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ON SOME RISK-REDUCING DERIVATIVES

Summary

In this paper, we propose some derivative designed for small stock investors. Using the Black-Scholes model we derive an explicit formula for the price of the derivative, computing its discounted expected payoff. The payoff is modelled on the payoff of the catastrophe bonds, random occurrence of a natural disaster is replaced by a random stock price falling. Different variants of the proposed derivative are obtained by introducing a parameter to the payoff of the derivative. By Monte Carlo method, to reduce the risk of large losses associated with the investment, indicated the variant of this instrument, appropriate to selected typical values of volatility of considered stock .

Keywords: Black-Scholes model, risk-reducing derivatives, Monte Carlo method, risk transfer

1. Introduction

Derivative is a financial instrument whose value is derived from an underlying asset. In this paper we use the Black-Scholes model with one risk-free asset and one risky asset. The risky instrument - a stock - is regarded as the underlying. We consider the simplest case of the model which is based on the following assumptions: security trading is continuous, there are no riskless arbitrage opportunities, there are no transaction costs and no dividends during the life of a derivative, the risk-free rate of interest and the volatility of an underlying asset are constant. Volatility, essential in the Black-Scholes model, can be computed as the standard deviation of the returns of an underlying asset for a period of one year (the annualized volatility). Assuming that there are 250 trading days in any given year, we obtain the annualized volatility, multiplying the standard deviation of the daily returns of a stock by $\sqrt{250}$ [Tarczyński, Zwolankowski, 1999, p. 280]. An estimate of future volatility of an underlying asset can be obtained by assuming that the recent realized level of volatility will continue in the future [Weron, Weron, 1998, p. 183]. Estimating volatility is a broad subject [Wiklund, 2012, p. 2] and goes beyond the scope of this work. The annualized volatility of stock, from now on called briefly volatility, is typically between 15% and 60% [Wiklund, 2012, p. 2]. The higher volatility, the greater risk of investing in stock. In the case of high volatility, to protect a holder of a stock against a large loss, we propose a financial instrument, paying an agreed amount

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of money when the value of the stock falls below a specified level. The issuer of this instrument could be an investor having large financial resources, willing to take more risk. An investor investing in risky stocks would be a buyer of this instrument. When the stock price falls below a specified level, determined in the prospectus of the derivative, the holder of the financial instrument exercises his right to sell the instrument at the agreed price. The issuer of this instrument is obliged to redeem the instrument for an amount of money, fixed in a contract. In this paper we obtain an analytical closed form formula to price the proposed derivative instrument. Using Monte Carlo simulations we consider the derivative in a few variants. Different variants of the derivative are obtained by introducing different parameter values to the function of the payoff of the instrument. Depending on volatility of a stock price, we indicate a proper variant of a derivative instrument for reducing risk of large losses associated with investing in a stock. The idea of this financial instrument is based on the idea of catastrophe bonds [Romaniuk, Ermolieva, 2004, p. 115].

2. Definition and pricing a risk-reducing derivative

Let r be the risk-free interest rate and $\sigma > 0$ be a stock price volatility. We assume the price S of the stock follows a geometric Brownian motion

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), t \in [0, T], \tag{1}$$

where $W = \{W_t, t \in [0, T]\}$ is a standard Brownian motion under the risk-neutral probability P, the stock price at time 0 is S_0 and T is the expiry date. Let us denote by E^P the expectation operator under the $P-$ measure and by $\{\mathcal{F}_t\}$ a filtration for Brownian motion W. Let us consider a financial derivative instrument dependent on parameter $a > 0$, with the following payoff function

$$f(S_T) = \begin{cases} S_T & if\ S_T \leq aS_0, \\ 0 & if\ S_T > aS_0. \end{cases} \tag{2}$$

