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Bayesian Estimation of Capital Stock and Depreciation in the Production Function Framework

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Abstract

We propose a Bayesian approach to estimating productive capital stocks and depreciation rates within the production function framework, using annual data on output, employment and investment only. Productive capital stock is a concept related to the input of capital services to production, in contrast to the more common net capital stock estimates, representing market value of fixed assets. We formulate a full Bayesian model and employ it in a series of illustrative empirical examples. We find that parameters of our model, from which the time-path of capital is derived, are weakly identified with the data at hand. Nevertheless, estimation is feasible with the use of prior information on the production function parameters and the characteristics of productivity growth. We show how precision of the estimates can be improved by augmenting the model with an equation for the rate of return.

Keywords: productive capital stock, depreciation rate, aggregate production function, Bayesian analysis

JEL Classification: C51, D24, E01, E22, E23

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1 Introduction

Studies of economic growth, as well as numerous empirical and applied economic models, rely on estimates of fixed capital stock and its depreciation (aggregate or industry-level). Appropriate measurement of these categories is a prerequisite to adequate assessment of, among other things, technical change and the impact of investment on production capacity. However, such measurement is subject to various conceptual and practical difficulties. As a result, it is often the case that either the available data are not best suited for productivity analysis, or the data are not available at all, particularly for detailed industry disaggregation used in some models. It is also a rather standard situation that the depreciation rate is not provided explicitly, along with the capital stock assessment. Our aim is to propose a practical econometric approach to estimation of capital stocks and depreciation rates that could be used by modelers in applied work even with limited data, as an alternative to relying on perhaps more arbitrary, *ad hoc* assumptions or questionable external estimates.

Measurement of capital stock and depreciation may relate to two distinct functions of capital – "*storage of wealth and a source of capital services in production*" (OECD, 2009, p. 11). The wealth aspect is captured by the net capital stock (net fixed asstes), representing the current market value of fixed assets. Accordingly, depreciation concept used in derivation of net capital stock relates to the decrease in market value of the assets during an accounting period, e.g., a year (Lequiller and Blades, 2014, p. 249–250).

A related measure is the gross capital stock, equal to accumulated acquisitions of fixed assets (investment) minus their withdrawals. In this case, depreciation concept relates to retirement of assets. Neither net nor gross capital stock is, in general, an appropriate representation of production capacity in a single year, which is the perspective taken by the production function. The actual contribution of capital to production is conceptualized in the measure of capital services. The flow of capital services is considered proportional to some kind of stock category which we shall call, following (OECD, 2009, p. 59–62), *productive capital stock* (although Oulton and Srinivasan, 2003, suggest that this category is better named as *volume index of capital services* – VICS).

In most cases, the data in public statistics, when available, concern net or gross capital stock (fixed assets). The latest version of the System of National Accounts, SNA 2008, recognizes the importance of measurement of capital services (and related capital stocks), by including a chapter devoted to that problem (SNA 2008, Chapter 20; see also Jorgenson and Schreyer 2013 for a discussion of productivity measurement within this new conceptual framework). Still, relevant estimates are not commonly provided by official sources. Data coverage (by country, industry etc.) from other sources, such as EU KLEMS (van Ark and Jäger, 2017), is usually limited. Worth noting, it is argued that "*rather than introducing a new concept into the SNA, capital services can, in theory, be identified within the existing accounts*" by "*improvements* *in the estimates of consumption of fixed capital (...) and of the values of capital stock*" (SNA 2008, p. 416).

The accounting perspective on capital measurement, and the related concepts, are discussed in detail Section 2.1. Another perspective, featured in the literature, is the econometric one, treating productive capital stock as an unobserved category. In particular, under the so called production function approach (laid down in Section 2.2), capital input is estimated (based, i.a., on data concerning past investments) simultaneously with the production function parameters. Further developments in this line – reviewed in Section 2.3 – involve framing capital utilization and withdrawals as a part of optimization problems underlying dynamic general equilibrium models.

Our paper draws upon the basic production function approach. Its contribution to the previous studies is in three areas. *First*, we treat the problem Bayesianly. One benefit is that the Bayesian approach allows to formulate prior knowledge explicitly and include it in the estimation process. This is particularly important in a situation in which informativeness of data is limited. Even if the estimation outcomes only modify the prior knowledge moderately, it might still be considered an improvement over the practice of using ad hoc assumptions given lack of data on capital stocks and depreciation rates. Another possibility is, for example, to use available net capital stock data as an initial proxy for productive capital stock characteristics, and then let our estimation procedure modify such prior assumptions in order to get a capital measure better suited for productivity analysis. In any case, the Bayesian approach provides a full picture of uncertainty of the estimates, illustrating how prior assumptions have been revised after confronting the data (conditional on the model under study). We propose a specification of full Bayesian model and estimate its variants using Markov-Chain Monte-Carlo (MCMC) techniques. *Second*, we augment the production function model with additional equation, relating observed capital income with the unknown stock, which improves precision of the estimates. *Third*, we assume a fairly flexible formulation of technical progress, allowing for stochastic shocks to total factor productivity.

The focus of this work is on methodology, whereas the presented empirical application for Poland should be treated as illustrative. In particular, we examine the feasibility of inference on capital stock and depreciation from relatively short time series of aggregate data on output, employment and investment (we use annual data for the years 1996–2017). From such a perspective, the assumption of fixed depreciation rate, treated as a proxy or average of time-varying rate, is a natural starting point. Also, capturing inter-industry differentials of capital characteristics, even approximately, is arguably the first need in many simulation studies based on inter-industry models, coming before theoretical advances. As we shall see in subsequent sections, estimation is already challenging for a fixed depreciation rate, due to weak informativeness of data. Therefore we treat introduction of time-varying or endogenous depreciation rate as topics for further research, beyond the scope of this paper.

2 Literature overview

2.1 Basic concepts of capital measurement

Extensive literature exists on concepts and methods of capital stock measurement. The topic has been covered comprehensively in the OECD manual titled *Measuring Capital* (OECD, 2009). Other systematic overviews are provided by Diewert (1996) and Oulton and Srinivasan (2003). The expositions start with the case of a single (homogeneous) asset type. A fundamental relationship is formulated, defining asset price in a given year as a sum of discounted future incomes from asset rental over its entire remaining lifetime. This leads to distinguishing between asset price, i.e., market value of an asset for resale, and rental prices, i.e., return from employing the asset in production in a given year. From this follows the distinction between asset depreciation and decay ("wear and tear"). Depreciation is defined as a decrease in asset value due to ageing which exhausts asset's remaining lifetime and efficiency – note that even when asset's efficiency is constant throughout entire lifetime (light bulb being the classic example), it still depreciates due to exhaustion. The term decay, on the other hand, relates directly to efficiency loss due to ageing. Accordingly, assets are characterized by age-price and age-efficiency profiles, only the latter of which corresponds directly with the contribution to production. These two profiles coincide only in the case of geometric decay pattern, i.e., when asset efficiency deteriorates at a constant rate (*Measuring Capital*, p. 97). Consequently, under geometric pattern of efficiency deterioration depreciation rate is equal to the rate of decay. We shall note at this point that most of the economics literature – unlike many of the works dealing directly with capital measurement problems – uses the term *depreciation* in a broader meaning. The actual interpretation then depends on the context of its use (e.g., it might also indicate decay, obsolescence, retirement etc.). In this paper we also adopt such a broader understanding of depreciation, leaning to the terminology traditionally used in economic modelling of the production process.

