at constant speed ($a > 0$). Because supply system is subject to random failure, resources are delivered by sender at constant speed ($c > a > 0$), which is greater than needed, and excess of the resources at speed ($c - a > 0$) is accumulated in the reservoir (marked "M") at constant capacity ($V > 0$), which is located near receiver. The main purpose of the reservoir is to secure continuous operation of the

receiver even if there is a supply failure. In that case, the receiver can get resources directly from the reservoir at speed $(a > 0)$. Frequency of failure occurrence in transport - subsystem can be characterized by parameter $q_1 > 0$, while frequency of failure elimination can be characterized by parameter $q_2 > 0$.

The systems of this type are observed in many branches of national economy, such as: coal processing, oil and earth gas processing, food processing, distribution of commodities etc.

Mathematical model of system operation can be described by the following stochastic partial differential equations:

$$
\begin{cases}
\dfrac{\partial}{\partial t} g_1(t, z) + (c - a) \dfrac{\partial}{\partial z} g_1(t, z) = -q_1(t)g_1(t, z) + q_2(t)g_2(t, z), \\[2mm]
\dfrac{\partial}{\partial t} g_2(t, z) - a \dfrac{\partial}{\partial z} g_2(t, z) = q_1(t)g_1(t, z) - q_2(t)g_2(t, z), \\[2mm]
\dfrac{d}{dt} P_n(t) = -q_1(t)P_n(t) + (c - a)g_1(t, V), \\[2mm]
\dfrac{d}{dt} P_d(t) = -q_2(t)P_d(t) + ag_2(t, 0),
\end{cases}
\tag{1}
$$

where:

t — is time;

z — is the stock level in the reservoir;

$g_1(t, z)$ — is density function of the probability distribution of the reservoir filling, when part "NM" of transport – subsystem is working;

$g_2(t, z)$ — is density function of the probability distribution of the reservoir filling, when part "NM" of transport – subsystem suffers failure;

$P_n(t)$ — is probability of resources excess in the system reservoir;

$P_d(t)$ — is probability of resources shortage in the system reservoir.

The differential equations take into consideration the dynamics of system operation (that means time variability of probability distribution of the reservoir filling, probabilities of resources shortage and excess as well as frequencies of failure occurrence and failure elimination). For such reasons it is very difficult to solve them in the analytical way.

So far, only analytical solutions for the systems of "transport–reserves" type are well known, which operate in stabilized conditions (the so-called "stationary variant"). For that variant the system essential parameters are time independent: $q_1(t) = q_1$, $q_2(t) = q_2$, $g_1(t, z) = g_1(z)$, $g_2(t, z) = g_2(z)$, $P_d(t) = P_d$, $P_n(t) = P_n$.

Moreover, analytical solutions for a "quasi-dynamic" variant of system operation can be obtained. That variant is characterized by equal and time independent frequencies of failure occurrence and failure elimination:

$q_1(t) = q_2(t) = q > 0$, stability of probability distribution of the reservoir filling between lower ($z = 0$) and upper ($z = V$) barrier: $g_1(t, z) = g_1(z)$, $g_2(t, z) = g_2(z)$ and variability in time of the probabilities of resources shortage and excess: $P_d(t)$, $P_n(t)$.

Obtained solutions depend in both variants on the values of the system characteristic parameter:

$$k = \frac{q_1 + q_2}{a(c - a)} (v_z - v_w) \qquad (2)$$

where:

$v_w = a \dfrac{q_1}{q_1 + q_2}$ – is average speed of reservoir emptying;

$v_z = (c - a) \dfrac{q_2}{q_1 + q_2}$ – is average speed of reservoir filling.

If parameter ($k = 0$), then we can say that system works regularly (i.e. that on average the same amount of resources is loaded to reservoir as is taken out of reservoir – reservoir is used effectively and acts as a buffer between sender and receiver).

If parameter ($k \neq 0$), then we can say that system works irregularly (more often than in regular variant the disadvantageous situations of resources shortage or resources excess will occur in reservoir).

