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REMARKS ON QUANTILES OF STATISTICAL DISTRIBUTIONS OF MULTIVARIATE NORMALITY TESTS BASED ON MOMENTS

Abstract. In the literature of the subject we can find a number of tests of the multivariate normality and rules for construction of their test statistics. A question arises here „Which test is the best in the sense of power?". The paper presents two categories of test statistics based on multivariate skewness and kurtosis coefficients worked out by Mardia and by Jarque and Bera, and six tests of multivariate normality based on these measures.

The aim of the paper is to verify the power of the tests at existing statistical distributions by applying the simulation-based Monte Carlo method for n = 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120; p = 2, 3, 4, 5. For tests which do not hold the required size we propose empirical quantiles, also obtained by Monte Carlo method.

Key words: quantiles, empirical, theoretical, significance level, power of the test, skewness and kurtosis.

I. INTRODUCTORY REMARKS

The main directions of evolution theories of multivariate normal distribution are connected with solving practical problems from social and economic life. This theory has become a convenient tool for analysis of empirical data, and, furthermore, statistical methods which are based on this foundation have mathematical conclusions that are easy to interpret.

There are many tests of multivariate normality and many principles of constructing their tests statistics.

In order to systematize tests of multivariate normality, Domański and Wagner divided them in 1984 into 5 basic types:

- randomization principle,
- measures of shapes,
- Uni's and Roy's section principle,

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- multivariate geometry concept,
- transformation of variables.

Many more methods of classification of multivariate normality tests can be found. They can be classified depending on critical regions or required size of the sample. However, these are second-class criteria.

Because of diversity of multivariate normality tests it is impossible to discuss them at length in one article. Multivariate measures of shapes are useful as statistics which describe a multivariate sample, or as a basis for multivariate normality tests. That is why, some research of multivariate normality tests based on measures of skewness and kurtosis was carried out.

I. BASIC TEST STATISTICS

Let us denote p-dimensional random vector as $\mathbf{x}$ and p-dimensional cumulative distribution function as $\Phi_p(\mathbf{x};\mu,\Sigma)$, where $x \in \Re^p$ is a certain point from $\Re^p$, and μ and Σ are p-dimensional expected value vector and pxp-dimensional covariance matrix, respectively. A notation $\mathbf{x}\sim\Phi_p(\mu,\Sigma)$ means that random vector $\mathbf{x}$ has distribution defined by distribution function $\Phi_p(\mathbf{x};\mu,\Sigma)$.

Let us denote simple hypothesis multivariate normality as:

$$\mathrm{HSMN}: F_p(\mathbf{x}) = \Phi_p(\mathbf{x};\mu_0,\Sigma_0) \qquad (1a)$$

which means that F_p ($\mathbf{x}$) is the distribution function of normal distribution $N_p(\mu_0,\Sigma_0)$, where μ_0 and Σ_0 are given parameters. In particular, let us assume that HSMN* is a simple hypothesis of the following form:

$$\mathrm{HSMN}^*: F_p(\mathbf{x}) = \Phi_p(\mathbf{x};\mathbf{0},\mathbf{I}) \qquad (1b)$$

which means that F_p ($\mathbf{x}$) is the distribution function of normal distribution $N_p(\mathbf{0},\mathbf{I})$. Furthermore, let us assume that composite hypothesis multivariate normality is of the form:

$$\mathrm{HCMN}: F_p(\mathbf{x}) = \Phi_p(\mathbf{x};\mu,\Sigma) \qquad (1c)$$

which means that F_p ($\mathbf{x}$) is the distribution function of normal distribution $N_p(\mu,\Sigma)$ with unknown parameters μ and Σ.