One of key topics of the theory of capital measurement is the problem of aggregation. Capital stock at either plant, industry or economy level is an aggregate of assets across vintages and types. Different weights of individual assets are used, dependent on whether the resulting aggregate stock is meant to represent wealth or input to production. To obtain wealth capital stock, asset prices are used as weights, while productive capital stock is based on weighting by rental prices. Notably, short lived assets are characterized by relatively high ratio of rentals to asset price. Therefore, they have a higher weight in productive capital stock than in wealth capital stock. For this reason aggregate net capital stock will in general evolve at a different rate than aggregate productive stock (*Measuring Capital*, p. 61). More specifically, if the aggregate is composed of individual assets characterized by different geometric decay rates, the aggregate age-efficiency pattern will not be geometric – with no new investment the stocks of short-lived assets will decrease more quickly, thus changing the shares of different asset types in the aggregate stock.

Estimation of capital stock is not trivial even for relatively homogeneous assets, based on plant level data. One reason is that asset prices of used assets, as well as rental prices cannot always be observed, since fixed assets are often used by their owners. Hulten and Wykoff (1981) argue that market prices might therefore not be representative of the value of non-marketed assets, as owners may prefer to keep assets of higher quality and sell only the assets of inferior quality (the so called lemons problem). This poses a challenge to econometric estimation of, e.g., depreciation rates, due to censored sample bias. Also, as Hulten (1991, p. 124) notes, age-efficiency profiles are rarely observed directly, and so they need to be determined using indirect methods. For example, Coen (1975) empirically tests plausibility of alternative schemes of capital efficiency deterioration, by looking at the fit of capital demand model. Likewise, Hulten and Wykoff (1981) perform econometric estimation of depreciation rates for various asset classes. Another approach, demonstrated, e.g., by Harris (2005), who derives productive capital stock in UK manufacturing not relying on econometric techniques, is to carefully review the source data and consider the consequences of alternative assumptions (e.g., the treatment of plant closures) underlying estimation procedure. Other authors highlight specific issues in capital measurement. Beaulieu and Mattey (1998) point to the workweek of capital as a factor influencing the flow of capital services, and present relevant estimates based on microdata. Oulton and Srinivasan (2003) discuss measuring depreciation related to obsolescence, considering the case of computers and software. Obsolescence means that although an asset does not decay physically, it is discarded. Whereas Cairns (2006) analyzes how capital measurement is affected by the problem of sunk capital. A lasting debate in the literature concerns the question whether geometric depreciation – a widely used assumption – is actually adequate. A thorough literature review and in-depth discussion of that issue is provided by Bitros (2010). One way to look at that problem is to treat the geometric pattern as a convenient approximation of more general convex age-efficiency or age-price profiles. It can be demonstrated that combining linear pattern of efficiency loss for individual assets with retirement profiles, characterized by probability distribution of asset retirement at each age, may well result in a convex pattern of productive stock deterioration (*Measuring Capital*, p. 42). Hulten and Wykoff (1981) show that geometric pattern is at least more plausible than other simple alternatives. While a constant rate may usefully approximate average depreciation over a longer period, it is argued that year-to-year rates of depreciation and retirement vary in response to economic forces (Bitros, 2010). We shall return to this point further in the section concerned with endogenous depreciation and capital utilization.

Abadir and Talmain (2001) propose a method of estimating capital stocks and time-varying depreciation rates from the series of gross and net investment (the difference between the two being defined by authors as depreciation charge – or consumption of fixed capital, in national accounts terms). The two investment series are not sufficient to reconstruct capital stocks and depreciation rates in a deterministic way. It would be possible, though, if only depreciation rate or capital stock was known for one period. The authors note that such an "initial condition" could be provided approximately if one could presume that depreciation rate is similar for two consecutive periods. In this line of thinking, they provide an objective criterion of identifying such a data point which leads to minimizing estimation error. The procedure proposed by Abadir and Talmain (2001) helps solve a practical problem of estimating capital stocks and time-varying depreciation rates in a specific data situation, defined by availability of gross and net investment series. However, this does not necessarily address the question of adequately measuring capital stock for productivity analysis, given that consumption of fixed capital, constituting the difference between gross and net investment, is still subject to conventionalities. Conceptually, the resulting capital stock measure leans towards the net capital stock, discussed above.

As a final point here, let us note that measurement of stocks and depreciation may relate to R&D capital, as well as physical capital. In this case, data availability is more restricted. One example of direct measurement is the study by de Rassenfosse and Jaffe (2018), who estimate R&D depreciation based on observed revenue streams from innovations.

2.2 Production function approach

Direct application of the concepts reviewed in the previous section require access to detailed micro data, including asset prices or rentals, allowing to compile aggregate series in a bottom-up fashion. The production function approach aims at identifying capital stocks in a more restricted data situation. It involves estimating a production function in which capital stock is treated as an unobserved (latent) variable, being a function of past investment, initial capital stock and depreciation rate(s). Initial capital stock and depreciation rate(s) are estimated along with the regular production function parameters. Some works in this line of research have been reviewed by Diewert (1996, p. 26–27). However, that approach did not feature particularly prominently in the literature.

It was adopted by Prucha (1995) who explains how to deal with the estimation in standard econometric packages, given that the number of explanatory variables increases with every observation (since capital stock in time t is a function of investment in the periods $1, \dots, t-1$). Nadiri and Prucha (1996) extend the approach to the estimation of R&D capital stock and depreciation, along with the physical capital characteristics. Hernández and Mauleón (2005) allow for variable depreciation rates, by regressing them on some explanatory variables. Diewert (1996, p. 26) considers a more general specification, with period-specific depreciation rates, indicating that this assumption may be restricted in some a priori fashion. Doms (1996) presents a particularly detailed empirical study based on the production function approach, using a rich sample of plant level data. The author focuses on producers using a common technology of steel production, representing it using the translog production function, in which he inserts "*a parametrized investment stream* *for a capital variable*" (Doms, 1996, p. 89). The model includes observed capacity utilization as an explanatory variable which, among other things, plays the role of "*transforming the measure of capital stock into a proxy for capital service*" (Doms, 1996, p. 84).

