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CHARACTERISTICS OF BIVARIATE BINOMIAL DISTRIBUTION

Abstract. In the paper there was defined the bivariate zero-one distribution and its use for deriving the bivariate binomial distribution. This distribution was described by the cumulative distribution function and the characteristic function which was used to derive appropriate normal moments of marginal distributions and the joint distribution. Moreover, there was given the vector-matrix form of the cumulative distribution function and the characteristic function and appropriate vectors of normal moments.

Key words: Bivariate zero-one distribution, bivariate binomial distribution, cumulative distribution function, characteristic function, moments.

I. INTRODUCTION

The notion of bivariate discrete and continuous random variables (X_1, X_2) is widely presented in many domestic (e.g. Fisz 1967, Pawłowski 1976, Firkowicz 1977, Wagner and Błażczak 1992, Hellwig 1994, Krzyśko 1996, Domański et. al. 1998) and foreign publications (e.g. Anderson 1958, Johnson and Kotz 1972, Seber 1984, Johnson et. al. 1997, Aczel 2000). The distributions of these variables are determined in the set of values Ω_{X_1,X_2} which is countable for discrete random variables and uncountable for continuous ones. In particular this set can be a rectangle, circle, ellipse, polygon or any other area limited on the plane. The cumulative distribution function is expressed by $f(x_1, x_2; \boldsymbol{\theta})$, where $(x_1, x_2) \in \Omega_{X_1,X_2}$, and $\boldsymbol{\theta}$ is the vector of parameters of values belonging to a certain set $\boldsymbol{\Theta}$. The graphical presentation of the function $f(x_1, x_2; \boldsymbol{\theta})$ for the set vector $\boldsymbol{\theta}$ is presented in the spatial graphs in which the basis on the plane OX_1X_2 is constituted by the area Ω_{X_1,X_2}, and over it there is a surface determined by the function $f(x_1, x_2; \boldsymbol{\theta})$.

Bivariate distributions of random variables are particular cases of multivariate distributions. They are derived by various methods (**Johnson** and **Kotz**

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1972). The majority of these methods come down to such a formulation of joint distribution of two random variables so that their marginal distribution belong to a given class of distributions. For example if there is determined the cumulative distribution function, of bivariate binomial distribution, then the distributions of particular random variables also have binomial distributions.

In the paper there was defined the bivariate zero-one distribution and its use for derivation of the bivariate binomial distribution. This distribution was described by the cumulative distribution function and the characteristic function which was used for derivation of appropriate normal moments of marginal distributions and the joint distribution. Moreover, there was given the vector-matrix form of cumulative distribution function and characteristic function and the appropriate vectors of normal moments.

II. BIVARIATE ZERO-ONE DISTRIBUTION

Let X_1, X_2, X_3 be random variable, each one of zero-one distribution (0–1), for which $P(X_i = 0) = 1 - p_i$ and $P(X_i = 1) = p_i$ at i = 1, 2 ,3. Their set of values consists of 8 (= 2^3) 3-element vectors, corresponding to Cartesian's product of three sets $\{0,1\}$, i.e.

$$\Omega_{X_1,X_2,X_3} = \{0,1\} \times \{0,1\} \times \{0,1\} = \{0,1\}^3 = \{(0,0,0), (0,01), \ldots, (1,1,1)\}.$$

The cumulative distribution function of the mentioned variables takes the form

$$f(k_1,k_2,k_3) = p_1^{k_1} p_2^{k_2} p_3^{k_3}, \text{ at } k_1,k_2,k_3 = 0,1,$$

so that $k_1 + k_2 + k_3 \le 1$. Let us notice that $X_1 + X_2 + X_3 = 1$, and therefore by determining $X_3 = 1 - X_1 - X_2$, the given distribution function is transformed into the form

$$f(k_1,k_2) = p_1^{k_1} p_2^{k_2} (1 - p_1 - p_2)^{1-k_1-k_2},$$

which for the specific k_1, k_2 leads to

$$f(k_1,k_2;p_1,p_2)=\begin{cases}1-p_1-p_2, & k_1=k_2=0,\\ p_1, & k_1=1,k_2=0,\\ p_2, & k_1=0,k_2=1.\end{cases}$$