Below some examples of obtained solutions for variant "quasi–dynamic" are presented:

Regular variant of the system operation ($k = 0$)

$$\begin{cases} g_1(z) = \omega, \ g_2(z) = \omega, \\ P_d(t) = \mu e^{-qt} + \theta\omega = \mu e^{-qt} + P_{dg}, \\ P_n(t) = -\mu e^{-qt} + \theta\omega = -\mu e^{-qt} + P_{ng} \end{cases} \qquad (3)$$

where:

P_{dg} – is probability of resources shortage for regular and stationary system, for $q = q_1 = q_2 > 0$;

P_{ng} – is probability of resources excess for regular and stationary system, for $q = q_1 = q_2 > 0$;

$\theta = \dfrac{a}{q_2}$, $\omega = \dfrac{1}{2(\theta + V)}$, $-\theta\omega \leqslant \mu \leqslant \theta\omega$ – are some parameters.

If parameter ($\mu = 0$), then we obtain known solutions for the stationary and regular variant with the same frequencies of failure occurrence and failure elimination: $q = q_1 = q_2 > 0$.

Irregular variant of the system operation ($k \neq 0$)

$$\begin{cases} g_1(z) = \omega e^{kz}, \ g_2(z) = \left(1 + \dfrac{c-a}{a}\theta k\right)\omega e^{kz}, \\[2ex] P_d(t) = \mu e^{-qt} + \theta\omega\left(1 + \dfrac{c-a}{a}\theta k\right) = \mu e^{-qt} + P_{dg}, \\[2ex] P_n(t) = -\mu e^{-qt} + \dfrac{c-a}{a}\theta\omega e^{kV} = -\mu e^{-qt} + P_{ng} \end{cases} \qquad (4)$$

where:

P_{dg} – is probability of resources shortage for stationary and irregular system, for $q = q_1 = q_2 > 0$;

P_{ng} – is probability of resources excess for stationary and irregular system, for $q = q_1 = q_2 > 0$;

$$\theta = \frac{a}{q_2}, \quad \omega = \frac{k}{2\left(e^{kV}\left(\dfrac{c-a}{a}\theta k + 1\right) - 1\right)}, \quad -\theta\omega\left(1 + \dfrac{c-a}{a}\theta k\right) \leqslant \mu \leqslant \dfrac{c-a}{a}\theta\omega e^{kV}$$

– are some parameters.

3. THE DEFINITION OF FUNCTION OF LOSSES

The effectiveness of the system operation (in economic terms) can be assessed on the basis of the so-called function of losses. The function of losses takes into account total system losses, which are connected with the following costs:

– Resource supply costs from sender;

– Storage costs in reservoir;

– Emergency supply costs from external suppliers in the situation of resources shortage;

– Function of losses also accounts for additional profits from the sale of the resources excess to external buyers.

$$S(t) = c\int\limits_0^t \xi(\tau)\frac{q_2(\tau)}{q_1(\tau) - q_2(\tau)}\,d\tau + \int\limits_0^t\left[\int\limits_0^V \psi(\tau, z)\,dF_\tau(z)\right]d\tau +$$

$$+ am\int\limits_0^t \xi(\tau)P_d(\tau)\,d\tau - (c - a)n\int\limits_0^t \xi(\tau)P_n(\tau)\,d\tau = \int\limits_0^t\left[c\xi(\tau)\frac{q_2(\tau)}{q_1(\tau) + q_2(\tau)} + \right. \tag{5}$$

$$\left. + \int\limits_0^V \psi(\tau, z)\,dF_\tau(z) + am\xi(\tau)P_d(\tau) - (c - a)n\xi(\tau)P_n(\tau)\right]d\tau$$

where:

$\xi(\tau)$ – is a unit purchase price of resources delivered by sender;

$\psi(\tau, z)$ – is a function which defines storage costs and depends on time and stock level;

$F_\tau(z)$ – is a distribution function of reservoir filling between its lower and upper barrier;

m – is a price parameter which defines price relations between unit price of purchase of resources from sender and unit price of emergency purchase in situation of shortage;

n – is a price parameter which defines price relations between unit price of purchase of resources from senders and unit price of their resale to external buyers in the situation of excess,

The function of storage costs will require a more detailed explanation. In practice, for a particular system this function may be determined using econometric methods, on the grounds of empirical observations of storage costs in dependence on time and stock level.

