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SOME NOTES ABOUT SINGULAR TWO PAN DESIGNS

Abstract. The paper deals with singular two pan design. We give the method of construction of nonsingular design based on singular one. The construction method of the design matrix is based on the incidence matrices of the balanced incomplete block design.

Key words: balanced incomplete block design, two pan design.

I. INTRODUCTION

Yates (1935) showed that if several light objects are weighed in groups rather than individually then the precision of the estimates increases considerably. In the scheme suggested by Yates the objects are always placed on the same pan in a chemical balance. Hotelling (1944) showed that Yates' scheme can be improved by distributing the objects on both the pans in a chemical balance. In this case, the precision of the estimates of the weights of the objects increases. Following Yates and Hotelling several authors provided methods of construction and analysis of such designs, together with investigation of the precision of such designs, see Banerjee (1950), Dey (1969). Present, in the literature, the design matrices for such experiments are called weighing matrices (see Dias, Giri (1986), Chadjiconstantinidis, Moyssiadis (1992), Chadjiconstantinidis, Chadjipadelis (1994)).

The scheme indicated above is called chemical balance weighing design as in chemical balance the objects are placed on one pan or on both for each weighing. Given p objects to be weighed in groups in n weighings and, let p_1 be the size of set of objects placed on one pan, say, left one, p_2 be the size of set of objects placed on the other pan and p_0 be the size of set of objects omitted in

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the weighing, $p = p_1 + p_2 + p_0$. If the objects are placed on one pan only, say, the left one, then $p_2 = 0$ and $p_0 = p - p_1$. If the objects are distributed one the two pans, then $p_0 = 0$ and $p_1 = p - p_2$. If the objects are distributed over both pans, we shall call the corresponding design two pan design or chemical balance weighing design. In two pans weighing the balancing weight may occur on any of the two pans. Let us denote the observation from i th weighing by y_i, $i = 1,2,...,n$, and the weight of the j th object by w_j, $j = 1,2,...,p$. The model for estimation of the weight for $i = 1,2,...,n$ is

$$y_i = \sum_{i=1}^{p} x_{ij} w_j + e_i \,, \tag{1}$$

where x_{ij} takes the values +1, -1 or 0, according as the j th object is placed on the left pan, right pan or does not occur in the i th weighing and e_i is the error component having a constant variance σ^2. Hence y_i denotes an observation including sign of the weights. The least squares estimates of the weight of the objects can be obtained by usual method when $n \geq p$. Using matrix notation (1) we can write as

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e} \,, \tag{2}$$

where $\mathbf{y}$ is the vector of the observation written as a column, $\mathbf{X}$ is the $n \times p$ design matrix, $\mathbf{w}$ is a $p \times 1$ vector representing unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random vector of the error component in the different observation. The normal equations estimating $\mathbf{w}$ are of the form $\mathbf{X}'\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{y}$, where $\hat{\mathbf{w}}$ is the vector of the weights estimated by the least squares method. We said that any two pan design is singular or not singular depending on whether the matrix $\mathbf{X}'\mathbf{X}$ is singular or not singular, respectively. Now, if $\mathbf{X}$ is of full column rank the least squares estimates of $\mathbf{w}$ are given by $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$ and the elements of the variance matrix of $\hat{\mathbf{w}}$: $\text{Var}\left(\hat{w}_j\right)$ are σ^2 times the j th element in the diagonal of $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$. In the case when $\mathbf{X}$ is not of full column rank then (2) is resolvable and we are not able to estimate unknown measurements of objects. In the case when $\mathbf{X}'\mathbf{X}$ is singular, the authors propose to add additionally weighing of all objects in order to obtain the design matrix of full column rank, i.e. all unknown weights are estimable.

From Hotelling (1944) we have

Definition 1. Any two pan design with the design matrix $\mathbf{X}$ is optimal if $\mathrm{Var}(\hat{w}_j)$ attains the lower bound for each j, $j = 1,2,...,p$.

Theorem 1. Any two pan design with the design matrix $\mathbf{X}$ is optimal if and only if

$$\mathbf{X}'\mathbf{X} = n\mathbf{I}_p, \tag{3}$$

where $\mathbf{I}_p$ is $p \times p$ identity matrix.

II. THE DESIGN MATRIX

Now, we give the definition of the balanced incomplete block design presented in Raghavarao (1971).

A balanced incomplete block (BIB) design there is an arrangement of v treatments in b blocks, each of size k in such a way, that each treatment occurs at most once in each block, occurs in exactly r blocks and every pair of treatments occurs together in λ blocks. The integers v, b, r, k, λ are called the parameters of the BIB design. Let $\mathbf{N}$ be the incidence matrix of this design. It is straightforward to verify that $vr = bk$, $\lambda(v-1) = r(k-1)$, $\mathbf{N}\mathbf{N}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$, $\mathbf{1}_v$ is $v \times 1$ vector of ones.