Let us now consider popular measures of multivariate skewness and kurtosis based on Mahalanobis distance and introduced by Mardia (1970)

The statistic from the sample for multivariate skewness developed by Mardia is defined by the following formula:

$$b_{1,p} = \frac{1}{n^2}\sum_{i,j=1}^{n}\{(X_i - \overline{X})' S^{-1}(X_j - \overline{X})\}^3 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^3 \tag{2}$$

Similarly, the statistic from the sample for multivariate kurtosis is defined as follows:

$$b_{2,p} = \frac{1}{n}\sum_{i=1}^{n}\{(X_i - \overline{X})' S^{-1}(X_i - \overline{X})\}^2 = \frac{1}{n}\sum_{i=1}^{n} d_{ii}^2 \tag{3}$$

where $X_1,.....X_n$ are n observations, $\overline{X}$ and S are unbiased etimators of μ, and Σ respectively, and d_{ij} are elements of matrix D =[d_{ij}] defined as follows:

$$D = (x_i - \overline{x})' S^{-1}(x_j - \overline{x}) \tag{4}$$

Measures proposed by Mardia are invariant.

Mardia proposed the following tests based on these test statistics:

$$M_1 = \frac{N}{6} b_{1,p} \tag{5}$$

If null hypothesis 1c is true, this test has asymptotic chi-square distribution with f degrees of freedom where $f = \frac{p(p+1)(p+2)}{6}$. For p >7 the following approximation may be used

$$\sqrt{2M_1} - \sqrt{2f-1}\,. \tag{6}$$

For a big sample size test of hypothesis $\beta_{1,p}$ = 0 needs determining a statistic M_1, and then rejecting the null hypothesis when $M_1 > \chi^2_{\alpha,f}$ for $p \le 7$. For $p > 7$ H_{01} is rejected when $\sqrt{2M_1} - \sqrt{2f-1} > u_\alpha$, where $\Phi(u_\alpha) = 1 - \alpha/2$, and $\Phi(u)$ is cumulative distribution function of standard normal distribution.

$$M_2 = \frac{N*(b_{2,p} - p(p+2))^2}{8p(p+2)} \tag{7}$$

If the null hypothesis 1c is true, this test has asymptotic chi-square distribution with 1 degree of freedom.

$$A_2 = \frac{\{b_{2,p} - p(p+2)*(n-1)\}/(n+1)}{\sqrt{(8p(p+2)/n}} \tag{8}$$

$$A'_2 = \frac{b_{2,p} - E(b_{2,p})}{D(b_{2,p})} \tag{9}$$

where

$$D(b_{2,p}) = \sqrt{D^2(b_{2,p})}\,, E(b_{2p}) = \frac{g(n-1)}{n+1}$$

and

$$D^2(b_{2p}) = \frac{8g(n-3)(n-p-1)(n-p+1)}{(n+1)^2(n+3)(n+5)}.$$

Statistics A2 and A2' have by virtue of central limit theorem asymptotic normal distribution N(0,1).

It is worth noticing that in contrast with statistics A2 and A2', a statistic M2 does not take into account a correction $\frac{n-1}{n+1}$. This correction is insignificant for a large n, so statistics A2 and A2' have asymptotic – not exact normal distribution. The null hypothesis is rejected when $|A_2| > u_\alpha$ or $|A'_2| > u_\alpha$ where $\Phi(u_\alpha) = 1 - \alpha/2$.

Besides directional tests another popular omnibus test was proposed by Mardia:

$$MSK = \text{M1} + \text{M2} \tag{10}$$

where statistics M1 and M2 are defined by formulas (5) and (7), respectively. Statistic MSK has a chi-square distribution with f degrees of freedom where $f = (n/6)(n+1)(n+2)+1$.

III. EXAMINATION OF TEST POWER ON THE BASIS OF THEORETICAL DISTRIBUTIONS

There are many tests of multivariate normality and principles of their construction. That is why, it seems natural to consider which of them are best in the sense of power, which have omnibus test properties, which are directional tests, and finally, which are best to be used in practice. Monte Carlo simulation experiments provide us with answers to these questions .