The given distribution function allows determining the characteristic function of the bivariate 0–1 distribution (bivariate zero-one distribution) by using successively the transformations at the condition $k_1+k_2\le n$:

$$\varphi_{X_1,X_2}(t_1,t_2)=E\{\exp[i(t_1k_1+t_2k_2)]\}=\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}e^{i(t_1k_1+t_2k_2)}\cdot f(k_1,k_2;p_1,p_2)=$$

$$=\sum_{k_1=0}^{1}\left\{\,e^{it_1k_1}f(k_1,0;p_1,p_2)+e^{i(t_1k_1+t_2)}f(k_1,1;p_1,p_2)\right\}=$$

$$=f(0,0;p_1.p_2)+e^{it_2}\cdot f(0,1;p_1.p_2)+e^{it_1}\cdot f(1,0;p_1.p_2)=1-p_1-p_2+p_1e^{it_1}+p_2e^{it_2}$$

III. DERIVATION OF CUMULATIVE DISTRIBUTION FUNCTION

The 0–1 distribution, given in chapter 2, allows determining the bivariate binomial distribution. For this purpose we introduce the assumptions:

- we will replace the variables in the given 0–1 distribution by $X_i\to X_{i,1}$ for i = 1,2,
- we assume that the variables $X_{i,1}$ at i = 1, 2 have bivariate 0–1 distributions,
- for each variable $X_{i,1}$, $X_{i,2}$ there occurs Bernoulli's pattern of series of n independent experiments,
- we determine the variables $X_i=X_{i,1}+X_{i,2}+\ldots+X_{i,n}$, at i = 1,2.

Taking the advantage of 0–1 independence of variables $X_{i,1}$ for i = 1, 2 and acting analogically to one-dimensional case, there is derived the probability function $P(X_1=k_1,X_2=k_2)$ of bivariate binominal distribution in the form

$$f(k_1,k_2;n,p_1,p_2)=\frac{n!}{k_1!k_2!(n-k_1-k_2)!}p_1^{k_1}p_2^{k_2}(1-p_1-p_2)^{n-k_1-k_2}, \qquad (1)$$

where $(k_1,k_2)\in\Omega_{X_1,X_2}=\{0,1,2,\ldots\}\times\{0,1,2,\ldots\}=\{0,1,2,\ldots\}^2$, so that $k_1+k_2\le n$ and $(p_1,p_2)\in\mathbf{\Theta}=(0,1)\times(0,1)=(0,1)^2$ and $p_1+p_2<1$. We symbolically write this distribution as $(X_1,X_2)\sim B_2(n;p_1,p_2)$.

For the set values of parameters of the distribution n, p_1, p_2 there are determined the values of the function (1). For example, for $n = 9,\ p_1 = 0{,}3,\ p_2 = 0{,}4$, such values are presented in table 1.

Table 1. Values of distribution function $B(9; 0{,}3, 0{,}4)$

k_1	k_2									
	0	1	2	3	4	5	6	7	8	9
0	0,0000	0,0002	0,0007	0,0017	0,0025	0,0025	0,0017	0,0007	0,0002	0,0000
1	0,0002	0,0019	0,0066	0,0132	0,0165	0,0132	0,0066	0,0019	0,0002	
2	0,0013	0,0088	0,0265	0,0441	0,0441	0,0265	0,0088	0,0013		
3	0,0039	0,0235	0,0588	0,0784	0,0588	0,0235	0,0039			
4	0,0078	0,0392	0,0784	0,0784	0,0392	0,0078				
5	0,0105	0,0418	0,0627	0,0418	0,0105					
6	0,0093	0,0279	0,0279	0,0093						
7	0,0053	0,0106	0,0053							
8	0,0018	0,0018								
9	0,0003									

Source: The author's elaboration

The quantities given in table 1 were used to make the spatial graph of the distribution $(X_1, X_2) \sim B(9; 0{,}3, 0{,}4)$ (fig. 1).

