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INTRODUCTION TO THE PROBLEM OF TRUNCATED POWER SERIES DISTRIBUTIONS

Abstract. In the paper there has been characterized a distribution of a truncated random variable of the class of power series distributions (abbrev.: PSD). From this class one can obtain some important distributions as special cases. The considerations relate to the case when the truncation is made at an arbitrary point c and to the special case when $c = 0$. In this special case one obtain formulae which are identical with those given in paper by W. Dyczka (1974).

Key words: power series, power series distributions, truncated distributions.

I. INTRODUCTION

Let

$$\{X(\Theta), \Theta \in T\}$$

stand for a one-parameter family of random variables $X(\Theta)$ which depend on a parameter Θ assuming values from some interval T . The $X(\Theta)$ stands for an element of this family, i.e. a random variable corresponding to a fixed value of the parameter Θ .

By $p_k(\Theta)$ we shall denote a probability that the random variable $X(\Theta)$ takes the value k ($k = 1, 2, \dots$), i.e.

$$p_k(\Theta) = P(X(\Theta) = k),$$

whereas by

$$\{p_k(\Theta), k = 0, 1, 2, \dots\} \quad (1)$$

a distribution of this random variable.

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The family of distributions, corresponding to (1), will be written down in the form

$$\{p_k(\Theta), \Theta \in T, k = 0, 1, 2, \dots\}.$$

Let

$$f(\Theta) = \sum_{k=0}^{\infty} a_k \Theta^k \quad (2)$$

be a power series with real coefficients a_k and $|\Theta| < R$, such that either the coefficients are non-negative (then $0 < \Theta < R$) or $(-1)^k a_k \geq 0$ (then $-R < \Theta < 0$).

We adopt the following definition of the family of random variables of class PSD (A. Noack (1950), 127).

Definition 1. One-parameter family (1) is called the family of random variables of type PSD if there exists a function of form (2) such that for $\Theta \in T$ (an interval of convergence)

$$p_k(\Theta) = \frac{a_k \Theta^k}{f(\Theta)}, \quad k = 0, 1, 2, \dots \quad (3)$$

The function $f(\Theta)$, connected with distributions of variables $X(\Theta)$ by formula (3), will be called a *determining function* of the family of PSD's.

In the case when the set of summation indices in (2) is a subset of the set $\{0, 1, 2, \dots\}$, PSD's bear the name of generalized distributions (G.P. Patil (1962-63), 179).

II. THE TRUNCATED PSD

The random variable $X(\Theta)$, truncated at an arbitrary point c , will be denoted by $Y(\Theta)$. Then

$$P(Y(\Theta) = k) = \frac{P(X(\Theta) = k)}{Q(c)}, \quad k = c+1, c+2, \dots, \quad (4)$$

where $Q(c)$ is the so-called *tail* of the distribution, that is,

$$Q(c) = 1 - \sum_{k=0}^c P(X(\Theta) = k).$$

It is easy to notice that

$$F(\Theta) = f(\Theta) - \sum_{k=0}^c a_k \Theta^k, \quad c = 0, 1, 2, \dots, \tag{5}$$

is a determining function of the family of distributions

$$\{P(Y(\Theta) = k, \Theta \in T, k = c + 1, c + 2, \dots)\}.$$

In the next paper we shall be concerned with the family of random variables $Y(\Theta)$ with distributions (4) and variables $Z(\Theta)$ which are sums of independent random variables belonging to the family $Y(\Theta)$, that is,

$$Z(\Theta) = \sum_{i=1}^n Y_i(\Theta). \tag{6}$$

The random variables Y_i have determining functions of form (5).

In particular, we shall consider the random variable $Z(\Theta)$ as two-addend sums

$$Z(\Theta) = Y_1(\Theta) + Y_2(\Theta)$$

with determining functions.

$$F_1(\Theta) = f_1(\Theta) - \sum_{k=0}^c a_k \Theta^k \quad \text{for} \quad f_1(\Theta) = \sum_{k=0}^{\infty} a_k \Theta^k, \tag{7}$$

$$F_2(\Theta) = f_2(\Theta) - \sum_{k=0}^c b_k \Theta^k \quad \text{for} \quad f_2(\Theta) = \sum_{k=0}^{\infty} b_k \Theta^k. \tag{8}$$

In order to employ determining functions as a tool for investigation of the random variables considered, one should, however, know whether the given family of variables (1) is of type PSD and be able to compute its determining function. Yet it turns out (G. P. Patil (1962–63), 182) that if

$$p_{k+1}(\Theta) / p_k(\Theta) = c_k \Theta, \tag{9}$$

where $c_k > 0$ and $p_0(\Theta)$ is an analytic function in a neighbourhood of $\Theta = 0$, then the given family is of type PSD with determining function equal to

$$f(\Theta) = \frac{a_0}{p_0(\Theta)}. \quad (10)$$

The determining function is defined to a precision of a constant factor, thus one may assume that in (10) $a_0 = 1$. It happens, however, that

$$p_{k+1}(\Theta)/p_k(\Theta) = c_k u(\Theta) \quad (11)$$

where $u(\Theta)$ is an injective function of the parameter Θ .

In this case, under the condition adopted, this family is not of type PSD. Regarding the situation, we shall introduce a definition of equivalent families of distributions of type PSD. It will be helpful for us in computing a determining function of, say, the binominal distribution. For the family of random variables with binomial distributions is not type PSD, but only equivalent to the family of type PSD (W. Dyczka, T. Świątkowski (1973), 7).

Definition 2. Two families $\{X(\Theta); \Theta \in T\}$ and $\{Y(u); u \in U\}$, where $u = u(\Theta)$ is an injective function, are said to be equivalent ones if

$$X(\Theta) = Y(u(\Theta)) \text{ for each } \Theta \in T.$$

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**WPROWADZENIE DO PROBLEMU UCIĘTYCH ROZKŁADÓW TYPU
SZEREGU POTĘGOWEGO**

W pracy zostaje podany rozkład uciętej zmiennej losowej klasy typu szeregu potęgowego (PSD). Z klasy tego typu można otrzymać ważne rozkłady prawdopodobieństwa jako szczególne przypadki. Rozważania tu podane dotyczą przypadku, gdy ucięcie jest dokonywane w dowolnym punkcie c , a w szczególnym przypadku, gdy $c = 0$. Ten przypadek był już rozpatrywany w pracy W. Dyczki (1974).