ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 228, 2009

Dominik Krężołek[*]

THE APPLICATION OF M-GARCH MODEL FOR EXAMINING THE VOLATILITY OF FINANCIAL ASSETS[†]

Abstract. The majority of econometric financial market models are based on well run and highly developed economies and available financial time series are very wide, numerous, reporting some specific features as clustering of variance and outliers. Thus, the application of classical methods of the stochastic processes analysis can be biased. The purpose of this paper is to present the review of M-GARCH model to examine the volatility of asset returns in financial market. The analysis includes both individual stocks and portfolios. The most popular approaches of multivariate GARCH models estimation are considered. As a result, the applicability assessment of this class of models within emerging markets will be presented.

Key words: M-GARCH, volatility.

I. INTRODUCTION

The financial time series analysis as well as the majority of econometric models are based on western highly developed economies and therefore the range of data available is quite large. These datasets require usually advanced statistical methods and tools to be precisely analyzed. A wide majority of econometric models have been developed under very strong assumption that the log-returns are normally distributed. Unfortunately, this assumption doesn't work in reality and the classical statistical models are inappropriate. Financial time series analysis is based on the empirical distributions analysis and mainly the first second moments are of interest. But these parameters are not constant over time, so the classical approach has to be rejected. The most relevant features to be observed in time series of market returns are as follow: high volatility, clustering, fat tails, leptokurtosis, leverage effects, serial correlations ect. As it's seen, the theory forms the foundation for making inferences, so this requires that the models and techniques have to be modified in some way.

---

[*] Master, Department of Statistics, Karol Adamiecki University of Economics in Katowice.
[†] Research supported by the KBN grant: N111 003 32/0262.

II. MULTIVARIATE GARCH MODELING

Under the empirical evidence the conditional mean can be estimated through the autoregressive (AR), moving average (MA) models and some its combination (ARMA, ARIMA, ect). For a good inference based on these models, time series have to be stationary and ergodic. If the volatility analysis is considered, this assumption is required as well. The basic models assume that the expected value of squared error terms is the same at any given point (homoskedasticity assumption) and if it's not carried out, the regression coefficient for an ordinary least squares regression are still unbiased, but not efficient, giving a false sense of precision (Engle, (2001)). Hence, the solution for modeling variance are ARCH/GARCH models. In this paper only the multivariate extension of GARCH is of interest and the univariate case is omitted.

Multivariate analysis of financial returns is presently of great importance, especially in the meaning of portfolio selection, assets and option pricing, hedging and risk management as all of these depend on the covariance matrix structure. The specified model should be flexible to represent the conditional variance and covariance between returns and should be parsimonious as well, to allow for easy estimation of the model. Although MGARCH models were developed over twenty years ago, its use still very rarely.

Generally the MGARCH model can be considered as follow. Let $[r_t]$ denotes $N \times 1$ vector of log-returns and has a form:

$$r_t = \mu_t + \varepsilon_t \tag{1}$$

where μ_t and ε_t represent the conditional mean vector generated by observed time series r_t based on information set $\Im$ available at time $t-1$ and error term vector respectively. These parameters satisfy $r_t \mid \Im_{t-1} \sim N(\mu_t, H_t)$ and $\varepsilon_t \mid \Im_{t-1} \sim N(0, H_t)$. Conditional mean vector μ_t can be obtained using VAR models (Piontek (2006), Tsay (2002)). For error term ε_t is assumed to satisfy $\varepsilon_t = H_t^{1/2} z_t$, where H_t represents a $N \times N$ positive definite matrix and furthermore z_t is a gaussian white noise. H_t is considered as a conditional covariance matrix of ε_t. A straightforward generalization of the univariate GARCH model is in the multivariate approach the VECH-GARCH(p,q) model of Bollerslev, Engle and Wooldridge (1988) and has a form Bollerslev T., Engle R., Nelson D. (1994):

$$vech(H_t) = W + \sum_{i=1}^{q} A_i vech(\varepsilon_{t-i}\varepsilon_{t-i}^T) + \sum_{j=1}^{p} B_j vech(H_{t-j}) \tag{2}$$

In this model $vech(\cdot)$ is defined as follow $vech\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = [a_{11} \quad a_{12} \quad a_{22}]^T$ and denote the half-vector operator which stacks the lower triangular element of an $N \times N$ matrix, W, A_i, B_j are an $0{,}5N(N+1)\times 1$ and $0{,}5N(N+1)\times 0{,}5N(N+1)$ matrices respectively.