If someone invests an amount S_0 in a risky stock at time 0, he obtains S_T at time T. If the investor is additionally the holder of the proposed derivative and value S_T of his investment falls below aS_0 at time T, he receives compensation equal to S_T, as the payoff of the instrument. If S_T exceeds the level of aS_0, he does not receive a compensation. Thus the instrument provides some protection against the collapse of the value of the stock and can be considered as an obligation transferring the risk from a holder of the derivative - an individual investor investing in stocks, to an issuer. The method of pricing the instrument is independent on a. Later in this work, we will analyze in more detail the instrument for selected values of a. Since $f(S_T)$ is positive, $\mathcal{F}_T-$ measurable and square-integrable with respect to measure P, the arbitrage price of the proposed instrument expresses as the expected value of its discounted payouts. This expectation is taken with respect to the risk-neutral measure P [Jakubowski et al., 2003, p. 180]. Then today's price of the instrument is

$$c = E^P\left(e^{-rT} f(S_T)\right). \tag{3}$$

By (2) and (3) we have:

$c = e^{-rT}E^P\left(S_T 1_{(S_T \le aS_0)}\right) = e^{-rT}\int_\Omega\ S_T\, 1_{(S_T \le aS_0)} dP = e^{-rT}\int_\Omega\ S_T\, 1_{(0 \le S_T \le aS_0)} dP.$

With the change of variables, denoting by g the probability density function of S_T, we have

$$c = e^{-rT}\int_0^{aS_0} xg(x)dx.$$

Let f denote probability density function of $X = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T$.

Since $S_T = S_0 e^X$, it follows that the probability density function g of S_T is expressed as

$$g(x) = \frac{1}{x} f\left(ln\frac{x}{S_0}\right) \text{ for } x > 0 \text{ and } g(x) = 0 \text{ for } x \le 0$$

and consequently we have

$$c = e^{-rT}\int_0^{aS_0} f\left(\ln\left(\frac{x}{S_0}\right)\right)dx.$$

Substituting $\ln\left(\frac{x}{S_0}\right) = u$ in last integral and taking into account that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{[x - \left(r - \frac{1}{2}\sigma^2\right)T]^2}{2\sigma^2 T}\right)$$

we have

$$c = \frac{S_0 e^{-rT}}{\sigma\sqrt{2\pi T}}\int_{-\infty}^{lna}\exp\left\{u - \frac{[u - \left(r - \frac{1}{2}\sigma^2\right)T]^2}{2\sigma^2 T}\right\}du.$$

But

$$u - \frac{\left[u - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T} = \frac{2\sigma^2 Tu - \left[u - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T} =$$

$$\frac{-u^2 + 2uT\left(r + \frac{1}{2}\sigma^2\right) - T^2(r - \frac{1}{2}\sigma^2)^2}{2\sigma^2 T} =$$

$$= -\left(\frac{u^2 - 2uT\left(r + \frac{1}{2}\sigma^2\right) + T^2(r + \frac{1}{2}\sigma^2)^2 + T^2(r - \frac{1}{2}\sigma^2)^2 - T^2(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2 T}\right)$$

$$= \frac{-[u - \left(r + \frac{1}{2}\sigma^2\right)T]^2 + 2r\sigma^2 T^2}{2\sigma^2 T}.$$

Hence

$$c = \frac{S_0 e^{-rT}}{\sigma\sqrt{2\pi T}}\int_{-\infty}^{lna} exp\left(\frac{-[u - \left(r + \frac{1}{2}\sigma^2\right)T]^2 + 2r\sigma^2 T^2}{2\sigma^2 T}\right)du = S_0 H(lna)$$

where H is cumulative distribution function of normal distribution with mean $\left(r+\frac{1}{2}\sigma^2\right)T$ and standard deviation $\sigma\sqrt{T}$.

Finally, we obtain the price of considered derivative

$$c = S_0 N\left(\frac{lna-\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \quad (4)$$

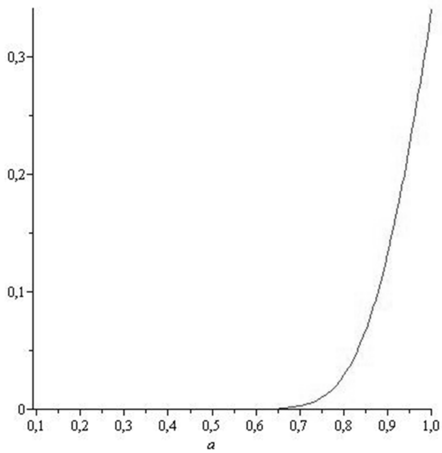
where N is the cumulative probability distribution function for a standardized normal distribution.