To simplify the analysis we have assumed, that the function of storage costs is a product of two functions:

$$\psi(\tau, z) = \eta(\tau) \cdot \varphi(z) \tag{6}$$

where:

$\varphi(z)$ – is a function of storage costs dependent on the level of stock accumulation,

$\eta(\tau)$ – is a function which describes changes of storage costs in time (it can be, for example inflation function).

To define a function of storage costs depending on the level of stock accumulation we used a well known in economics coefficient of elasticity for storage costs:

$$E_z[\varphi(z)] = \lim_{\Delta z \to 0} \frac{\varphi(z + \Delta z) - \varphi(z)}{\varphi(z)}\frac{z}{\Delta z} = \frac{z}{\varphi(z)}\frac{d\varphi(z)}{dz} \tag{7}$$

We assumed that this coefficient is proportional to stock level in the reservoir and also proportional to its storage costs:

$$E_z[\varphi(z)] = \lambda z \varphi^{\delta - 1}(z) \tag{8}$$

where:

λ – is a proportionality coefficient;

δ – is a certain numerical coefficient.

As a result the function of storage costs is described by the following differential equation:

$$\frac{d\varphi(z)}{dz} = \lambda \varphi^{\delta}(z) \tag{9}$$

The solution of this equation at adequately well-chosen boundary conditions will lead to obtaining various analytical forms of function of storage costs.

4. PRACTICAL CONCLUSIONS

The detailed econometric analysis of function of losses for different variants of the system operation resulted in a lot of practical conclusions, which makes it possible (by means of appropriate choice of parameters) to control the system in such a way that its operation is the most effective.

The econometric analysis of efficiency of the system "transport–reserves" operation included the following stages:

1. First of all we investigated the influence of reservoir capacity on the amount losses by system working in stabilized conditions (the so-called stationary variant). We obtained relations between parameters characterizing the economic conditions in which the system operates (such as: storage cost parameters, price parameters etc.), which makes it possible to examine (in an easy way) the influence of reservoir capacity on the quantity of system losses. The obtained conditions enabled the determination of the optimal values of reservoir capacity at which the system losses are the least. Practical use of obtained results is very important, because they make it possible to determine proper storage capacity (at the stage of planning a system), so that the system will work possibly most effectively. Besides the corrections for the systems already working in practice are made possible. Below we present some examples of obtained practical conclusions.

Example:

– When parameters characterizing the dynamics of changes of storage costs (parameter λ and parameter γ_0 – determining constant maintenance cost of empty reservoir) and parameters which determine the price relations assume the following values: $\lambda > 0$, $m - n > \dfrac{\gamma_0}{a\,v_z} > 0$, then stationary system always ends in losses (independently on reservoir capacity). The function of losses (Figure 1) in that case possesses minimum for the certain optimal storage capacity (the value of optimal capacity can be estimated numerically). From the point of view of the system operation efficiency it will work, in such a case, most effectively at determined optimal reservoir capacity. Too small, or too large reservoir capacities in relation to optimal capacity will lead to large losses (the system will work less effectively).
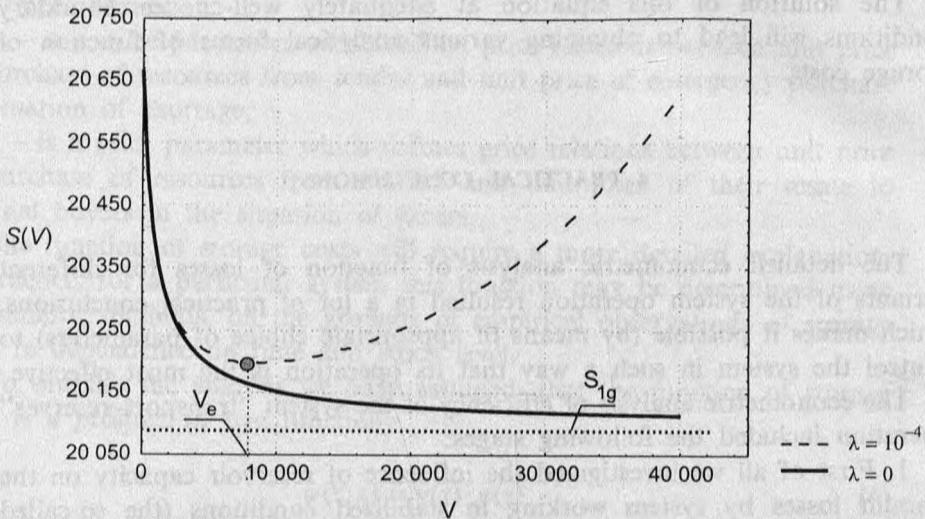