In the conducted Monte Carlo simulation experiment the power of the following six tests was studied: M1, M2, MSK, A2, A2a

In experiment 10 000 repetitions of multivariate normal distribution were taken into account. This distribution was generated in accordance with the method proposed by Wieczorkowski and Zieliński („Komputerowe generatory liczb losowych" *p*. 105). for 20,30,40,50,60,70,80,90,100,110 and 120; p =2,3,4,5; α = 0,5;

The results are presented in tables 1–4 and on charts. Red numbers mean that the test is based on a given statistic, under the assumption that the null hypothesis is true for specific n, p and alfa is higher than a given significance level.

Table 1. Empirical power of chosen tests of multivariate normal distribution for p = 2,3,4,5, α =0,05; i n= 20,30, 40, 50, 60, 70, 80, 90,100, 110, 120.

Test statistics	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
1	2	3	4	5	6	7	8	9	10	11	12
	$p = 2$										
M1	0.0137	0.0239	0.0306	0.0352	0.0356	0.0392	0.0414	0.0431	0.0443	0.0425	0.0455
M2	0.002	0.0079	0.0141	0.0172	0.0214	0.028	0.0265	0.0279	0.0306	0.0322	0.0347
MSK	0.0089	0.0191	0.0254	0.0293	0.0306	0.0342	0.0362	0.0383	0.0398	0.0385	0.0418
A2	0.0027	0.008	0.0129	0.0161	0.0179	0.0251	0.0234	0.0252	0.0286	0.0277	0.0319
A2'	0.0349	0.0314	0.0335	0.0333	0.0362	0.0402	0.0383	0.0388	0.041	0.0397	0.0441
	$p = 3$										
M1	0.0076	0.0169	0.0263	0.0329	0.0344	0.0376	0.0405	0.0395	0.0398	0.0431	0.0425
M2	0.0335	0.0362	0.035	0.0397	0.0379	0.0422	0.0402	0.0446	0.0411	0.0447	0.0431
MSK	0.0052	0.0132	0.0217	0.0281	0.0302	0.0341	0.0389	0.0355	0.0373	0.0398	0.0394
A2	0.0008	0.0046	0.0094	0.0146	0.0172	0.0237	0.0228	0.0276	0.0285	0.0294	0.0319
A2'	**0.0916**	**0.0591**	0.0472	0.0488	0.0462	0.0474	0.0444	0.0479	0.0454	0.0466	0.0457
	$p = 4$										
M1	0.0041	0.0139	0.0217	0.0271	0.0296	0.0336	0.0365	0.0371	0.0392	0.0397	0.0382
M2	**0.1425**	**0.1033**	**0.0843**	**0.0767**	**0.0699**	**0.0631**	**0.0636**	**0.062**	**0.0659**	**0.0634**	**0.0609**
MSK	0.0026	0.0114	0.0183	0.025	0.0268	0.0305	0.0328	0.0343	0.0346	0.0363	0.0359
A2	0.0001	0.004	0.0105	0.0166	0.0193	0.0195	0.0248	0.0289	0.0317	0.0312	0.0324
A2'	**0.1958**	**0.1013**	**0.0717**	**0.0633**	**0.0578**	**0.0544**	**0.0524**	**0.053**	**0.057**	**0.0519**	**0.0533**

Table 1. (cont.)

1	2	3	4	5	6	7	8	9	10	11	12
	$p=5$										
M1	0.0014	0.0098	0.016	0.0219	0.025	0.0289	0.0321	0.0347	0.0364	0.036	0.0366
M2	**0.3402**	**0.2123**	**0.1576**	**0.1287**	**0.1159**	**0.1003**	**0.0971**	**0.089**	**0.0877**	**0.0818**	**0.0783**
MSK	0.0012	0.0085	0.0143	0.0189	0.0228	0.0262	0.0292	0.0308	0.0339	0.0343	0.0341
A2	0.0011	0.007	0.0134	0.0168	0.0242	0.0224	0.0278	0.0306	0.0324	0.0319	0.004
A2'	**0.3363**	**0.1658**	**0.1103**	**0.087**	**0.079**	**0.0647**	**0.0642**	**0.0652**	**0.0645**	**0.0592**	0.009

Source: Authors' calculations.