From (4) it follows that for every fixed σ the price of the considered derivative is an increasing function of the coefficient a and does not exceed S_0. Our purpose is to propose a variant of the derivative (i.e. to propose value of a) with a payoff given by (2) according to the volatility of the stock. To this end, let us examine more closely how the price c varies according to parameters a and σ. Since the instrument is designed to protect against a decline in the stock price i.e. against the event $S_T \leq aS_0$, we consider parameter a in the interval [0,1].

Example. Let $S_0 = 1, T = 1, r = 5\%$. Figures 1 and 2 plot today's price c against the coefficient a.

FIGURE 1.

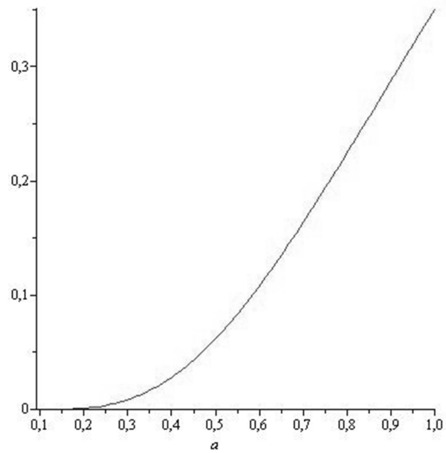
The price c as a function of a, for fixed σ=15%



Source: own study.

FIGURE 2.

The price c as a function of a, for fixed σ=60%



Source: own study.

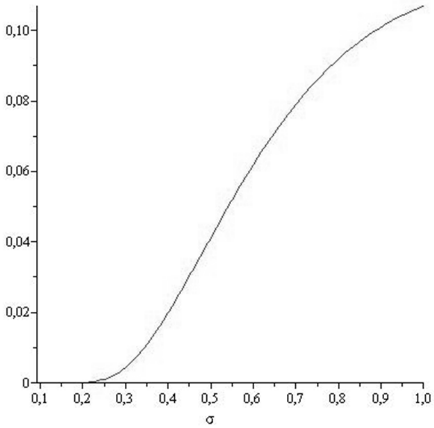
A growth rate of the price depends on the volatility of the stock price. When the volatility increases, the price of the proposed instrument is positive for smaller and smaller values of parameter a. It can be explained by the fact that the greater volatility implies greater probability of a large decline in the stock price and consequently greater probability of payoff from the derivative.

In Figure 3 we present dependence of the price c on the volatility σ changing in the range $[10\%, 100\%]$, with three fixed values of a.

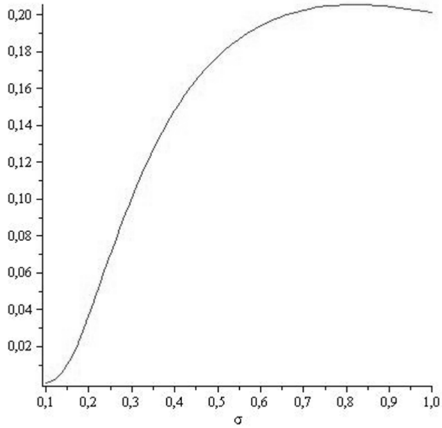
One can see the derivative price not always increases with stock price volatility. The graphs are significantly different, it can be assumed that the choice of the parameter a will have significantly different consequences for the investor. The above cursory analysis of the derivative price does not yet provide decision-making rule which variant of the considered derivative to choose (from an investor point of view). The graphs motivate the need for further analysis of the problem. We proceed with the study of the problem in the next section.

FIGURE 3.

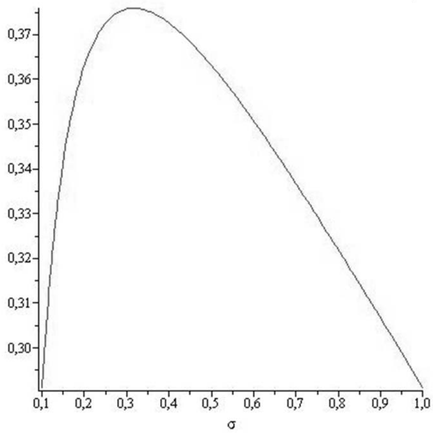
Plots of c against σ, for different but fixed values of a



a) $a = 0.5$



b) $a = 0.75$



c) $a = 1$

Source: own study.