Although VECH-GARCH model seems to be very attractive for volatility analysis, it still has disadvantages as a number of parameters to be estimated or the positive definiteness of H_t. Even for low dimensions of N the number of parameters is very large (it depends on p and q as well) and results $[0{,}5N(N+1)][1+0{,}5(p+q)N(N+1)]$ parameters to be estimated. If N = 3, $p=q=1$ VECH-GARCH model contains 78 parameters which in unmanageable in practice (Bollerslev T., Engle R., Nelson D. (1994)).

To avoid the problem with a burdensome estimation of parameters and ensure that the conditional covariance matrices are positive definite VECH model has been modified. The matrices A_i, B_j are assumed to be diagonal. This condition is satisfied if the Hadamard product $\otimes$ is used and the model has a form:

$$H_t = W^* + \sum_{i=1}^{q} A_i^* \otimes (\varepsilon_{t-i}\varepsilon_{t-i}^T) + \sum_{j=1}^{p} B_j^* \otimes (H_{t-j}) \tag{3}$$

where $A_i = diag[vech(A_i^*)]$, $B_j = diag[vech(B_j^*)]$. This model is called DVECH-GARCH and is an extension of previous one. The algebra of Hadamard products provides the condition for W^* to be positive definite and for A_i^*, B_j^* to be positive semi-definite. In this model the number of estimated parameters is reduced up to $0{,}5N(N+1)(1+p+q)$ Bollerslev T., Engle R., Wooldridge (1988).

The attractive model which provides, by construction, the positive definiteness of H_t is BEKK model, proposed by Baba, Engle, Kraft and Kroner [see in 7]. In this model the equality $H_t = H$ is satisfied, and model has a form:

$$H_t = V^T V + \sum_{k=1}^{K}\sum_{i=1}^{q} A_{ki}^T \varepsilon_{t-i}\varepsilon_{t-i}^T A_{ki} + \sum_{k=1}^{K}\sum_{j=1}^{p} B_{kj}^T H_{t-j} B_{kj} \tag{4}$$

where V, A_{ki}, B_{kj} are $N \times N$ matrices and V is upper triangular. In this case the number of parameters is $N[NK(p+q)+0{,}5(N+1)]$ or $N[K(p+q)+0{,}5(N+1)]$ if A_{ki}, B_{kj} are diagonal. Even if BEKK model satisfies the condition for H_t, numerical difficulties still appear.

As the volatility analysis is strongly connected with financial returns, it's necessary to find out the roots of returns changes. If someone looks at the concept of volatility through the economic theory, it's obvious that is caused by some factors (usually unknown). Therefore the expected return can be considered as a linear combination of unknown factors and can be written as $r_t = M\varpi_t + \varepsilon_t$, where M is a $N \times K$ matrix and ϖ_t is a $K \times 1$ vector of factors (it's assumed for ε_t and ϖ_t to be uncorrelated). Thus in factor GARCH model (F-GARCH) the conditional covariance matrix H_t has a form:

$$H_t = G + \sum_{k=1}^{K} m_k m_k^T \lambda_{kt} \tag{5}$$

where G is an $N \times N$ semi-definite matrix, m_k are linearly independent vectors of factor weights and λ_{kt} are vectors of factors (elements of conditional covariance matrix Λ_t of factors ϖ_t). Moreover λ_{kt} has a GARCH(1,1) form. The parameter K represents the number of assets included, so r_t can be considered as a portfolio and should satisfy $K < N$. In literature have been proposed modifications of basic F-GARCH, as orthogonal GARCH (O-GARCH) or full factor GARCH (FF-GARCH) and all of those are supplementary.