IV. EXAMINATION OF TEST POWER ON THE BASIS OF EMPIRICAL DISTRIBIUTIONS

Power of none of the tests proved to be satisfying, so we constructed some tables of critical values calculated empirically on the basis of quantiles. Then our studies on power of the tests M1, M2, MSK, A2 and A2' were repeated on the basis of new tables. Empirical and theoretical critical values for all the tests are as follows:

Table 2. Theoretical critical values for test of multivariate normality M1. (The same values for all *n*)

p	α			
	0.1	0.05	0.01	0.001
2	7.7794	9.4877	13.2767	18.4668
3	15.9872	18.307	23.2093	29.5883
4	28.412	31.4104	37.5662	45.3147
5	46.0588	49.8018	57.3421	66.6188

Source: statistical tables.

Table 3. Empirical critical values for test of multivariate normality M1

n	$p=2$				$p=3$			
	α				α			
	0.1	0.05	0.01	0.001	0.1	0.05	0.001	0.0001
1	2	3	4	5	6	7	8	9
20	5.4503	6.8761	10.0423	15.7585	11.2433	13.3524	17.5464	23.8255
30	6.1213	7.7673	11.6046	18.9404	12.888	15.0612	20.2444	29.237
40	6.5319	8.2712	12.303	19.7391	13.6388	15.9353	21.6074	29.8379
50	6.8368	8.6502	13.0164	19.4981	14.1956	16.7526	22.8001	30.6286
60	6.9514	8.6324	12.694	18.7501	14.5701	16.9702	22.7402	31.7306
70	7.0293	8.7174	12.962	19.7522	14.9543	17.362	22.8631	31.8318

Table 3 (cont.)

1	2	3	4	5	6	7	8	9
80	7.2356	9.0594	13.1628	22.1401	14.9237	17.5376	23.2642	30.9363
90	7.2816	9.0386	13.5814	19.7794	15.0492	17.4762	23.3093	31.6237
100	7.3752	9.1197	13.4266	19.9388	15.2591	17.5959	23.5386	30.2018
110	7.3347	9.0175	13.2114	19.7071	15.351	17.8836	23.2416	33.5868
120	7.3385	9.2274	13.1968	18.4505	15.185	17.7955	22.8487	30.3847
			$p=4$			$p=5$		
20	20.3362	22.6869	28.4182	35.2574	32.7922	35.8416	42.0542	51.3343
30	23.015	25.985	32.7105	43.1343	32.2763	41.313	49.7101	60.6111
40	24.3223	27.4876	35.5197	45.1439	39.6746	43.4886	52.3749	63.557
50	25.2516	28.7087	36.4566	47.2764	41.1904	44.9854	53.2593	67.1056
60	25.9241	28.8764	36.3625	45.7613	42.0466	45.837	54.8192	73.7732
70	26.3011	29.4378	36.6402	46.0206	42.877	46.9107	56.1353	68.8663
80	26.6483	29.9744	37.2842	47.5261	43.087	47.3082	55.763	67.515
90	26.6866	30.0337	37.5528	48.5573	43.389	47.7209	56.4541	67.4357
100	27.0689	30.4308	36.8814	46.7495	43.9014	48.0245	56.1498	66.6664
110	27.1421	30.3332	37.0678	47.5899	44.1838	48.3511	56.4163	67.0873
120	27.2844	30.2292	37.0345	45.9632	44.3511	48.3073	55.9935	66.9324

Source: Authors' calculations.

Table 4. Theoretical critical values for test of multivariate normality M2.
(Values are the same for all n and p)

$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.001$
2.7055	3.8415	6.6349	10.8276

Source: statistical tables.