3. Return on investment

To examine the usefulness of proposed derivative instrument, let us compare two investment portfolios:

1. The portfolio is composed of one stock with value S_0 at time 0. The discounted profit from the portfolio at time T equals
$$S_T e^{-rT} - S_0.$$
We will simulate related, discounted profit from the portfolio, expressed in percentage

$$U = \frac{S_T e^{-rT} - S_0}{S_0} * 100\% \tag{5}$$

2. An investor decides to buy one stock with value S_0 at time 0 and additionally one risk reducing derivative at price c. The portfolio consists of one stock and the purchased derivative. The discounted gain from the portfolio is
$$(S_T + f(S_T))e^{-rT} - (\,S_0 + c).$$
The related, discounted percentage of profit from the portfolio equals
$$V = \frac{\left(S_T + f(S_T)\right)e^{-rT} - (\,S_0 + c)\big)}{S_0 + c} * 100\%. \tag{6}$$

We will compare the two portfolios, computing U and V. We use formula (4) to calculate c and Monte Carlo method to calculate random variables occurring in formulas (5) and (6). Namely, we simulate a sample $s_1, \dots, s_n$ of $n = 10^5$ values of random variable S_T. Then we calculate, substituting s_i in place of S_T:

$$f_i = f(s_i), i = 1, \dots, n \text{ by (2)},$$
$$U_i = U(s_i), i = 1, \dots, n \text{ by (5)},$$
$$V_i = V(s_i), , i = 1, \dots, n \text{ by (6)}.$$

Let $q(U)$ denote quantile of order 0.05 of the sample $U_i, i = 1, \dots, n$ and let $q(V)$ be the same order quantile of the sample $V_i, i = 1, \dots, n$. Hence $P\big(U \le q(U)\big) = P\big(V \le q(V)\big) = 0.05$ and each of the quantiles indicates the potential loss of the respective portfolio, expressed in percentage over time horizon T for a given confidence level 0.95. Let us denote by $Q(U)$ and $Q(V)$ quantile of order 0.95 of the samples of $U_i,\ i = 1, \dots, n$ and $V_i,\ i = 1, \dots, n$ respectively. Then $P\big(U > Q(U)\big) = P\big(V > Q(V)\big) = 0.05$. The gain of the portfolio I, expressed in percentage, over time horizon T, does not exceed $Q(U)$ with probability 0.95. The analogous gain of the portfolio II, impassable with probability 0.95, is equal to $Q(V)$.

The results of our calculations for $T = 1, r = 5\%,\ S_0 = 1,\ \sigma = 0.1 * k, k = 1, \dots, 10$ and $a \in \left\{\frac{1}{2}, \frac{3}{4}, 1\right\}$ are presented below:

TABLE 1.

Quantiles of order 0.05 and 0.95 of U and V, for $a = 1/2$

σ	$q(U)$	$Q(U)$	$q(V)$	$Q(V)$
0.1	-15.57	17.18	-15.57	17.18
0.2	-29.52	36.00	-29.5	35.99
0.3	-41.60	56.57	-40.30	55.90
0.4	-52.19	78.30	-46.12	74.83
0.5	-61.22	101.19	-49.46	93.21
0.6	-68.87	125.78	-52.41	112.61
0.7	-75.20	147.22	-55.46	129.11
0.8	-80.48	172.28	-64.24	149.35
0.9	-84.90	191.81	-72.58	165.03
1	-88.35	214.87	-78.95	184.46

Source: own study.

As one can see, for a=1/2 and volatility 0.1, the quantiles $q(U)$ and $q(V)$ are equal, for volatility 0.2 and 0.3 the difference between the quantiles $q(U)$ and $q(V)$ is not significant. The biggest difference, about 20%, is observed for volatility 0.7.

TABLE 2.

Quantiles of order 0.05 and 0.95 of U and V, for $a = 3/4$

σ	$q(U)$	$Q(U)$	$q(V)$	$Q(V)$
0.1	-15.64	17.34	-15.64	17.36
0.2	-29.42	36.52	-26.33	34.75
0.3	-41.75	56.38	-31.61	42.02
0.4	-52.20	78.34	-34.94	55.32
0.5	-60.93	100.96	-38.31	70.68
0.6	-68.83	126.43	-47.80	89.62
0.7	-75.33	146.70	-58.97	105.14
0.8	-80.46	169.74	-67.58	123.75
0.9	-84.73	196.49	-74.65	146.12
1	-88.21	211.41	-80.38	159.27

Source: own study.

