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THE MINORITY GAME AND QUANTUM GAME THEORY

Abstract. This paper builds up on proposed convincing procedures telling how to quantize well-known games from the classical game theory. The thesis introduces some simple models of quantum games. At first, the prisoners' dilemma in classical and quantum version is described. This simple model has many practical applications in economics, one example being frauds in cartel agreements. The next model of quantum game is the quantum market game, described with the help of the quantum harmonic oscillator. It is known that quantum algorithms may be thought of as the games between classical and quantum agents; therefore, as the last example the quantum minority game is introduced.

Key words: minority game, quantum game theory, Nash equilibrium.

I. THE QUANTUM GAME – DEFINITION

In recent years there appeared many articles on the subject of quantum games. As it turned out many situations in quantum theory can be reformulated in terms of game theory. The quantum game can be defined as strategic manoeuvering of quantum system; the essential elements of quantum game are:

– a definition of the physical system which can be analyzed using the tools of quantum mechanics,

– existence of one or more players (particles) who are able to manipulate the quantum system,

– players' knowledge about the quantum system on which they will make their moves or actions,

– a definition of what constitutes a strategy for a player,

– a definition of strategy space for the players, which is the set of all possible actions that players can take on the quantum system,

– a definition of the pay-off functions or utilities associated with the players' strategies.

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For example, a two-player quantum game is a set consisting of a Hilbert space H of the physical system, the initial state ρ, the sets S_A and S_B of allowed quantum operations for two players, and the pay-off functions or utilities P_A and P_B. This set can be written as

$$\Gamma = (H, \rho, S_A, S_B, P_A, P_B). \tag{1}$$

II. THE CLASSICAL MODEL OF THE PRISONERS' DILEMMA

Prisoners' dilemma is a classical game and the rules of this game are as follows. Two criminals are arrested after having committed a crime together and wait for their trial. Each suspect is placed in a separate cell and they have no contact with each other. In this way no one knows whether the other confesses his guilt or not. But we have no conclusive evidence that prisoners are guilty. So we have to view the case as circumstantial one, and a size of sentence depends on whether they will talk or not. If neither suspect confesses, they are sentenced to three years' imprisonment. If one prisoner confesses and tells about complicity of the other, then the priosoner who confesses is sentenced to two years' imprisonment, while the prisoner who did not confess goes to prison for 15 years. If both prisoners confess, then both are sentenced to eight years' imprisonment.

In accordance with this, each suspect may choose between two strategies:

- confessing (D – defection)
- not confessing (C – cooperation).

The following matrix of payoffs can represent the game:

$$\begin{array}{cc} & \text{prisoner A} \\ & \begin{array}{cc} C & D \end{array} \\ \text{prisoner B} \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (2,15) \\ (15,2) & (8,8) \end{pmatrix} \end{array} \tag{2}$$

The dilemma is that for either choice of the opponent it is hence advatageous to confess (D) but on the other hand, if both confess (D, D) the payoff remains less than in the case when both cooperate (C, C).

We will use a generalized matrix which is given as:

$$\begin{array}{cc} & \text{prisoner A} \\ & \begin{array}{cc} C & D \end{array} \\ \text{prisoner B} \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (r,r) & (s,t) \\ (t,s) & (u,u) \end{pmatrix} \end{array} \tag{3}$$

where $s < r < u < t$.

III. THE QUANTUM MODEL OF THE PRISONERS' DILEMMA

The physical model of the prisoners' dilemma was given by Eisert, Wilkens and Lewenstein (1999). In this model the authors suggested that the players can escape the dilemma if they both resort to quantum strategies. The physical model consists of: a source making available two bits, one for each player; physical instruments enabling the players to manipulate their own bits; a measurement device that determines the players' payoffs from the final state of the two bits. The classical strategies C and D are assigned two basis vectors $|C\rangle$ and $|D\rangle$ in Hilbert space. The state of the game is described by a vector which is spanned by the classical game basis $|CC\rangle$, $|CD\rangle$, $|DC\rangle$ and $|DD\rangle$.