Although not explicitly a production function analysis, a similar general idea underlies an earlier work by Pakes and Griliches (1984), who regress operating profits on a distributed lag of past investment, in order to identify depreciation pattern. Their paper is primarily concerned with developing econometric methodology for estimation of distributed lags models from panel data – the mentioned application serves as an illustrative example. In turn, Linnemann (2016) estimates capital utilization rate, inferring its "*empirical characteristics (...) from assumptions concerning the production function*". Even though the cited study refers to a different problem than the one considered in our paper, that latter statement illustrates well the essence of the production function approach.

2.3 Endogenous depreciation in general equilibrium models

Depreciation is often, or even typically, treated as an exogenous characteristic of capital, related to technical properties of the assets alone. Notwithstanding, in the field of dynamic equilibrium modelling a broader view is sometimes taken, by recognizing that observed capital depreciation and utilization should in fact be treated as a result of the optimizing decisions by producers. When new, more efficient capital becomes available or maintenance cost becomes high, producers might consider quicker replacement of obsolete assets. Note that, in principle, such an approach does not neglect existence of age-efficiency schedules of individual assets related to physical properties of the equipment etc. (intensity of capital use may change asset's lifetime and efficiency, though). Rather, producer decisions may lead to earlier retirement of assets or making them idle.

A thorough review of endogenous depreciation theory, accompanied by a critique of the so called proportionality theorem, which underlies the assumption of a constant rate of depreciation, is given by Bitros (2010). An early example of estimation of the model of capital utilization and depreciation dependent on profit maximization is the work by Epstein and Denny (1980). Boucekkine and Ruiz-Tamarit (2003) formulate a theoretical model, in which depreciation rate is a function of the ratio of maintenance cost to capital, and capital utilization rate. They find, among other things, that maintenance and investment expenditure are complementary, i.e. they move together in response to changes in either unit maintenance cost or price of new capital, or in response to technical improvement. Chatterjee (2005) summarizes some findings from empirical literature, such as that (i) capital is typically utilized less than fully, (ii) capital utilization rate and depreciation rate are positively correlated with per capita income, (iii) capital utilization rates vary significantly across industries, and (iv) convergence rates of per capita income between countries are considerably smaller than implied by standard neoclassical growth models. Based on one-sector theoretical model the author shows that treating utilization as an optimization variable (also, with depreciation sensitive to utilization) helps reconcile the theory with the evidence. Further theoretical developments are provided by Boucekkine et al. (2009). In their formulation, the authors distinguish between neutral and investment-specific technical progress (embodied – increasing productivity of new investment, and disembodied reducing the cost of new investment), as well as between use-related depreciation and obsolescence of capital. One of the findings is that accelerated investment-specific technical progress leads to an increase in use-related depreciation and the scrapping rate. From this follows a faster economic depreciation (reduction of the value of assets), consistent with observations related to computer equipment and software. The model also helps explain some additional stylized facts on capital and depreciation. The positive link between capital-embodied (investment-specific) technical progress and depreciation (related to obsolescence) is confirmed empirically by Barañano and Romero-Ávila (2015). Escribá-Pérez et al. (2018) apply a model with endogenous depreciation rate, resulting from dynamic optimization, in order to measure aggregate capital stock for the non-financial business sector in Spain. The outcome, termed 'economic' capital stock, is compared with 'statistical' capital stock reconstructed from official data, showing marked differences particularly during the years of the recent financial crisis. Accordingly, 'economic' depreciation rate fluctuates around 'statistical' rate.

When applied empirically, dynamic general equilibrium models reviewed in this section have a common point with the production function approach. Namely, they allow for inference on capital stock and depreciation as latent variables, linked to observable variables through theoretical formulations (such as production function or factor demand equations resulting from optimization). Still, this literature takes a broader perspective and is primarily concerned with theoretical extensions that allow to better explain the phenomena observed in the data, rather than measurement (of capital stock and depreciation) per se (the exception being the work by Escribá-Pérez et al., 2018). Notably, the models considered in this line of research are typically highly aggregate (comprising one or two sectors), with continuous time.

3 The basic model and estimation results

3.1 The economic model and statistical specification

Consider Cobb-Douglas production function, with output (Y) produced from capital (K) and labour (L):

$$Y_t = A_t K_t^{\alpha} L_t^{\beta} \qquad (t = 1, 2, ..., T). \tag{1}$$

With elasticities $\alpha, \beta > 0$, the specification allows for non-constant returns to scale. We assume growth rates of total factor productivity (A) to be stochastic, independent and identically normally distributed:

$$\Delta \log A_t \sim ii\mathrm{N}\left(\mu, \sigma^2\right) \tag{2}$$

with unknown mean (μ) and standard deviation (σ). Changes in A capture technological improvements, variations in production efficiency and all other shocks. Thus, no additional error term applies. However, in order to be fully realistic, changes in A should be more directly related to business cycle. This would require generalising (2) to some non-trivial stochastic process through, e.g., an autoregressive specification. Since our Bayesian approach to capital stock estimation is now presented within the simplest possible framework, such generalisation is not considered here and left for future research.
Capital stock is accumulated according to:

$$K_t = K_{t-1} \cdot \rho + I_{t-1},$$

which yields, after a series of substitutions:

$$K_t = \rho^t \cdot K_0 + \sum_{i=1}^{t} \rho^{i-1} \cdot I_{t-i}, \tag{3}$$

where I is investment (gross fixed capital formation), K_0 is the initial capital stock, and ρ is the depreciation coefficient (defined as $\rho \equiv 1-\delta$, where δ is the rate of capital depreciation). Recall that in the context of our study the term "depreciation" relates to decay (efficiency deterioration), while capital refers to productive capital stock.
In the model composed of equations (1)–(3), the core parameters (to be estimated) are: α, β, $\rho \in \langle 0, 1\rangle$, μ, $\sigma > 0$, $K_0 \geq 0$, whereas K_t $(t = 1, 2, ..., T)$ are functions of core parameters (K_0, ρ) and data on investment.
In terms of $y_t = \log Y_t$ and $a_t = \log A_t$, formulas (1) and (2) can be written as:

$$y_t = \alpha \log K_t + \beta \log L_t + a_t \qquad (t = 1, 2, ..., T), \tag{4}$$

$$a_t = a_{t-1} + \mu + \varepsilon_t, \qquad \varepsilon_t \sim ii\mathrm{N}\left(0, \sigma^2\right). \tag{5}$$

Elimination of latent variables a_t is achieved by inserting $a_t = y_t - (\alpha \log K_t + \beta \log L_t)$ and its lagged value into both sides of (5). This leads to the following normal nonlinear regression equation for production growth:

$$\Delta y_t = \alpha \Delta \log K_t + \beta \Delta \log L_t + \mu + \varepsilon_t, \qquad \varepsilon_t \sim ii\mathrm{N}\left(0, \sigma^2\right), \tag{6}$$

where unobserved K_t is replaced by the right hand side of (3). This simple univariate specification leads to the conditional normal distribution of y_t (given L_t and past data) with mean $m_t = y_{t-1} + \alpha g(I_0, ...I_{t-1}; K_0, \rho) + \beta \Delta \log L_t + \mu$ and variance σ^2, where $g(I_0, ...I_{t-1}; K_0, \rho)$ stands for $\Delta \log K_t$. The likelihood function is based on the product of the T densities of these normal conditional distributions, each denoted as