Table 5. Empirical critical values for test of multivariate normality M2

n	$p=2$				$p=3$			
	α				α			
	0,1	0,05	0,01	0,001	0,1	0,05	0,001	0,0001
1	2	3	4	5	6	7	8	9
20	2.1801	2.5551	3.2291	4.2934	3.1132	3.6269	4.4794	5.2981
30	2.2177	2.7156	3.644	9.0623	3.0032	3.5924	4.7152	5.8654
40	2.297	2.8633	4.2557	12.3339	2.9112	3.5296	4.9424	6.9875
50	2.282	2.8794	4.426	10.7925	2.8451	3.5873	5.0808	7.5882
60	2.3296	2.9982	4.9467	13.9842	2.8492	3.5458	5.1722	10.7756
70	2.3783	3.1592	5.1878	14.198	2.881	3.6529	5.3986	9.7708
80	2.3457	3.0655	5.4328	14.7006	2.8315	3.6195	5.3587	9.436
90	2.4468	3.1808	5.4875	13.1721	2.7327	3.6609	5.726	11.4355
100	2.4015	3.2803	5.5025	14.6172	2.7918	3.6295	5.6786	10.3116
110	2.4077	3.2575	5.5835	15.3422	2.8004	3.6738	5.7104	13.5113
120	2.4841	3.3613	6.1926	14.6784	2.7708	3.6656	5.9222	10.9509

Table 5 (cont.)

1	2	3	4	5	6	7	8	9
	$p = 4$				$p = 5$			
20	4.1585	4.6993	5.6859	6.6661	5.3181	5.8811	6.9427	8.1032
30	3.8665	4.5505	5.7721	6.8414	4.7898	5.538	6.9136	8.2826
40	3.6508	4.348	5.6885	7.2664	4.4643	5.3673	6.8787	8.564
50	3.4901	4.2708	5.9273	7.5043	4.1989	5.0272	6.8373	8.748
60	3.3927	4.2556	6.1383	9.3257	4.0605	4.9893	6.7722	8.9199
70	3.2675	4.1213	5.7537	9.1746	3.8547	4.7501	6.589	9.8604
80	3.2596	4.1505	5.9784	9.0598	3.7988	4.7204	6.7025	9.2735
90	3.1921	4.1367	6.0901	9.8431	3.6629	4.718	6.895	9.5099
100	3.2161	4.2015	6.3987	10.7964	3.6522	4.6507	6.6635	9.7732
110	3.2379	4.1712	6.2836	10.2408	3.5438	4.5661	6.6678	8.9492
120	3.1247	4.1037	6.2215	9.749	3.5111	4.5085	6.7818	8.826

Source: Authors' calculations.

Table 6. Theoretical critical values for test of multivariate normality MSK (the same values for all n)

p	α			
	0,1	0,05	0,01	0,001
2	9.2364	11.0705	15.0863	20.515
3	17.275	19.6751	24.725	31.2641
4	29.6151	32.6706	38.9322	46.797
5	47.2122	50.9985	58.6192	67.9852

Source: statistical tables.

Table 7. Empirical critical values for test of multivariate normality MSK

n	$p = 2$				$p = 3$			
	α				α			
	0.1	0.05	0,01	0,001	0,1	0,05	0,001	0,0001
1	2	3	4	5	6	7	8	9
20	5.817	7.1113	10.7028	19.574	11.8569	13.6063	17.9709	25.3694
30	6.5557	8.2292	13.3743	27.3185	13.3845	15.3503	21.3235	33.1057
40	7.0774	8.8699	14.8527	30.536	14.1508	16.324	22.9615	36.3832
50	7.4342	9.331	15.0585	30.4172	14.7758	17.2979	24.5455	36.7573
60	7.633	9.522	15.5345	31.8589	15.1669	17.6494	24.5158	38.8303
70	7.7821	9.725	16.0674	30.7164	15.6426	18.0798	25.4224	38.1957
80	8.0354	10.0817	16.4795	34.4569	15.7385	18.4424	25.1987	34.7706
90	8.1611	10.225	16.6579	29.6774	15.8187	18.3311	25.5444	38.3485
100	8.2781	10.2872	16.1181	29.288	16.0075	18.4455	25.7506	41.4584
110	8.1424	10.2614	16.3685	30.1199	16.1132	18.8665	25.4954	39.2812
120	8.3109	10.5223	16.321	27.7641	16.0534	18.847	25.3457	36.7143

Table 7 (cont.)