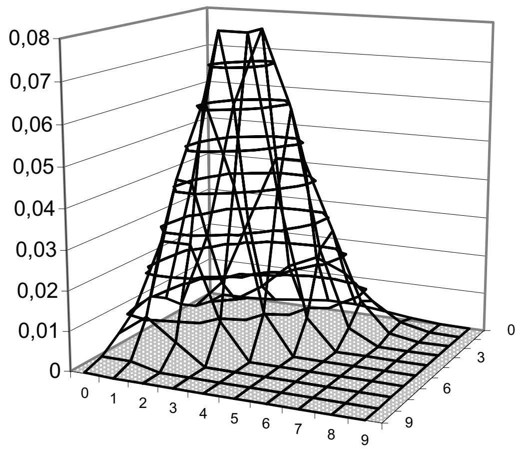

Fig. 1. Graph of bivariate distribution function $B(9;\ 0{,}4\ ,0{,}3)$

Source: The author's elaboration.

The data given in the table lead to the motions:

(a) the table of values of the distribution function takes the form of upper-triangular table, which results from the inequality $k_1 + k_2 \leq n$,

(b) the distribution function $f(k_1, k_2; n, p_1, p_2)$) can take the maximum value for a few different combinations ((k_1, k_2) at set values p_1, p_2 ,

(c) for set k_2 the highest values of probability appear among two middle values k_1 and vice versa,

(d) the cumulative distribution function takes the lowest values on the edges of the upper-triangular table,

(e) the distribution $B(n; k_1, k_2)$ is asymmetrical in the direction of the longest side of the upper-triangular table at $p_1 < p_2$.

IV. CHARACTERISTIC FUNCTION AND MOMENTS

The characteristic function is derived directly from the function given in table 2 for the 0–1 distribution and it takes the form

$$\varphi_{X_1,X_2}(t_1,t_2) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \exp\left[i(t_1k_1 + t_2k_2\right] f(k_1,k_2;n,p_1,p_2) =$$

$$= \left(1 - p_1 - p_2 + p_1e^{it_1} + p_2e^{it_2}\right)^n, \tag{2}$$

at the same time the summation is performed at the condition $k_1 + k_2 \leq n$. The function (2) is determined by the parameters of the distribution n, p_1, p_2. It also meets the condition $\varphi_{X_1,X_2}(0,0) = 1$.

The expected values and standard deviations for the marginal distributions equal

$$E(X_i) = np_i, \; D(X_i) = \sqrt{np_i(1 - p_i)}, \; i \; = \; 1, \; 2.$$

The covariance of the joint distribution is $Cov(X_1, X_2) = -np_1p_2$. Its derivation is carried out in the following way. We use the characteristic function (2) which we express in the form

$\varphi_{X_1,X_2}(t_1,t_2) = Q^n(t_1,t_2)$, where $Q(t_1,t_2) = (1 - p_1 - p_2 + p_1e^{it_1} + p_2e^{it_2})$. Here $Q(0,0) = 1$. We calculate the derivative in relation to t_1 :

$\frac{\partial\varphi}{\partial t_1} = nQ^{n-1}(t_1,t_2)ip_1e^{it_1}$, and hence $\frac{\partial\varphi}{\partial t_1}\Big|_{t_1=t_2=0} = inp_1$, i.e. $\alpha_{10} = np_1$. From the symmetry we have $\alpha_{01} = np_2$. Now we determine the mixed partial derivative: $\frac{\partial^2\varphi}{\partial t_1 \partial t_2} = n(n-1)Q^{n-2}(t_1,t_2)\cdot\left(ip_1e^{it_1}\right)\cdot\left(ip_2e^{it_2}\right)$, and hence

$$\frac{\partial^2\varphi}{\partial t_1 \partial t_2}\Big|_{t_1=t_2=0} = i^2 n(n-1)p_1p_2\text{, i.e. } \alpha_{11} = n(n-1)p_1p_2.$$

Finally we calculate the covariance $\mu_{11} = \alpha_{11} - \alpha_{10}\cdot\alpha_{01}$, obtaining

$$\mu_{11} = n(n-1)p_1p_2 - n^2p_1p_2 = n[n-1-n]p_1p_2 = -np_1p_2\,.$$

Assuming the denotations for central moments $\mu_{20} = np_1q_1$, $\mu_{02} = np_2q_2$, where $q_i = 1 - p_i$ and the product moment $\mu_{11} = -np_1p_2$, we obtain the correlation coefficient

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\cdot\mu_{02}}} = -\sqrt{\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}}\,,$$

which is the root of the product of quotients of probability of occurrence and non-occurrence of success for each of the variables X_1 and X_2.