For two pans design b blocks of BIB correspond to $n = b$ weighings and v treatments of BIB design correspond to $p = v$ objects. Moreover, each block of BIB design indicates k objects. Let these k objects be placed on the left pan and the remaining $v-k$ objects be placed on the right pan. Next, balancing is made and the balancing weight is the observation corresponding to this block. From b blocks we become b observations. Such a scheme of weighing is called two pan weighing. The two sets of objects for each weighing are obtained from one from a block of BIB design and the other from its complementary design. If $\mathbf{N}$ is the incidence matrix of the BIB designs with the parameters v, b, r, k, λ, then the design matrix of two pan weighing may be in the form

$$\mathbf{X} = 2\mathbf{N}' - \mathbf{1}_b\mathbf{1}_v'. \tag{4}$$

Lemma 1. Any two pan design with the design matrix $\mathbf{X}$ given in (4) is nonsingular if and only if

$$v \neq 2k. \tag{5}$$

Proof. For $\mathbf{X}$ in (4) we have

$$\mathbf{X}'\mathbf{X} = a\mathbf{I}_v + d\mathbf{1}_v\mathbf{1}_v', \tag{6}$$

where $a = 4(r-\lambda)$, $d = b - a$. Thus $\det(\mathbf{X}'\mathbf{X}) = \frac{r}{k}(v-2k)^2 \cdot [4(r-\lambda)]^{v-1}$. Because $r \neq \lambda$ then $\det(\mathbf{X}'\mathbf{X}) = 0$ if and only if $v = 2k$. Hence, the Theorem.

Theorem 2. Any two pan design with the design matrix $\mathbf{X}$ given in (4) is optimal if and only if

$$b = 4(r-\lambda) \tag{7}$$

Proof. For $\mathbf{X}$ in (4), comparing (6) and (3) one can easily see, that the design $\mathbf{X}$ is optimal if and only if the condition (7) is fulfilled.

BIB designs for which (7) is fulfilled are called A – family and the parameters of such designs are given in Shirkhande (1962). Now, we consider the case when the design matrix $\mathbf{X}$ given in (4) is not of full column rank, i.e. the parameters of BIB design satisfy the condition $v = 2k$.

Let us consider the design matrix of two pan design $\mathbf{X}$ in the form

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}' - \mathbf{1}_b\mathbf{1}_v' \\ \mathbf{x}' \end{bmatrix}, \quad \mathbf{x} = \theta\mathbf{1}_v, \quad \theta = 1 \text{ or } -1. \tag{8}$$

For the design matrix $\mathbf{X}$ in the form (8) we have $\mathbf{X}'\mathbf{X} = a\mathbf{I}_v + (d+1)\mathbf{1}_v\mathbf{1}_v'$ and moreover $\det(\mathbf{X}'\mathbf{X}) = \frac{r}{k}[4(r-\lambda)]^{v-1}\left[(v-2k)^2 + v\right]$. Because $v = 2k$ thus $\det(\mathbf{X}'\mathbf{X}) = b \cdot [4(r-\lambda)]^{v-1} > 0$, i.e. $\mathbf{X}$ in the form (8) is of full column rank. On the other hand $(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{a}\left[\mathbf{I}_v - \frac{d+1}{a+v(d+1)}\mathbf{1}_v\mathbf{1}_v'\right]$, thus the design is optimal if and only if $d + 1 = 0$. Hence $2r = 4\lambda + 1$. This condition is never fulfilling. Thus we obtain

Theorem 3. Any two pan design with the design matrix $\mathbf{X}$ given in (8) is nonsingular and this design is not optimal.

Let us consider the design matrix of two pan design $\mathbf{X}$ in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}' - \mathbf{1}_b\mathbf{1}_v' \\ \mathbf{x}' \\ \mathbf{z}' \end{bmatrix}, \quad \mathbf{x} = \theta\mathbf{1}_v, \quad \mathbf{z} = \tau\mathbf{1}_v, \quad \theta, \tau = +1 \text{ or } -1. \tag{9}$$

For the design matrix $\mathbf{X}$ in the form (9) we have $\mathbf{X}'\mathbf{X} = a\mathbf{I}_v + (d+2)\mathbf{1}_v\mathbf{1}_v'$ and $\det(\mathbf{X}'\mathbf{X}) = \frac{r}{k}[4(r-\lambda)]^{v-1}\left[(v-2k)^2 + 2v\right]$. Because $v = 2k$ thus $\det(\mathbf{X}'\mathbf{X}) = 2b \cdot [4(r-\lambda)]^{v-1} > 0$, i.e. $\mathbf{X}$ in the form (9) is of full column rank. Because $(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{a}\left[\mathbf{I}_v - \frac{d+2}{a+(d+2)v}\mathbf{1}_v\mathbf{1}_v'\right]$ then the design is optimal if and only if $d+2=0$. Thus $k = \lambda + 1$. Hence we obtain

Theorem 4. Any two pan design with the design matrix $\mathbf{X}$ given in (9) is optimal if and only if $d+2=0$.

Based on the paper of Kageyama (1987) we are able to formulate.

Theorem 6. If the parameters of BIB design are equal to $v = 2k, \quad b = 2l(2k-1), \quad r = l(2k-1), \quad k, \quad \lambda = l(k-1), \quad k$ being even, $l = 1,2,...,$ then the two pan design with the design matrix $\mathbf{X}$ given in (9) is optimal.

Proof. It is obvious that for given parameters and the design matrix in the form (9) condition (3) is true.

III. EXAMPLE

Let us consider the balanced incomplete block design with the parameters $v = 6, \quad b = 10, \quad r = 5, \quad k = 3, \quad \lambda = 2$ given by the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

For $\mathbf{x} = \mathbf{1}_6$ and $\mathbf{z} = -\mathbf{1}_6$ we have

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}. \quad (10)$$

Based on the Theorem 6 the two pan design with the design matrix in (10) is optimal.

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