$f_N(y_t|m_t, \sigma^2)$, which depend on six parameters: α, β, μ, ρ, K_0 and σ^2. Thus, the likelihood function is defined on the six-dimensional parameter space. Linearity of the conditional mean of y_t (i.e., m_t) in α, β and μ (given ρ and K_0) means that in (6) we have the partly linear regression model, Bayesianly treated in Osiewalski (1988). Our specification is, to some extent, in line with Osiewalski, Wróblewska, and Makieła (2020). In fact, the simple model (1)–(2) gives economic interpretation to the best models in Osiewalski, Wróblewska, and Makieła (2020), where co-integration specifications (based on the aggregate production function concept) fail empirical comparisons with simple vector auto-regression (VAR) structures for first differences, which describe the three aggregates (i.e., logs of output, capital and labour) by three stochastic trends. Our formulation here, with total factor productivity driven by a simple stochastic trend process, reconciles the aggregate production function framework with seemingly a-theoretical VAR structures that assume three stochastic trends for three observable aggregates. Of course, our framework and purpose of the analysis are different than in Osiewalski, Wróblewska, and Makieła (2020); here we do not treat K_t as an observable process and we stay within uni-variate framework, assuming exogenous labour and investment. This simplifying assumption, not essential for our idea of estimation of the K_t path, could be relaxed in some future research.

3.2 Prior distributions of the parameters

The parameter space of our partly linear regression model (6), with K_t defined by (3), is six-dimensional. We adopt the following weakly informative prior distributions for (groups of) model parameters, which are assumed independent a priori:

i) (α, β) are jointly normal with means 0.5, variances 0.0225 and covariance -0.018 (so $\alpha + \beta \sim \mathrm{N}(1, 0.009)$),

ii) $\mu \sim \mathrm{N}(0, 0.02^2)$,

iii) $\tau = \sigma^{-2}$ is gamma(100,0.01), with mean 10000 and standard deviation 1000,

iv) ρ is beta with mean 0.92 and standard deviation 0.03,

v) K_0 is exponential with mean K_m.

Prior mean K_m of initial capital stock (K_0) is determined as five times the observed capital income in year 0, implying an assumed gross rate of return of 20%.
Sensitivity analysis with respect to prior assumptions is provided in Appendix A.

3.3 The Bayesian model and simulation from the posterior distribition

The full Bayesian model for our problem is composed of (i) the T conditional sampling densities $f_N(y_t|m_t, \sigma^2)$, defined below (6), (ii) priors for six model parameters, listed in Section 3.2. Given the available data, inference on the model parameters is based on their 6-dimensional posterior distribution, characterised by the density function proportional to the product of the joint prior density and the likelihood function:

$$p(\theta|y) \propto p(\theta) \prod_{t=1}^{T} f_N(y_t|m_t, \sigma^2), \tag{7}$$

where $y = (y_1, ..., y_T)'$ and $p(\theta)$ is the joint prior density of $\theta = (\alpha, \beta, \mu, \rho, K_0, \sigma^2)'$. Obtaining samples of six basic parameters from their posterior distribution is easy within any Monte Carlo (MC) simulation scheme. Markov-Chain Monte-Carlo (MCMC) sampling from the posterior distribution was performed using Stan modeling language (Stan Development Team, 2021). Model code is available on request. Each result discussed in the paper is based on $N = 200000$ draws (derived from mixing four chains, each composed of 100000 draws, initial 50% of which were discarded as warm-up samples). Even much smaller samples led to virtually the same posterior characteristics, confirming convergence of the MCMC chain.
However, our main goal is to make inferences on the path of K_t $(t = 1, 2, ..., T)$, the unobserved variables described by the capital accumulation identity (3). Also, we would like to infer about the path of A_t $(t = 1, 2, ..., T)$. Note that K_t is a transformation of two core parameters (K_0, ρ) and the investment data, and A_t is a function of the model parameters and the available data: $A_t = \exp[y_t - (\alpha \log K_t + \beta \log L_t)]$. Thus, for each MCMC draw from the posterior distribution, we obtain the corresponding path of K_t first, and then, using it, we compute the corresponding path of A_t. So finally, after obtaining N draws from the posterior distribution, we also have N bundles of simulated paths of K_t and A_t, which enable us to calculate their posterior characteristics.

3.4 Data

Estimation results presented in this paper are based on annual time series from Eurostat, including constant price output (gross value added) and investment (gross fixed capital formation), as well as employment measured in hours worked – at aggregate and sector (NACE Rev. 2, 1-digit industry) level. Where available, we also use data on gross fixed assets and net fixed assets (in constant prices) to compare against the estimated path of capital stocks. However, those two series are not used in the estimation or prior formulation. Since estimation results in this paper are meant to be illustrative of the method, the presentation is limited to a few selected cases. We show the results for Poland, but also comment on the outcomes for other countries. The data for Poland feature 22 observations, spanning the years 1996–2017.

3.5 Results: Total economy

Consider first the estimation results for the total economy. Table 1 reports prior and posterior means and standard deviations of the six core parameters; in the case of posterior, they are computed from samples from the marginal posterior distributions. Whereas Figure 1 characterizes those distributions in terms of histograms (light gray for prior, and dark gray for posterior distributions). The histograms have been scaled in such a way that the areas of posterior density plots are comparable across the parameters. As a consequence, prior density plots have been truncated in a number of cases.

Apart from the core parameters, Figure 1 illustrates the time-path of mean estimated capital stocks (solid line), with the ribbon representing ± one standard deviation of the posterior distribution. It also presents gross (dashed line) and net (dotted line) fixed assets from Eurostat data.

Table 1: Summary of prior and posterior distributions: Total economy

	Prior		Posterior	
	Mean	St. dev.	Mean	St. dev.
μ	0.000	0.020	0.012	0.006
σ	0.010	0.0005	0.010	0.0005
α	0.500	0.150	0.391	0.098
β	0.500	0.150	0.545	0.081
$\alpha+\beta$	1.000	0.095	0.937	0.090
ρ	0.920	0.031	0.949	0.021
K_0	1910	1910	1052	303