Marginal distributions in the considered case are one-dimensional binomial distributions, which results directly from the function (2) and the replacement $t_2 = 0$, which leads to

$$\varphi(t_1,0) = \left(1 - p_1 - p_2 + p_1e^{it_1} + p_2\right)^n = \left(1 - p_1 + p_1e^{it_1}\right)^n,$$

i.e. there is obtained the characteristic function of the marginal distribution X_1. It is similar for the marginal distribution X_2. The presented deliberations lead to the following properties of the bivariate binomial distribution:

(a) if $X_1 \sim B(n_1, p)$, $X_2 \sim B(n_2, p)$ are two independent random variables of indicated binomial distributions, then the random variable $Y = X_1 + X_2$ has the binomial distribution $Y \sim B(n_1 + n_2, p)$, i.e. the probability distribution $P(X = k)$, for $k = 0, 1, 2, \ldots, n_1 + n_2$,

(b) if $X_1 \sim B(n_1, p),\ X_2 \sim B(n_2, p)$ are two independent random variables of indicated binomial distributions, then the distribution X_1, on condition $X_1 + X_2 = m$, has the form

$$P(X_1 = k \mid m) = \frac{\binom{n_1}{k} p^k q^{n_1-k} \binom{n_2}{k} p^{m-k} q^{n_2-m+k}}{\binom{n_1+n_2}{m} p^m q^{n_1+n_2-m}} = \frac{\binom{n_1}{k}\binom{n_2}{m-k}}{\binom{n_1+n_2}{m}},$$

where $\max\{0, m-n_2\} \le k \le \min\{n_1, m\}$, which gives the hypergeometrical distribution,

(c) at the assumptions (b) the cumulative distribution function of the difference $X_1 - X_2$ is expressed by

$$P(X_1 - X_2 = k) = \sum_j \binom{n_1}{j}\binom{n_2}{j-k} p^{2j-k} q^{n_1+n_2-2j+k},$$

where the summation is performed within the scope $\max\{0, k\} \le j \le \min\{n_1, n_1+n_2\}$,

(d)if $(X_1, X_2) \sim B(n_1; p_1, p_2),\ (Y_1, Y_2) \sim B(n_2; p_1, p_2)$ are two bivariate independent random variables of indicated distribution, then the bivariate random variable $(X_1 + Y_1, X_2 + Y_2)$ has the distribution $B(n_1 + n_2; p_1, p_2)$.

V. VECTOR–MATRIX DESCRIPTION

Now we will carry out the characteristics of bivariate binominal distribution by the matrix calculus. We introduce the denotations:

- $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ – vector of random variables of bivariate binomial distribution,
- $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ – vector of probability of successes,
- $\mathbf{q} = \mathbf{1} - \mathbf{p} = \begin{bmatrix} 1-p_1 \\ 1-p_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ – vector of probability of failures (defeats),

where **1** is a bivariate vector of ones,

- ❖ $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \end{bmatrix} = n\mathbf{p}$ – vector of expected values,
- ❖ $\boldsymbol{\Sigma} = \begin{bmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{bmatrix} = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 \\ -np_1p_2 & np_2(1-p_2) \end{bmatrix} = n[diag(\mathbf{p}) - \mathbf{pp}']$ – matrix of variance – covariance, where $diag(\mathbf{p}) = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ is a diagonal matrix with elements on the main diagonal of vector **p,**
- ❖ $tr(\boldsymbol{\Sigma}) = n(p_1q_1 + p_2q_2) = n\mathbf{p}'\mathbf{q}$ – trace of the matrix $\boldsymbol{\Sigma}$,
- ❖ $|\boldsymbol{\Sigma}| = n^2 p_1p_2(q_1q_2 - p_1p_2) = n^2p_1p_2(1-p_1-p_2) = \frac{1}{2}n^2[(\mathbf{1}'\mathbf{p})^2 - \mathbf{p}'\mathbf{p})(1-\mathbf{1}'\mathbf{p})$

–determinant of matrix $\boldsymbol{\Sigma}$, which on the basis of earlier assumption $p_1 + p_2 < 1$ on the distribution $B_2(n, p_1, p_2)$ is always positive, at the same time there occurs the inequality $\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} < 1$.