The initial state is written as:

$$\hat{J}\,|CC\rangle \tag{4}$$

where $\hat{J}$ is a unitary operator known to both players. All strategic decisions belong to a strategic space S. The prisoners' strategic decisions (moves) are associated with unitarity operators $\hat{U}_A$ and $\hat{U}_B$. So when players make a decision the state of the game changes to:

$$\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{J}\,|CC\rangle \tag{5}$$

while the final state of the game is given by

$$|\Psi_f\rangle = \hat{J}^{+}\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{J}\,|CC\rangle \tag{6}$$

The payoff of player A will be defined as

$$P_A = r\,\left|\langle CC\,|\,\Psi_f\rangle\right|^2 + s\,\left|\langle CD\,|\,\Psi_f\rangle\right|^2 + t\,\left|\langle DC\,|\,\Psi_f\rangle\right|^2 + u\,\left|\langle DD\,|\,\Psi_f\rangle\right|^2 \tag{7}$$

where the quantities r, s, t and u are from the classical prisoner dilemma matrix. The payoff of player B is obtained by interchanging $s \leftrightarrow t$ in equation (7).

Eisert J., Wilkens M. (2000) use the following matrix representations of unitary operators of their two-parameter strategies:

$$U(\theta,\phi)=\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} \tag{8}$$

where $0 \le \theta \le \pi$ and $0 \le \phi \le \frac{\pi}{2}$.

The classical dilemma must be represented by quantum model, to this end the authors imposed additional conditions on $\hat{J}$:

$$\left[\hat{J},\hat{D}\otimes\hat{D}\right]=0 \qquad \left[\hat{J},\hat{D}\otimes\hat{C}\right]=0 \qquad \left[\hat{J},\hat{C}\otimes\hat{D}\right]=0 \tag{9}$$

where $\hat{C}$ and $\hat{D}$ are the operators corresponding to the strategies of confessing and not confessing, respectively.

A unitary operator satisfying the conditions (9) is

$$\hat{J}=\exp\left\{ i\gamma\,\hat{D}\otimes\hat{D}/2\right\} \tag{10}$$

where $\gamma\in[0,\pi/2]$. The operator $\hat{J}$ can be called a measure of the game's entanglement. The quantum game reduces to its classical form for $\gamma = 0$. For $\gamma = \pi/2$ the game is maximally entangled and the classical Nash equilibrium $\hat{D}\otimes\hat{D}$ is replaced by a different unique equilibrium $\hat{Q}\otimes\hat{Q}$ with $\hat{Q}\sim\hat{U}(0,\pi/2)$. The new equilibrium is also found to be the Pareto optimal, while in the classical game the Pareto optimal and the Nash equilibrium are not the same. That is why Eisert, Wilkens and Lewenstein claimed that in its quantum version the prisoner dilemma disappears from the game, and quantum strategies give a superior performance if entanglement is present.

IV. QUANTUM MARKET GAMES

The quantum market games provide the next example of application of quantum games to economy. Many papers on this subject were written by Piotrowski and Sładkowski (2002). They proposed a guantum-like description of markets and economies where players' strategies belong to Hilbert space. The quantum model of market is the following.

For simplicity the authors assume that there is only one asset on the market. The asset is distributed in units $1G$ of price c. Then the k-th player implements a strategy, which is dentoted by $|\Psi\rangle_k$ and declares the participation of his whole capital in the game. The whole capital consists of s_k units of asset and d_k monetary units ($s_k \geq 0$, $d_k \geq 0$). An arbiter considers the data $\{|\Psi\rangle_k, s_k, d_k\}$ coming from all traders and decides on the actual participation of the k-th trader in the market turnover. The authors consider an inclination of player to reach a transaction with different prices.

Let the real variable q

$$q = \ln c - E(\ln c) \tag{11}$$

denote a value of logarithm of the resignation price over which the k-th player gives up the purchase of asset. E(ln *c*) denotes the expectation value of (ln *c*). The variable *p*

$$p = E(\ln c) - \ln c \tag{12}$$

concerns the situation in which the *k*-th player sells the asset according to his strategy $|\Psi\rangle_k$. Suppose the strategies $|\Psi\rangle_k$ belong to Hilbert spaces H_k and the only attempts of influance on quantum phenomena from outside are possible by means of classical objects.