Summarizing models which has been presented above it's easy to divide it into two groups. The first one includes VECH, DVECH and BEKK models can be defined as a generalization of simple univariate GARCH model, and the second one includes a set of factor models and can be defined as a linear combination of GARCH models. Additionally another group can be defined, namely nonlinear combination of GARCH [1]. This group is very attractive in the meaning of estimation procedures and includes constant and dynamic conditional correlation models (CCC-GARCH, DCC-GARCH), flexible dynamic conditional correlation models (FDCC-GARCH) ect.

The first nonlinear model is the Constant Conditional Correlation model of Bollerslev (CCC-GARCH) where conditional correlations are constant but conditional covariances are proportional to the product of the corresponding conditional standard deviations. Hence, H_t has a form:

$$H_t = D_t P D_t \tag{6}$$

where P is a symmetric positive definite matrix with dimension $N \times N$ satisfying $\rho_{ii} = 1$, $i = 1,...,N$ and $D_t = diag\left(\sqrt{h_{11,t}}, \sqrt{h_{22,t}}, ..., \sqrt{h_{NN,t}}\right)$. A condition for $h_{ii,t}$ to be defined in terms of univariate GARCH is required. In CCC model a total number of parameters is considerably less than in VECH or BEKK models and contains $0{,}5N(N+5)$ parameters. Moreover, to ensure positive definiteness of H_t all the conditional variances in D_t have to be positive and the matrix P has to be positive definite as well. As the condition of constant conditional correlation in an empirical applications is usually rejected, the model has to be modified. This re-specification assumes that the conditional correlation matrix is time-dependent. Then the model proposed above becomes dynamic. Therefore the new model looks similarly like (6) but yet the correlation matrix is indexed by time:

$$H_t = D_t P_t D_t \tag{7}$$

where P_t is an $N \times N$ time-varying correlation matrix at time t. In literature exist a vast number of how to specify P_t. One of the most popular representation of DCC model is that given by Tse and Tsiu [see in 1]. In this model P_t satisfies:

$$P_t = (1 - \pi_1 - \pi_2)P + \pi_1 S_{t-1} + \pi_2 P_{t-1} \tag{8}$$

where π_1, π_2 are non-negative parameters ($\pi_1 + \pi_2 < 1$), P is an $N \times N$ positive definite matrix with ones on the diagonal, S_{t-1} is a sample correlation matrix of the past M standardized residuals ε_τ, $\tau = t - M$, $t - M + 1,...,t-1$ and moreover:

$$\varepsilon_{ij,t-1} = \left(\sum_{m=1}^{M} u_{i,t-m} u_{j,t-m}\right)\left[\left(\sum_{m=1}^{M} u_{i,t-m}^2\right)\left(\sum_{m=1}^{M} u_{j,t-m}^2\right)\right]^{0,5}$$

where $u_{i,t}. = \frac{\varepsilon_{it}}{\sqrt{h_{ii,t}}}$. In this model the conditional correlation matrix is time-varying and is a function of the conditional correlations of the previous periods. The number of parameters to be estimated is $0{,}5N(N-1)+2$. The DCC models are very attractive to financial applications (if portfolio analysis is of interest), but what have to be satisfied is the same dynamic structure for all correlations.