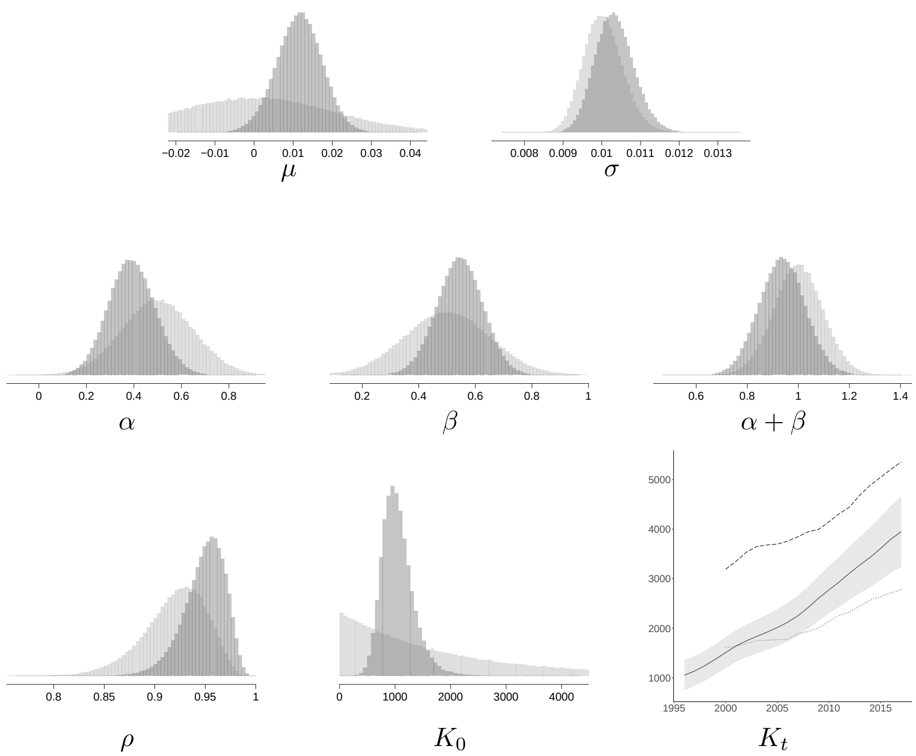
Inspection of the results leads to the following observations:

i) The informativeness of data is limited – they allow to revise and narrow down prior distributions only moderately (although it depends on the parameter in question).

ii) Output elasticity with respect to capital (posterior mean 0.39) is lower than the elasticity with respect to labour (posterior mean 0.55). The former is estimated with somewhat smaller precision, given that the capital stock is unobserved. Constant returns to scale are not rejected by the results.

iii) Posterior mean rate of total factor productivity growth is rather low, equal to 1.2%. This corresponds with a rather high depreciation coefficient, and thus a fairly low depreciation rate, $0.05 = 1 - 0.95$. Consequently, capital

Figure 1: Prior (light gray) and posterior (dark gray) distributions of model parameters: Total economy



stock grows faster than shown by conventional capital measures (net and gross fixed assets). Hence, according to estimation results, investment and capital embody more technical change than under conventional capital measurements, thus diminishing the estimated rate of disembodied productivity growth. The "tradeoff" between ρ and μ can be seen in Figure 2 which shows negative correlation of these two quantities in the posterior distribution. Negative correlation in the posterior distribution can also be seen between α and μ (Figure 3), that is, the higher the estimated capital elasticity, the lower the estimated productivity growth rate.

iv) Mean posterior capital time path fits between official net and gross fixed assets estimates, except in the very first years.

We could not find – in the literature or public data sources – estimates of productive capital stocks or capital services for Poland that could be used for a systematic

Figure 2: Posterior distribution: ρ and μ

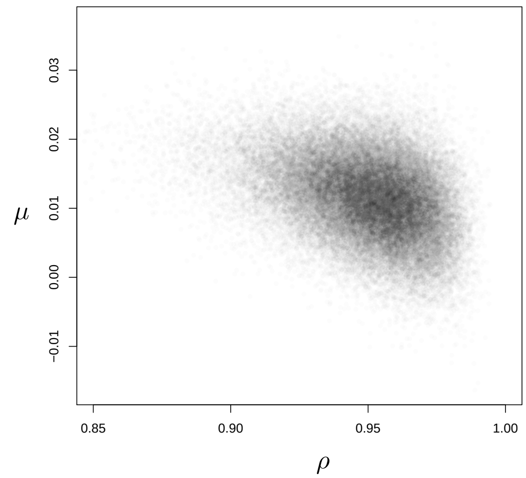
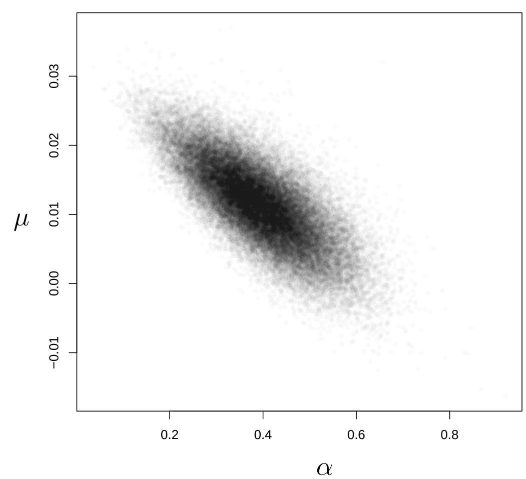


Figure 3: Posterior distribution: α and μ



comparison with our results. In particular, EU KLEMS database (van Ark and Jäger, 2017) does not include such estimates for Poland, due to missing data. Gradzewicz et al. (2018) have estimated the volume of capital services for Poland (total economy) as a part of a growth accounting study. In that case, capital services are derived by aggregating and accumulating investment in different asset types, using capital rental rates as weights, assuming asset-specific depreciation rates. The authors provide growth rates of capital services for the years 1997–2013. Considering that capital services can be interpreted as a flow proportional to the productive capital stock, we can compare the results from Gradzewicz et al. (2018) with posterior mean growth rates of productive capital stocks. In most years our estimates are higher by 1–2 percentage points than the ones from the cited work. In the years of most intensive investment, this difference increases to 2–3 percentage points. Otherwise, the pattern of changes in capital growth rates in time is similar between the two studies, because of the reliance on analogous investment data (although we only used aggregate series) and the assumption of geometric depreciation scheme. We conclude that our estimation yields smaller depreciation rates than the more conventional (non-estimated) values, and/or lower initial productive capital stock (which we could not compare directly, as it was not reported in the cited study).