The characteristic values λ_1, λ_2 of the set matrix of variance-covariance are determined from the determinantal equation $|\boldsymbol{\Sigma} - \lambda\,\mathbf{I}| = 0$, which after transformation becomes the quadratic equation $\lambda^2 - b\lambda + c = 0$, where $b = tr(\boldsymbol{\Sigma})$ and $c = |\boldsymbol{\Sigma}| = \det(\boldsymbol{\Sigma})$. The discriminant of this equation is $\Delta = b^2 - 4c = n^2[(p_1q_1 - p_2q_2)^2 + 4p_1^2p_2^2]$. Since there is fulfilled the inequality $\Delta > 0$, therefore for the given quadratic equation there exist two real roots $\lambda_1 = \frac{b - \sqrt{\Delta}}{2}$ and $\lambda_2 = \frac{b + \sqrt{\Delta}}{2}$. These roots fulfil Viéte's formulas for the sum $\lambda_1 + \lambda_2 = tr(\boldsymbol{\Sigma})$ and the product $\lambda_1\lambda_2 = \det(\boldsymbol{\Sigma})$ of the roots of the quadratic equation.

Now we will deal with the matrix description of the properties of the characteristic function (2). Let us write its new form $\varphi_{\mathbf{X}}(\mathbf{t}) = Q^n(\mathbf{t})$, where $Q(\mathbf{t}) = 1 - \mathbf{p}'\mathbf{1} + \mathbf{p}'\mathbf{w}(\mathbf{t})$, at the same time

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix},\ \mathbf{w}(\mathbf{t}) = \begin{bmatrix} e^{it_1} \\ e^{it_2} \end{bmatrix} = \mathbf{w}_1(t_1) + \mathbf{w}_2(t_2),\ \text{where}$$

$$\mathbf{w}_1(t_1) = \begin{bmatrix} e^{it_1} \\ 0 \end{bmatrix}, \mathbf{w}_2(t_2) = \begin{bmatrix} 0 \\ e^{it_2} \end{bmatrix}.$$

For the vector $\mathbf{t}=\mathbf{0}$, we have $\mathbf{w}(\mathbf{0})=\mathbf{1}$, $\mathbf{w}_1(0)=\boldsymbol{\varepsilon}_1$ i $\mathbf{w}_2(0)=\boldsymbol{\varepsilon}_2$, where $\boldsymbol{\varepsilon}_j$ is the unit vector_with the one element for *j*-th coordinate, and hence $Q(\mathbf{0}) = 1$.

We use the characteristic function, given in the new form, to determine the first two normal moments of marginal distributions and joint distribution.

(a) determination of normal moments of the *1st* row.

- $\dfrac{\partial\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1} = nQ^{n-1}(\mathbf{t})\cdot\dfrac{\partial Q(\mathbf{t})}{\partial t_1}$,
- $\dfrac{\partial Q(\mathbf{t})}{\partial t_1} = \mathbf{p}'\dfrac{\partial \mathbf{w}(\mathbf{t})}{\partial t_1} = i\mathbf{p}'\mathbf{w}_1(\mathbf{t})$,
- $\dfrac{\partial\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1}\Big|_{\mathbf{t}=\mathbf{0}} = i\cdot n\cdot\mathbf{p}'\mathbf{w}_1(\mathbf{0}) = i\cdot n\cdot\mathbf{p}'\boldsymbol{\varepsilon}_1 = inp_1$,
- $\alpha_{10} = \dfrac{\dfrac{\partial\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1}\Big|_{\mathbf{t}=\mathbf{0}}}{i} = np_1$, at the same time on the basis of the symmetry $\alpha_{01} = np_2$.

The given procedure_can be expressed more generally, at the same time taking into consideration both variables, which we will write as

$\dfrac{\partial\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_j} = nQ^{n-1}(\mathbf{t})\cdot\dfrac{\partial Q(\mathbf{t})}{\partial t_j} = nQ^{n-1}(\mathbf{t})\cdot i\cdot\mathbf{p}'\mathbf{w}_j(t_j)$ and

$\dfrac{\partial\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_j}\Big|_{\mathbf{t}=\mathbf{0}} = i\cdot n\cdot\mathbf{p}'\mathbf{w}_j(\mathbf{0}) = i\cdot n\cdot\mathbf{p}'\boldsymbol{\varepsilon}_j = inp_j$, for $j = 1,2$.