1	2	3	4	5	6	7	8	9
	$p = 4$				$p = 5$			
20	21.2547	23.2971	28.6504	35.3816	34.4123	36.8047	42.5852	51.4063
30	23.6368	26.3003	33.2629	44.4505	38.177	41.6853	49.9058	62.0913
40	24.9572	27.8849	36.0435	49.1196	40.3908	43.8938	52.6136	64.0018
50	25.8586	29.1743	38.0197	51.6469	41.8176	45.4738	54.103	70.6069
60	26.5043	29.432	37.9061	51.3971	42.6255	46.3897	55.5639	78.7786
70	26.9156	30.0588	37.9823	50.9915	43.4985	47.4636	57.1916	72.2677
80	27.3922	30.7343	38.2768	55.5621	43.7969	47.949	56.5886	70.5583
90	27.3994	30.7876	38.6482	54.9837	44.0476	48.342	57.4999	71.4491
100	27.8593	31.2367	39.2223	55.0881	44.6902	48.6787	57.8754	71.359
110	27.9884	31.2247	38.9153	52.0469	44.8854	49.0182	57.5729	70.2078
120	28.0207	31.0465	38.6977	52.0664	45.0766	48.9277	57.3643	69.5063

Source: Authors' calculations.

Table 8. Empirical power of some tests of multivariate normality for p=2, $\alpha = 0.1; 0.05; 0.01$ and $n = 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120$ on the basis of quantiles

Test statistic	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
	$\alpha = 0,1$										
M1	0.1005	0.1013	0.1007	0.1005	0.1011	0.1011	0.0995	0.0992	0.0989	0.0991	0.1016
M2	0.1005	0.0988	0.0993	0.1005	0.1	0.1003	0.0989	0.0989	0.0992	0.1003	0.1011
MSK	0.1009	0.1017	0.1	0.1004	0.1008	0.1004	0.0982	0.0989	0.0993	0.1006	0.1016
	$\alpha = 0.05$										
M1	0.0495	0.0502	0.0498	0.05	0.05	0.0514	0.0499	0.0492	0.0494	0.0483	0.0499
M2	0.0513	0.0492	0.05	0.0507	0.0491	0.0498	0.0498	0.0502	0.0497	0.0509	0.0506
MSK	0.0495	0.0505	0.0493	0.0496	0.0507	0.0502	0.0497	0.0486	0.0488	0.0489	0.0492
	$\alpha = 0.01$										
M1	0.01	0.0101	0.0098	0.0105	0.0097	0.0095	0.0103	0.01	0.0096	0.0092	0.0102
M2	0.0107	0.0097	0.0097	0.0099	0.009	0.0099	0.0097	0.0099	0.0098	0.0104	0.0099
MSK	0.01	0.0106	0.0098	0.0102	0.0097	0.0099	0.0103	0.01	0.0095	0.0099	0.0098
	$\alpha = 0.001$										
M1	0.0009	0.0007	0.0009	0.001	0.001	0.001	0.0011	0.0009	0.001	0.0009	0.001
M2	0.0011	0.0008	0.0007	0.0009	0.001	0.001	0.0011	0.0009	0.001	0.001	0.0008
MSK	0.001	0.0007	0.0008	0.001	0.0008	0.0008	0.0011	0.0009	0.001	0.0007	0.0008
JB	0.0008	0.0012	0.0009	0.001	0.0009	0.001	0.001	0.0009	0.0011	0.0009	0.001

Source: Authors' calculations.