Fig. 1. Graphs of function of losses for the system of "transport–reserves" type, when values of parameters characterizing the system operation are equal: $q_1 = 1$, $q_2 = 2$, $c = 3000$, $a = 2000$, $\alpha = 10$, $\gamma_0 = 100$, $\theta = 1000$, $v_z = 660.7$, $\delta = 0.95$, $m = 1.1$, $n = 1$. The numeric value of optimal reservoir capacity, for which system gets minimal losses and the numeric value of losses for that optimal capacity are equal: $V_e = 8707$, $S(V_e) = 20\,197$

2. Moreover, for the stationary systems we compared the efficiency of system operation in a "basic system" with efficiency of analogous system in which storage costs depend on the stock level. Comparative analysis provided interesting conclusions which make it possible to control the system in such a way (by appropriate influence on the storage costs parameters) that its operation is the most effective.

The conclusions (Figure 2) are as follows:

– If the system is characterized by the following property: storage costs increase when the stock level increases (which is expressed by positive values of parameter λ), then its losses will always be greater than analogous losses of the basic system (for which parameter λ is equal zero). The basic system will in that situation always work more efficiently. Moreover, if storage costs increase more quickly when the stock level rises (which is expressed by large positive values of parameter λ), then the system will suffer greater losses.

– If the storage costs decrease when the stock level increases (which is expressed by negative values of parameter λ), then its losses will always be smaller than analogous losses of the basic system. The basic system will in that situation always work less efficiently. Moreover, if storage costs decrease more quickly when the stock level rises (which is expressed by large negative values of parameter λ), then the system will make smaller losses.

– If the values of parameters characterizing the storage costs accomplish the following relations: $(\delta \to +\infty)$ and $(\gamma_0 < 1)$ or $(\delta \to -\infty)$ and $(\gamma_0 > 1)$, then losses of such a system will be comparable to losses in the basic system.

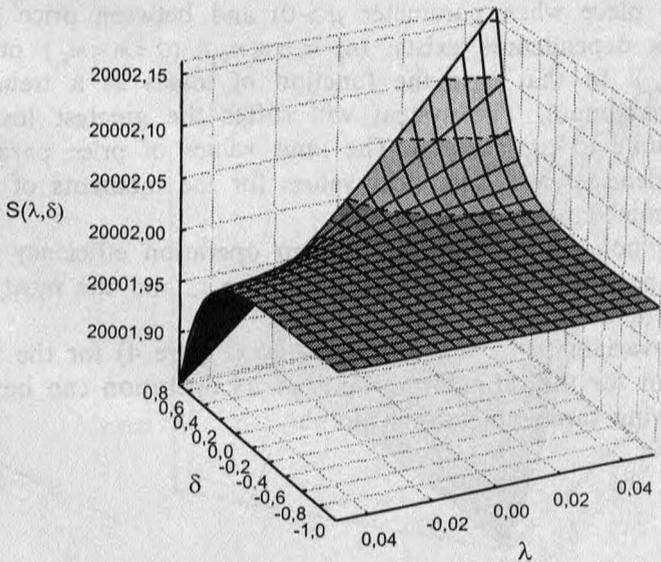The obtained conclusions show that storage costs essentially influence the efficiency of the system operation.