The state of the game we can write as

$$|\Psi\rangle_{in} := \sum_k |\Psi\rangle_k . \tag{13}$$

Piotrowski and Sładkowski define the hermitan operators of demand Q_k and supply P_k, which act in Hilbert spaces. The operators Q_k and P_k are of counterpart's of position and momentum. The capital flows resulting from an ensemble of simultaneous transactions correspond to the physical process of measurement. A transaction consists in a transition from the state of players strategies $|\Psi\rangle_{in}$ to the describing the capital flow state

$$|\Psi\rangle_{out} := \tau_\sigma |\Psi\rangle_{in} \tag{14}$$

where

$$\tau_\sigma := \sum_k |q\rangle_{k_d\,k_d} \langle q| + \sum_k |p\rangle_{k_s\,k_s} \langle p| \tag{15}$$

is the projective operator given by the division σ of the set of players $\{k\}$ into two separate subsets $\{k\}=\{k_d\}\cup\{k_s\}$, that is those buying at the price $e^{q_{k_d}}$ and those selling at the price $e^{-p_{k_s}}$ in the round of transaction in question. The algorithm A (an arbiter) determines the division of the market σ, the set of price parameters $\{q_{k_d}, p_{k_s}\}$ and the set of values of capital flows.

The set of values of capital flows is fixed according to the interpretation of cumulative distribution function

$$\int_{-\infty}^{\ln c} \frac{\left|\langle q|\Psi\rangle_k\right|^2}{{}_k\langle\Psi|\Psi\rangle_k} dq \tag{16}$$

as the probability that the player $|\Psi\rangle_k$ is willing to buy the asset at the transaction price c or lower. The cumulative distribution function which describes the probability of selling by the player $|\Psi\rangle_k$ at the price c or greater is defined as

$$\int_{-\infty}^{\ln\frac{1}{c}} \frac{\left|\langle p|\Psi\rangle_k\right|^2}{{}_k\langle\Psi|\Psi\rangle_k} dp. \tag{17}$$

The risk inclination operator is defined by the use of the equation of quantum harmonic oscillator:

$$H(P_k, Q_k) := \frac{(P_k - p_{k0})^2}{2m} + \frac{m\omega^2 (Q_k - q_{k0})^2}{2} \tag{18}$$

where

$$p_{k0} := \frac{{}_k\langle\Psi|P_k\Psi\rangle_k}{{}_k\langle\Psi|\Psi\rangle_k}, \qquad q_{k0} := \frac{{}_k\langle\Psi|Q_k\Psi\rangle_k}{{}_k\langle\Psi|\Psi\rangle_k}, \qquad \omega = \frac{2\pi}{\theta}. \tag{19}$$

In equation (19) θ denotes the characteristic time of transaction which is equal to the average interval beetwen two opposite transactions of one player; $m > 0$ measures the risk asymmetry between buying and selling positions. The characterization of quantum market games is described with the help of quantum harmonic oscillator. The constant h_E describes the minimal inclination of the player to risk. In view of an uncertainty relations the constant h_E is equal to the product of the lowest eigenvalue of $H(P_k, Q_k)$ and the minimal interval 2θ during which it makes sense to measure the profit.

Because in general case the players observe moves of other players and often act accordingly, the operators Q_k do not commute. Therefore, Piotrowski and Sładkowski consider noncommutative quantum space and assume that

$$[\, Q_j, Q_k \,] = i\Theta\varepsilon_{jk}. \tag{20}$$

The analysis of the solution of multidimensional harmonic oscillator [6] implies that the parameter θ modifies the constant $h_E \to \sqrt{h_E^2 + \Theta^2}$ and, accordingly, the eigenvalues of $H(P_k, Q_k)$. It means that moves performed by other players can diminish or increase one's inclination to take risk.

V. THE QUANTUM MINORITY GAME

The minority game is a model of speculative trading in financial market where agents buy and sell asset shares with the only goal of profiting from price fluctuations [1]. The basic idea is that when most traders are buying it is profitable to sell and vice-versa, so the minority group always wins. We have N agents and each of them formulates at every time step "t" a binary bid (sell/buy) $a_i(t) = \{0,1\}$.