Table 9. Empirical power of some tests of multivariate normality for $p = 3$, $\alpha = 0.1$; 0.05; 0.01 and n = 20, 30, 40, 50, 60, 70, 80, 90,100, 110, 120 on the basis of quantiles

Test statistic	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
	$\alpha = 0{,}1$										
M1	0.1013	0.102	0.0999	0.1008	0.0984	0.0971	0.1005	0.1013	0.1002	0.0991	0.0989
M2	0.1002	0.1	0.1012	0.0984	0.0998	0.0982	0.0988	0.1017	0.0988	0.0974	0.0996
MSK	0.1015	0.1015	0.1003	0.1012	0.0985	0.0973	0.0999	0.1008	0.1001	0.0993	0.0997
	$\alpha = 0.05$										
M1	0.0496	0.0506	0.0502	0.05	0.0498	0.0478	0.05	0.0514	0.0497	0.049	0.0494
M2	0.0498	0.052	0.0512	0.0504	0.0499	0.0485	0.049	0.0504	0.0491	0.0484	0.0492
MSK	0.05	0.0505	0.05	0.0501	0.0503	0.0483	0.0498	0.0522	0.0498	0.049	0.05
	$\alpha = 0.01$										
M1	0.0097	0.0097	0.0102	0.0099	0.0097	0.0098	0.0101	0.0103	0.0096	0.0101	0.0107
M2	0.0103	0.0098	0.0108	0.01	0.0106	0.0095	0.0098	0.0101	0.0096	0.0098	0.0092
MSK	0.0096	0.0097	0.0102	0.0099	0.0097	0.0092	0.01	0.0101	0.0096	0.0103	0.01
	$\alpha = 0.001$										
M1	0.0009	0.0008	0.0011	0.001	0.0012	0.0008	0.0009	0.0009	0.001	0.0008	0.001
M2	0.0009	0.0009	0.001	0.001	0.001	0.0009	0.001	0.001	0.0013	0.0008	0.0011
MSK	0.0009	0.0008	0.0011	0.001	0.001	0.0008	0.0008	0.001	0.0008	0.001	0.001

Source: Authors' calculations.

Table 10. Empirical power of some tests of multivariate normality for p=4, α = 0,1;0,05;0,01 and n = 20,30, 40, 50, 60, 70, 80, 90,100, 110, 120 on the basis of quantiles

Test statistic	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
	$\alpha = 0.1$										
M1	0.1003	0.1027	0.1002	0.1005	0.1004	0.0976	0.1003	0.101	0.099	0.1018	0.1016
M2	0.0996	0.0985	0.0984	0.0969	0.1	0.103	0.1021	0.0972	0.0969	0.0992	0.0991
MSK	0.0994	0.1031	0.1	0.0999	0.1008	0.0978	0.0997	0.1003	0.0986	0.1015	0.1017
	$\alpha = 0.05$										
M1	0.0498	0.051	0.0508	0.05	0.0503	0.0487	0.0501	0.05	0.0501	0.0509	0.0517
M2	0.0496	0.0492	0.0494	0.0486	0.05	0.0513	0.0507	0.0491	0.0471	0.049	0.0479
MSK	0.0496	0.0513	0.0505	0.0499	0.0505	0.0488	0.0499	0.0502	0.0502	0.0511	0.0513
	$\alpha = 0.01$										
M1	0.0098	0.0098	0.01	0.0096	0.0098	0.0096	0.0096	0.0097	0.01	0.011	0.0102
M2	0.0094	0.0095	0.0101	0.0097	0.0102	0.0101	0.0098	0.0095	0.0094	0.0105	0.0106
MSK	0.0099	0.0097	0.0101	0.0095	0.0099	0.0101	0.01	0.0093	0.0098	0.011	0.0103
	$\alpha = 0.001$										
M1	0.0009	0.001	0.001	0.0011	0.001	0.0009	0.001	0.0009	0.001	0.0012	0.0012
M2	0.0009	0.001	0.0009	0.001	0.0011	0.0012	0.0009	0.001	0.001	0.0013	0.0011
MSK	0.001	0.001	0.001	0.0011	0.0011	0.0008	0.001	0.0009	0.001	0.0012	0.0012

Source: Authors' calculations.