When a = 3/4 and volatility equals 0.1 quantiles $q(U)$ and $q(V)$ are equal, for volatility 0.2 the difference is about 3% only. The biggest difference between $q(U)$ and $q(V)$, over 20%, is observed for volatility 0.5 and 0.6.

TABLE 3.

Quantiles of order 0.05 and 0.95 of U and V, for $a = 1$

σ	$q(U)$	$Q(U)$	$q(V)$	$Q(V)$
0.1	-15.46	17.23	-25.33	45.16
0.2	-29.69	35.87	-28.46	36.04
0.3	-41.71	56.33	-28.92	34.55
0.4	-52.13	78.11	-30.60	36.84
0.5	-61.21	100.33	-43.09	46.96
0.6	-68.93	123.77	-54.00	65.66
0.7	-75.25	147.70	-62.97	85.30
0.8	-80.68	171.17	-70.77	105.15
0.9	-84.92	189.79	-76.92	121.79
1	-88.30	212.98	-81.87	142.40

Source: own study.

In case when a =1, if volatility is 0.1, $q(U)$ and $q(V)$ differ by about 10%. For volatility 0.2, the difference between the quantiles $q(U)$ and $q(V)$ is not significant. The difference is significant, over 20%, for volatility 0.4. Knowing the value of the parameter σ (calculated on the base of observations of market prices of stock) an investor can choose the optimum value of the parameter a to minimize the risk of the investment in the stock. Namely, for each of the typical value of the volatility we indicate for an investor the parameter and consequently the derivative which reduces the risk of a large loss by more than 10% on confidence level 95%. Precisely, if $\sigma = 0.3$ then choosing the derivative with parameter $a = 1$, an investor obtains reduction of the potential loss of his portfolio from 41.71% to 28.92%, for confidence level 0.95. This means that probability that loss of portfolio U exceeds 41.71% (gain is less than -41.71%) equals 5% while for portfolio V, probability of 5% refers to the loss greater than 28.92% only. If $\sigma = 0.4$ and $a = 1$, potential loss, on 0.95 confidence level, decreases from 52.13% to 30.6%. If $\sigma = 0.5$ or $\sigma = 0.6$, an investor gets the largest reduction of the potential loss, over 20%, at a confidence level of 0.95, for the derivative with parameter a = 3/4. If $\sigma = 0.7$, taking into account reduction of potential loss of his portfolio, the most preferred purchase of the derivative corresponds to the value of the parameter $a = 1/2$. If the stock price volatility σ does not exceed 0.2 the proposed derivative has no effect on the potential loss of a portfolio. As one can see, generally $q(U) < q(V)$ which means that the proposed derivative instrument reduces the risk associated with investing in stocks.

4. Conclusions

To summarize, we propose a risk reducing derivative designed for small stock investors. We obtain an analytical closed form formula to price the proposed derivative instrument. To analyze the applicability of the proposed instrument, we compare the two investment portfolios, one composed of a stock with the other composed of a stock and the proposed risk reducing derivative. Using Monte Carlo simulation we compute and compare discounted profit and corresponding quantiles of various order for both portfolios. The obtained results demonstrate that generally, the portfolio including our proposed derivative has associated greater quantile indicating potential loss and consequently smaller risk.

The payment of the proposed derivative depends on a parameter. The derivative is precisely determined by fixing the parameter value. Using Monte Carlo, for each of the typical value of the volatility we indicate for investors the parameter and consequently the derivative which reduces the risk of a large loss by more than 10% on confidence level 95%. To conclude, our proposed derivative instrument reduces the risk of large losses associated with investing in stocks. Reducing the risk of ruin can be important for small investors. Note that the lower risk of a large loss goes hand in hand with limited chance for a big profit. On the other hand, the possibility of large profits, even fraught with greater risk, may be attractive for large market players, holders of the proposed instrument.

In future work, a more general Black-Scholes model could be used to investigate the proposed derivative.

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