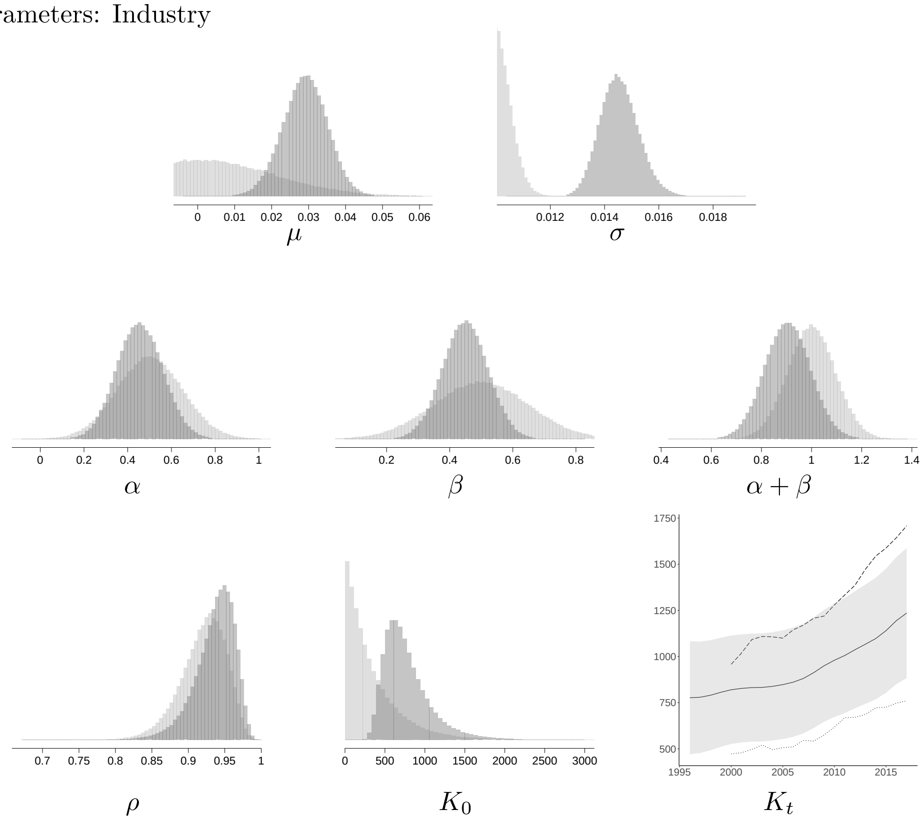
3.6 Results: Industrial production

Estimation results for the industrial sector, comprising mining, manufacturing, energy, and utilities, are shown in Table 2 and Figure 4. Uncertainty of the estimated capital path is distinctively higher than in the case of the economy as a whole. This is also the case of the depreciation coefficient, although to a lesser extent. Otherwise the posterior uncertainty is similar to the total economy case. The estimated mean posterior rate of total factor productivity growth for industry is 2.9%, compared with 1.2% for the total economy. This result perhaps explains the relatively high variance of posterior capital stocks – we could hypothesize that substantial contribution from exogenous technical change to output growth makes it difficult to identify the contributions of capital increments.

Table 2: Summary of prior and posterior distributions: Industry

	Prior		Posterior	
	Mean	St. dev.	Mean	St. dev.
μ	0.000	0.020	0.029	0.006
σ	0.010	0.0005	0.015	0.0007
α	0.500	0.150	0.455	0.106
β	0.500	0.150	0.449	0.072
$\alpha+\beta$	1.000	0.095	0.904	0.091
ρ	0.920	0.031	0.936	0.028
K_0	351	351	775	305

Figure 4: Prior (light gray) and posterior (dark grey) distributions of model parameters: Industry



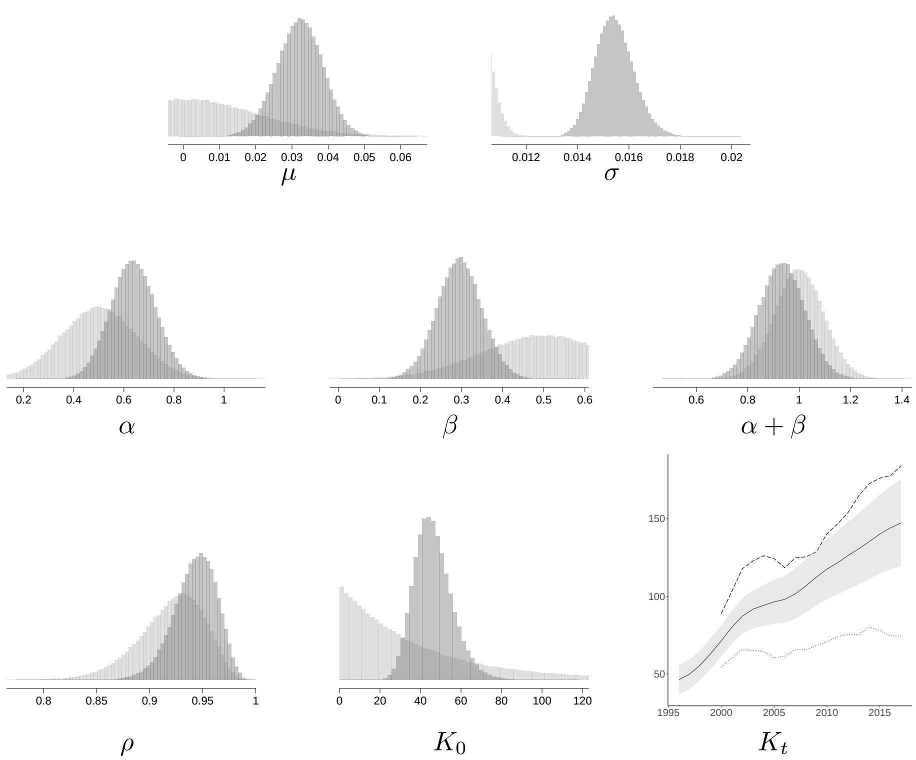
3.7 Results: Information and Communication

Three results stand out in the case of the Information and Communication sector (see Table 3 and Figure 5). First, elasticity of capital is relatively high (with posterior mean equal to 0.64), and elasticity of labour is low (0.30). Second, the elasticities (labour elasticity in particular) are estimated with relatively high precision, compared to industry and total economy. Third, precision of the posterior capital path is relatively high, especially in the first part of the data sample, as well as it exhibits structural change – initial rapid capital growth is followed by marked slowdown. The latter reflects the fact that investment-to-output ratio was initially high, and then it decreased permanently after the year 2001. Such a shift was perhaps helpful for identification of the capital stock – in this case the rate of exogenous technical change, similar to the one found in the industry sector, did not overwhelm the contribution of capital accumulation to production changes.

Table 3: Summary of prior and posterior distributions: Information and Communication

	Prior		Posterior	
	Mean	St. dev.	Mean	St. dev.
μ	0.000	0.020	0.032	0.006
σ	0.010	0.0005	0.015	0.0007
α	0.500	0.150	0.638	0.091
β	0.500	0.150	0.295	0.055
$\alpha+\beta$	1.000	0.095	0.933	0.090
ρ	0.920	0.031	0.943	0.020
K_0	39	39	46	10

Figure 5: Prior (light gray) and posterior (dark gray) distributions of model parameters: Information and Communication



3.8 Results: Real Estate

The real estate sector encompasses, among other things, the services of renting real estate, including rents of owner-occupied dwellings. Therefore, the capital stock in this sector mostly consists of properties. Worth to note, investment in this sector covers, i.a., purchases of property by private households.
Interestingly, in the light of posterior distributions of model parameters, the real estate sector is likely characterized by decreasing returns to scale (see Table 4 and Figure 6). This result can be straightforwardly explained by the omission of land in the production function. Mean posterior elasticity of output with respect to labour is only about 0.08, whereas the mean posterior elasticity with respect to capital equals 0.60 – the latter being subject to substantial uncertainty, though. The posterior variance of the initial capital stock is rather high too. The estimated depreciation rate is lower than in the other individual sectors considered (industry, and information and communication), with posterior mean of 3.7%, consistent with the prevailing type of capital (properties). As a result, the path of productive capital stock is increasing relatively fast – faster than the official net and gross fixed assets estimates. Worth noting, the results show negative rate of total factor productivity growth (posterior mean equal to -1.4%).