Table 11. Empirical power of some tests of multivariate normality for p=5, α = 0,1; 0,05; 0,01 and n = 20,30, 40, 50, 60, 70, 80, 90,100, 110, 120 on the basis of quantiles

Test statistic	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
$\alpha = 0,1$											
M1	0,1015	0,1012	0,0972	0,0994	0,1016	0,0976	0,1016	0,1007	0,1013	0,1011	0,1021
M2	0,1007	0,1029	0,1008	0,1005	0,1002	0,0996	0,1003	0,1003	0,099	0,1011	0,0991
MSK	0,1004	0,1021	0,097	0,0992	0,102	0,0977	0,1015	0,1002	0,1008	0,101	0,1019
$\alpha = 0,05$											
M1	0,0508	0,05	0,0495	0,0509	0,0509	0,0486	0,0499	0,0494	0,0513	0,0508	0,0509
M2	0,0514	0,0501	0,0499	0,0505	0,0501	0,0511	0,0505	0,0502	0,0495	0,0509	0,0507
MSK	0,051	0,0499	0,0496	0,0509	0,0509	0,0483	0,0498	0,0504	0,0512	0,0512	0,0514
$\alpha = 0,01$											
M1	0,0104	0,0097	0,0105	0,01	0,0101	0,0101	0,01	0,0102	0,0103	0,0105	0,0105
M2	0,0103	0,0098	0,0095	0,0101	0,0097	0,0103	0,0099	0,0099	0,0097	0,01	0,0106
MSK	0,0102	0,0098	0,0104	0,0101	0,01	0,0101	0,0101	0,0102	0,0101	0,0107	0,0105
$\alpha = 0,001$											
M1	0,0009	0,0006	0,0012	0,001	0,0009	0,001	0,001	0,0011	0,0011	0,001	0,0013
M2	0,001	0,0009	0,001	0,0011	0,0011	0,001	0,001	0,0008	0,0009	0,0011	0,001
MSK	0,001	0,0006	0,0011	0,0011	0,0008	0,001	0,0011	0,001	0,0011	0,0009	0,0012

Source: Authors' calculations.

CONCLUSIONS

The power of tests M1, M2, and MSK calculated with the use of critical values based on quantiles is close to the assumed significance levels. The power of these tests is high, and does not depend on the size and dimension of the sample. This leads us to the conclusion that they should be used especially for small sample sizes ($n < 80$), and for low alfas (alfa = 0,01 and alfa = 0,001) because power of the tests calculated with the use of critical values based on distributions is much lower.

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UWAGI O KWANTYLACH ROZKŁADU STATYSTYK TESTÓW WIELOWYMIAROWEJ NORMALNOŚCI OPARTYCH NA MOMENTACH

W literaturze przedmiotu możemy spotkać wiele testów wielowymiarowej normalności i zasad konstrukcji statystyk testowych. Powstają więc pytania, które z nich są najlepsze w sensie mocy. W artykule tym przedstawione zostaną miary skośności i spłaszczenia dla rozkładów wielowymiarowych opracowane przez Mardię (1970).

Celem artykułu jest weryfikacja mocy testów przy istniejących rozkładach statystyk na podstawie eksperymentu symulującego metodę Monte Carlo dla n = 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120; p = 2, 3, 4, 5. Dla testów, które nie utrzymują wymaganego rozmiaru zaproponowane zostaną kwantyle empiryczne, uzyskane metodą Monte Carlo.