The last model proposed in this paper is a generalization of the previous one. The modification concerns to the correlation structure. Billo, Caporin and Gobbo (2006) propose flexible dynamic conditional correlation model (FDCC-GARCH), where the constraint of the same dynamic structure is unnecessary. Therefore the structure of conditional correlation matrix is assumed to be equal only among groups of variables, hence a block-diagonal form has been proposed. The model (6) can be re-written as:

$$H_t = D_t P_t^* D_t \tag{9a}$$

$$P_t^* = \left(Q_t^*\right)^{-1} Q_t \left(Q_t^*\right)^{-1} \tag{9b}$$

The symmetric matrix P_t^* represents the correlation dynamics, where $Q_t^* = diag\left(\sqrt{q_{11,t}}, \sqrt{q_{22,t}}, \ldots, \sqrt{q_{NN,t}}\right)$. Moreover Q_t is of the form:

$$Q_t = (\Delta - \Pi_1 - \Pi_2)P + \Pi_1 \otimes S_{t-1} + \Pi_2 \otimes Q_{t-1} \tag{9c}$$

where Δ, Π_1, Π_2 are $N \times N$ matrices and notation $\circ$ denotes the Hadamard product.

In FDCC model is assumed that dynamics are common among a group of assets. The estimation is similar as for DCC model and consists two steps, the first one based on univariate GARCH estimation and second one based on correlations. The block-diagonal FDCC model is an attractive tool for volatility analysis if different areas are considered (geographical, sectorial, ect.) cause the dynamics can differ in these sectors significantly.

III. CONCLUSIONS

This paper presents a review of models which are used for volatility analysis. The volatility is considered in a multivariate case, therefore the extension of univariate GARCH model is presented (the multivariate GARCH models). These models differ not only in parameterization, but also in estimation procedures, number of parameters, complexity, ect. Comparing M-GARCH models presented in this paper the clear lead of diagonal models is confirmed (especially in the estimation and parameterization case). Moreover if portfolio selection is of interest, DCC and FDCC models can be used. The last one seems to be very attractive, comparing models and its applicability in emerging and developed markets.

REFERENCES

Bauwens L., Laurent S., Rombouts J. V. K. (2006), Multivariate GARCH Models: A Survey, *Journal of Applied Econometrics*, 21, p. 79–109.

Billio M., Caporin M., Gobbo M. (2006), Flexible Dynamic Conditional Correlation multivariate GARCH models for asset allocation, *Applied Financial Economics Letters*, 2, p. 123–130.

Bollerslev T., Engle R., Nelson D. (1994), ARCH Models, *Handbook of Econometrics*, Volume IV, p. 2959–3038.

Bollerslev T., Engle R., Wooldridge J. (1988), A Capital Asset Pricing Model with Time-varying Covariances, *Journal of Political Economy*, vol. 96, no. 1, p. 116–131.

Engle R. (2001), GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics, *Journal of Economic Perspectives*, Volume 15, Number 4, p. 157–168.

Piontek K. (2006), Niektóre praktyczne wyzwania w modelowaniu wielowymiarowych procesów GARCH, PN AE we Wrocławiu Nr 1126, *Taksonomia 13*, Wrocław, s. 194–205.

Silvennoinen A., Teräsvirta T. (2007), Multivariate GARCH Models, *SSE/EFI Working Paper Series in Economics and Finance* No. 669.

Tsay R. S. (2002), *Analysis of Financial Time Series*, John Wiley & Sons. Inc, Chicago.

Dominik Krężołek

ZASTOSOWANIE MODELI KLASY M-GARCH DO BADANIA ZMIENNOŚCI AKTYWÓW FINANSOWYCH

Większość ekonometrycznych modeli rynków finansowych konstruowanych jest w oparciu o wielkie i rozwinięte gospodarki światowe. Podejście takie nie zawsze znajduje zastosowanie w przypadku młodych i wschodzących rynków. Wynika to po pierwsze z dostępności, a po drugie z charakteru danych tworzących finansowe szeregi czasowe (skupiska danych, grube ogony, autokorelacja). Celem pracy jest zastosowanie modelu M-GARCH do analizy poziomu zmienności stóp zwrotu aktywów finansowych w przypadku, gdy badaniu poddane są portfele inwestycyjne (o więcej niż dwóch składnikach). Przedstawione zostaną różne podejścia do analizy warunkowej wariancji (modyfikacje M-GARCH). Wynikiem będzie ocena stosowalności tej klasy modeli.