Table 4: Summary of prior and posterior distributions: Real Estate

	Prior		Posterior	
	Mean	St. dev.	Mean	St. dev.
μ	0.000	0.020	-0.014	0.008
σ	0.010	0.0005	0.015	0.0008
α	0.500	0.150	0.596	0.129
β	0.500	0.150	0.084	0.047
$\alpha+\beta$	1.000	0.095	0.680	0.114
ρ	0.920	0.031	0.963	0.017
K_0	200	200	197	120

4 The use of return rate on capital – Model 2

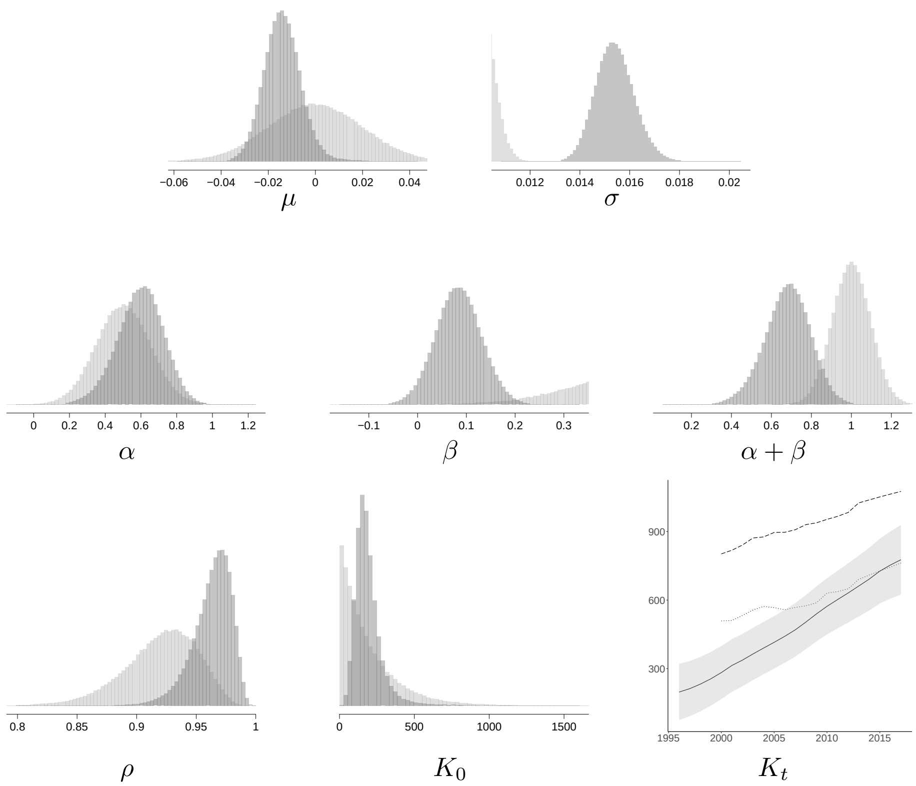
In this section we take up the problem of little informativeness of data, and extend the model with an additional observed variable and a related equation. Consider the following relationship, augmenting the basic model (Model 1):

$$C_t \sim ii\mathrm{N}\left(\phi K_t, (\omega K_t)^2\right), \tag{8}$$

i.e.:

$$C_t/K_t \sim ii\mathrm{N}\left(\phi, \omega^2\right), \tag{9}$$

Figure 6: Prior (light gray) and posterior (dark gray) distributions of model parameters: Real Estate



where C_t is capital income (in constant prices), and C_t/K_t is gross rate of return on capital. Therefore, rate of return is stochastic, normally distributed with mean ϕ and variance ω^2. Capital income, C_t is approximated by gross operating surplus, available in the national accounts data. The rationale behind this specification is that, short-run fluctuations aside, capital income is proportional to the (unobserved) productive capital stock. Here we do not consider the possibility of trends in the rate of return, which seems a natural further model extension. The following prior distributions have been adopted:

i) $\phi \sim \mathrm{N}(0, 0.2^2)$,

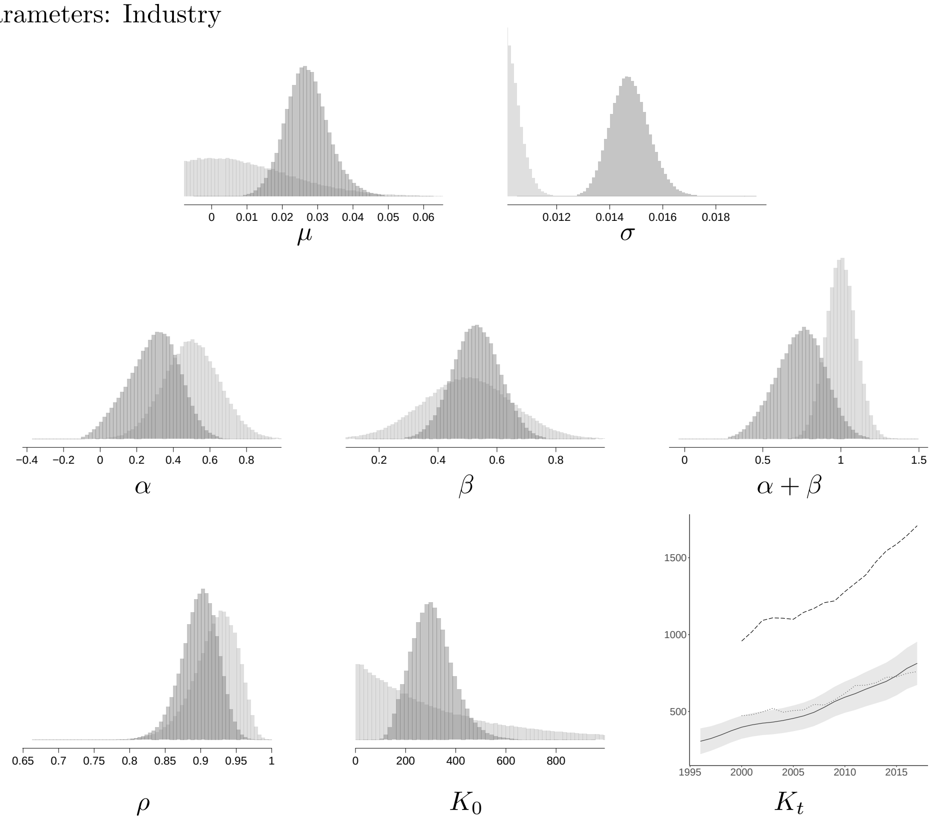
ii) $\psi = \omega^{-2}$ is gamma(5, 0.05), with mean 100 and variance 2000.