Fig. 2. The function of losses for the system of "transport–reserves" type, when the parameters characterizing the system operation are equal: $q_1 = 1$, $q_2 = 2$, $c = 3000$, $a = 2000$, $a = 10$, $v_z = 666.7$, $\theta = 1000$, $\gamma_0 = 100$, $V = 10$, $m = 1$, $n = 1$

3. In econometric evaluation of system operation efficiency we also carried out the time – analysis of efficiency of "quasi-dynamic" system (taking into consideration time variability of the economic conditions in which the system operates). Such type analysis is especially interesting, because it enables the examination of system efficiency in a long time horizon (range). We obtained the conditions for the numerical values of parameters, which characterize the dynamics of time variability of probability of resources excess and shortage and for the parameters characterizing price relations. These conditions make the investigation of trend of function of losses easier. It is very useful to determine such time intervals in the assumed time horizon (range) in which the system makes losses or brings the so-called additional profit (negative values of function of losses). The practical significance of obtained results is very essential for the system operation, because they make it possible to select properly the parameters characterizing the system operation (at the stage of system designing), so that the system will operate efficiently for as long as possible. Below we present some examples of obtained conclusions.

Example:

– The system makes losses in time interval $(0 < t < t_s)$ and then gains profit (which increases without time limit), when probability of resources shortage decreases and probability of resources excess increases in time, (which takes place when parameter $\mu > 0$) and between price parameters the following dependences exists: $(n_{g_1} < n \leqslant n_{g_2})$, $(0 < m < m_{q_1})$ or $(n > n_{g_2})$, $(m_{g_2} < m < m_{g_1})$. In that case the function of losses as a trend function reaches its maximum. The system will suffer the greatest losses at the moment of time $(0 < t = t_e < t_s)$. The limit values of price parameters (in above dependencies) and numerical values for the moments of time t_e, t_s can be determined numerically.

From the point of view of the system operation efficiency (Figure 3) the system operates more efficiently after time t_e and the most efficiently after time t_s (additional profit).