(b) determination of normal moments of *2nd* row.

We will use the general formula of determination of moments of the second row for marginal distributions. We have respectively ($j = 1, 2$):

- $\dfrac{\partial^2\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_j^2} = i\cdot n\left[(n-1)Q^{n-2}(\mathbf{t})\cdot i\cdot(\mathbf{p}'\mathbf{w}_j(t_j))^2 + Q^{n-1}(\mathbf{t})\cdot i\cdot\mathbf{p}'\mathbf{w}_j(t_j)\right]$,
- $\dfrac{\partial^2\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_j^2}\Big|_{\mathbf{t}=\mathbf{0}} = i\cdot n\left[(n-1)Q^{n-2}(\mathbf{t})\cdot i\cdot(\mathbf{p}'\boldsymbol{\varepsilon}_j)^2 + i\cdot\mathbf{p}'\boldsymbol{\varepsilon}_j\right] = i^2[n(n-1)p_j^2 + np_j]$,
- $\alpha_{20} = n(n-1)p_1^2 + np_1$ and from the symmetry $\alpha_{02} = n(n-1)p_2^2 + np_2$

Now we will proceed to determination of the product moment:

- $\dfrac{\partial^2\varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1\partial t_2} = i\cdot n(n-1)Q^{n-2}(\mathbf{t})\cdot\mathbf{p}'\mathbf{w}_1(\mathbf{t})\cdot i\cdot\mathbf{p}'\mathbf{w}_2(\mathbf{t})$,

- $\frac{\partial^2 \varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1 \partial t_2}\big|_{\mathbf{t}=\mathbf{0}} = i^2 \cdot n(n-1) \cdot \mathbf{p}'\boldsymbol{\varepsilon}_1 \cdot \mathbf{p}'\boldsymbol{\varepsilon}_2 = i^2 \cdot n(n-1) p_1 p_2$,
- $\alpha_{11} = \frac{\frac{\partial^2 \varphi_{\mathbf{X}}(\mathbf{t})}{\partial t_1 \partial t_2}\big|_{\mathbf{t}=\mathbf{0}}}{i^2} = n(n-1) p_1 p_2$.

From the given normal moments in a direct way there are determined central moments which were mentioned earlier.

SUMMARY

In the paper there was given one of many possible ways of determining the bivariate binominal distribution. Other possibilities are listed in the paper of Johnson et. al. (1997, p. 31–92). They are connected with the multinomial distribution. The approach proposed in the paper is a natural extension of the one-dimensional zero-one distribution and the binomial distribution. In the bivariate case, the form of characteristic function, proposed in the paper, in the transformed form and the vector form, allowed derivation of formulas for normal and product moments.

The bivariate binomial distribution at the number of samples *n* approaching infinity becomes Poisson bivariate distribution, and at some *n, p*$_1$*, p*$_2$ it becomes the limiting bivariate normal distribution.

The bivariate binomial distribution can be extended to the multivariate case (Johnson et. al. 1997, pp.105–113).

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CHARAKTERYSTYKA DWUWYMIAROWEGO ROZKŁADU DWUMIANOWEGO

W pracy podano jeden z wielu możliwych sposobów określenia dwuwymiarowego rozkładu dwumianowego. Inne możliwości są wymieniane w pracy Johnsona i in. (1997, s. 31–92). Mają one związek z rozkładem wielomianowym. Podejście proponowane w pracy jest naturalnym rozszerzeniem jednowymiarowego rozkładu zero-jedynkowego i rozkładu dwumianowego. W przypadku dwuwymiarowym proponowana w pracy postać funkcji charakterystycznej w formie rozpisanej i wektorowej pozwoliła na wyprowadzenie wzorów na momenty zwykle i mieszane.

Dwuwymiarowy rozkład dwumianowy przy liczbie prób n dążących do nieskończoności przechodzi w dwuwymiarowy rozkład Poissona, a przy pewnych n, p_1, p_2 przechodzi w graniczny dwuwymiarowy rozkład normalny.

Dwuwymiarowy rozkład dwumianowy daje się rozszerzyć na przypadek wielowymiarowy (Johnson i in. 1997, s. 105–113).