This model can be considered as information processing system where the players' strategies are the input and the payoffs are the output [4]. The player can choose between two strategies, so the choice can be encoded in the classical case by a bit. In the quantum theory, the bit is altered to a qubit, with the basic states $|0\rangle$, $|1\rangle$ representing the classical strategies [2]. The initial state is described as

$$|\Psi_0\rangle = |00...0\rangle \tag{21}$$

so it consists of one qubit for each player. If the entangling operator $\hat{J}$ acts on $|00...0\rangle$, the initial state is converted into an entangled Greenberger-Horne-Zeilinger state (GHZ state). Pure quantum strategies are local unitary operators acting on a player's qubit. The state before the measurement in the N-player case can be computed by

$$|\Psi_1\rangle = \hat{J}|\Psi_0\rangle$$
$$|\Psi_2\rangle = \left(\hat{M}_1 \otimes \hat{M}_2 \otimes \ldots \otimes \hat{M}_N\right)|\Psi_1\rangle \tag{22}$$
$$|\Psi_f\rangle = \hat{J}^{+}|\Psi_2\rangle$$

where $\hat{M}_k$ (k = 1, …, N) is an unitary operator representing the move of the player k. In the same way as in the quantum prisoners' dilemma the entangling operator $\hat{J}$ commutes with any direct product of classical moves, so the classical game is simply reproduced if all players select a classical move.

Assume that the players do not make use of their knowledge of past behaviour. The classical pure strategies are then "always choose 0" or "always choose 1". The unitary operator, which provides a counterpart of pure quantum strategy, is written as

$$\hat{M}(\theta,\alpha,\beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix} \tag{23}$$

where $\theta \in [0,\pi]$ and $\alpha, \beta \in [-\pi, \pi]$. The entanglement operator is given by

$$\hat{J}(\gamma) = \exp\left(i\frac{\gamma}{2}\sigma_x^{\otimes N}\right), \tag{24}$$

with $\gamma \in [0, \pi/2]$ and $i\hat{\sigma}_x = \hat{M}(\pi, 0, 0)$. For $\gamma = \pi/2$ we have a maximal entanglement in GHZ state. Operators of the form $\hat{M}(\theta, 0, 0)$ are equivalent to classical mixed strategies, with the mixing controlled by θ, since when all players use these strategies the quantum game reduces to the classical one.

It has been found (Flitney A., Hollenberg L. (2007)) that in the four player quantum minority game an optimal strategy exists:

$$\hat{s}_{NE} = \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{16}\right)\left(\hat{I} + i\hat{\sigma}_x\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{16}\right)\left(i\hat{\sigma}_y + i\hat{\sigma}_z\right) = M\left(\frac{\pi}{2}, \frac{-\pi}{16}, \frac{\pi}{16}\right), \tag{25}$$

here $\hat{I}$ is the identity operator, $\hat{I} = \hat{M}(0,0,0)$. The strategy $\hat{s}_{NE}$ is seen to be a Nash equilibrium by observing the payoff to the first player as a function of his general strategy $\hat{M}(\theta, \alpha, \beta)$ while the others play $\hat{s}_{NE}$.

VI. CONCLUSION

This paper shows that in the context of the quantum Prisoners' Dilemma, the players escape the dilemma if they both resort to quantum strategies. Quantum computing gives rise to new Nash equilibria, which belong to several different classes. Classical games assume new dimensions, generating a new strategy continuum, and new optima within and tangential to the strategy spaces as a function of quantum mechanics. A number of quantum games can also be mathematically distinguished from their classical counterparts and have Nash equilibria different than those arising in the classical games.

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GRA MNIEJSZOŚCIOWA I KWANTOWA TEORIA GIER

Artykuł zawiera propozycje procedur kwantowania gier dobrze znanych w klasycznej teorii gier. Przedstawiono proste modele gier kwantowych. Jako pierwszy został omówiony dylemat więźnia zarówno w wersji klasycznej jak i kwantowej, który ma wiele praktycznych zastosowań w ekonomii, jednym z przykładów są oszustwa w porozumieniach kartelowych. Kolejnym modelem gry kwantowej jest kwantowa gra rynkowa opisana za pomocą kwantowego oscylatora harmonicznego. Jako ostatni został omówiony model kwantowej gry mniejszościowej.