The time variability of function of losses (Figure 4) for the "quasi–dynamic" system for the different variants of its operation can be illustrated by the following three-dimensional graph.
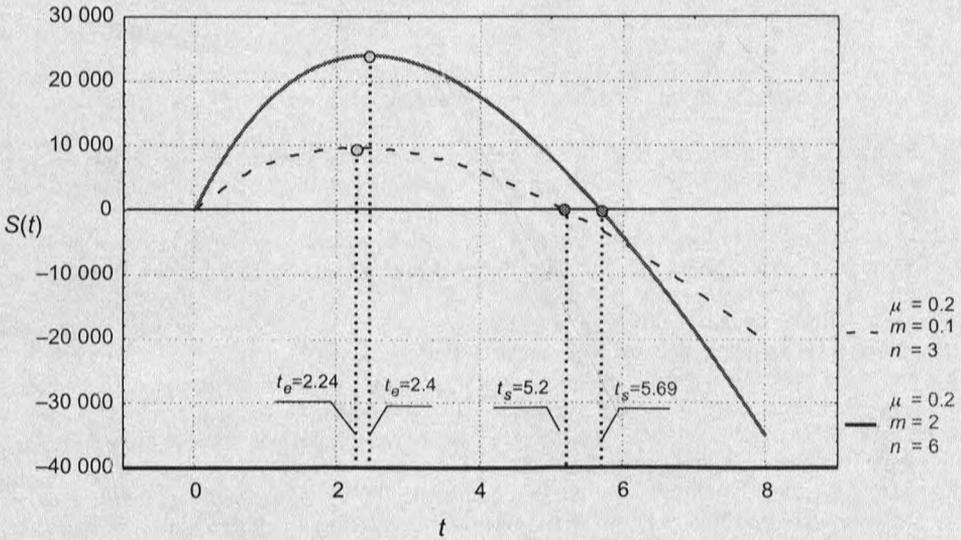
Fig. 3. The graphs of function of losses for the "quasi–dynamic" system of "transport–reserves" type for the different values of the parameters characterizing its operation, when $c = 4000$, $a = 2000$, $a = 10$, $S_u = 2550$, $q = 0.5$, $P_{dg} = 0.4$, $P_{ng} = 0.4$, $i = 0.12$
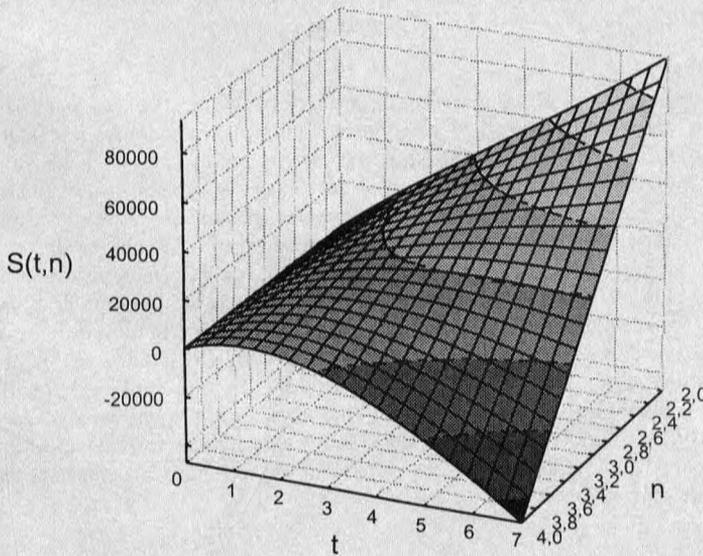

Fig. 4. The function of losses for the system of "transport–reserves" type, when the parameters characterizing the system operation are equal: $c = 4000$, $a = 2000$, $a = 10$, $S_n = 2550$, $q = 0.5$, $P_{dg} = 0.4$, $P_{ng} = 0.4$, $i = 0.12$, $m = 0.1$, $\mu = 0.2$

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EKONOMETRYCZNA OCENA EFEKTYWNEGO STEROWANIA SYSTEMEM TYPU „TRANSPORT–ZASOBY"
(Streszczenie)

W pracy przedstawiono ekonometryczną ocenę możliwości efektywnego sterowania pewnym systemem typu „transport–zasoby" z awaryjnym układem transportowym, składającym się z pojedynczego nadawcy zasobów i pojedynczego ich odbiorcy oraz z magazynem zasobów jako buforem, zapewniającym ciągłą pracę odbiorcy, nawet w sytuacji awarii dostaw zasobów od nadawcy. Za miarę efektywności funkcjonowania systemu (w aspekcie ekonomicznym) przyjęto tzw. funkcję strat, uwzględniającą sumaryczne straty systemu związane z kosztami pozyskiwania zasobów od nadawcy, kosztami magazynowania ich zapasów oraz kosztami awaryjnego pozyskiwania brakujących zasobów od innych kontrahentów zewnętrznych (w sytuacji ich deficytu). W funkcji strat uwzględniono także ewentualne dodatkowe zyski systemu, wynikające ze sprzedaży nadwyżki zasobów magazynowych (w sytuacji ich nadmiaru) kontrahentom zewnętrznym. Szczegółowa ekonometryczna analiza funkcji strat dla różnych wariantów funkcjonowania systemu pozwoliła na uzyskanie szeregu wniosków o istotnym znaczeniu praktycznym, które umożliwiają takie sterowanie funkcjonowaniem systemu (poprzez odpowiedni dobór jego parametrów), aby funkcjonował on najbardziej efektywnie (ponosił relatywnie niskie straty).