Consider again the Industry sector. Table 5 and Figure 7 show estimation results obtained using the extended model. The new observational equation significantly reduces posterior standard deviation of the initial capital stock K_0 (from 305 to

Table 5: Summary of prior and posterior distributions: Industry

	Prior		Posterior	
	Mean	St. dev.	Mean	St. dev.
μ	0.000	0.020	0.027	0.006
σ	0.010	0.0005	0.015	0.0007
α	0.500	0.150	0.291	0.138
β	0.500	0.150	0.530	0.079
$\alpha + \beta$	1.000	0.095	0.820	0.108
ρ	0.920	0.031	0.898	0.026
K_0	351	351	306	84
ϕ	0.000	0.200	0.260	0.048
ω	0.108	0.027	0.072	0.012

Figure 7: Prior (light gray) and posterior (dark gray) distributions of model parameters: Industry



84), and, accordingly, uncertainty of the capital path. At the same, time the level (posterior mean) of the initial capital stock is revised downwards considerably (from 775 to 306), whereas the (posterior mean) depreciation rate increases from 6.4% to 10.2%. In this case, the level of capital stock is in fact close to the net fixed assets found in the data, albeit it exhibits a slightly higher growth rate. Changes in elasticities are significant, with (posterior mean) capital elasticity reduced from 0.46 to 0.29, and labour elasticity increased from 0.45 to 0.53. On the contrary, TFP growth rate is very similar between the two models.

Narrowing the posterior distributions of capital stocks (as well as shifting their means) is certainly obtained here using a rather strong economic assumption of fixed long-run rate of return, also leading to a significant revision of production function elasticities. Nevertheless it illustrates the point that broadening the set of economic relationships in the model may, in principle, improve inference on the quantity of interest.

5 Discussion of outcomes for other countries and estimation problems

Apart from the results reported in this paper, we have also conducted a series of estimations for other EU Member States, based on the same data source and sectoral classification. A relatively frequent problem was non-convergence of the MCMC sampling process, undermining inference. This is related to the occurrence of the so called divergent transitions, "*that indicate the Hamiltonian Markov chain has encountered regions of high curvature in the target distribution which it cannot adequately explore*" (Betancourt, 2017). In our estimations divergences were often encountered for near-zero capital elasticities, α. Note that $\alpha = 0$ implies that capital has no effect on output, so it cannot be identified based on output variation. In most cases decreasing prior standard deviations of elasticities α and β (under bi-variate normal distribution) from 0.15 to 0.1 resolved the problem.

Interestingly, the above issue was more frequently found for the more developed EU economies, than for the emerging economies ("new" EU Member States, including Central and Eastern European economies; although it also appeared in, e.g., the agricultural sector in Poland). Two broad explanations can be hypothesized. First, emerging (catching-up) economies are likely characterized by higher capital growth and more variation in investment, thus leading to more informative data. Second, unlike for example Poland, "old" Member States recorded sharp output drops in 2009, due to global financial crisis. These drops were perhaps difficult to reconcile with the model which (at least explicitly) assumes full capacity utilization. Output decline can be accommodated by a decline in total factor productivity. However, since the model accumulates TFP changes in a random walk process, it does not account for the likely recovery after the period of capacity under-utilization.

The above points certainly require further exploration, and no binding conclusions can be drawn at this stage. Two broad directions include revising prior distributions to exclude economically implausible values, and revising the model to allow for more flexibility in adjusting to data.

6 Conclusions

We have proposed two variants of a Bayesian model, allowing to estimate unobserved capital stocks and depreciation rate from output, employment and investment data only, based on the production function approach. The exposition was complemented with several illustrative applications for Poland. The resulting estimates allude to the concept of productive capital stock, as opposed to, e.g., net and gross fixed assets, largely dependent on the underlying accounting conventions. The idea to estimate productive capital stocks by reference to the production function, has been proposed and employed in the literature, although definitely not as the mainstream approach to capital measurement. We add to that literature by framing the problem in the Bayesian methodology, by modeling technical change as a random walk process, as well as by extending the production function based model with a relationship involving observed returns to capital (as a source of supplementary information). The Bayesian approach allows to explicitly display uncertainty and correlations between parameters. It is also flexible in terms of formulation of the prior knowledge regarding the economic relationships under study.

Although the annual (sectoral and macro) series proved to be rather little informative, the proposed estimation approach does allow to obtain estimates of capital stocks (along with production function parameters) based on output, employment and investment data. Often, the resulting (posterior mean) path of capital stocks lies between net and gross fixed assets, available from official statistics, at least for sub-periods of the data sample. At the same time, in most cases the resulting capital growth rate is higher than the one of net or gross fixed assets (implying that our estimated depreciation rate is smaller than the one underlying official statistics). The results point to positive correlation, in the posterior distribution, of depreciation rate $(1 - \rho)$ and total factor productivity growth rate (μ). There is also negative correlation between the estimates of capital elasticity (α) and TFP growth rate (μ). Consequently, compared to production function estimates based on conventional capital stock measures, our estimated productive capital stock would either imply lower exogenous TFP growth rates (meaning that technical progress is to a larger extent embodied in investment and capital) or a lower capital elasticity. The key to distinguish between the contributions of exogenous productivity changes and latent capital increments to the output changes was the prior assumption of a small variance of productivity growth rates.

Uncertainty of capital estimates varied significantly across sectors, according to data informativeness. It was typically the average total factor productivity growth rate for which prior uncertainty was reduced most. Extending the model with an observational equation for capital income substantially reduced posterior uncertainty of capital and depreciation coefficient, although at the cost of a rather strong economic assumption of fixed average long-run capital rental rate.

We believe one area in which the proposed method could be particularly useful is the field of dynamic computable general equilibrium (CGE) modeling. CGE models typically rely on external estimates of parameters, such as production function elasticities or demand elasticities, while calibrating other parameters – such as share or scale parameters – to single period benchmark data. Such an "outsourcing" practice is justified by high cost or infeasibility (due to data or other constraints) of dedicated econometric estimation, in the otherwise complex, highly disaggregated simultaneous equation models, used primarily for simulation analyses of various policies' effects. However, representation of capital accumulation within the CGE models is perhaps an area, in which country and industry specific estimates are particularly desired. For example, the specification proposed by Dixon and Rimmer (2009, pp. 189–195) requires, *inter alia*, average growth rates of fixed capital stocks from past several years, characterizing individual industries. More basic specifications require at least capital stock estimates for the base year, and the depreciation rates by industry (see, e.g., van der Mensbrugghe, 2019, pp. 46–48 and 117–118). If adequate data are not available, a common proceeding is to calculate capital stock from base-year capital income, assuming an arbitrary rate of return (often uniform across industries); alternatively capital stock is calculated from base-year investment, assuming an arbitrary capital growth rate. From such a perspective, the proposed estimation approach is arguably a step forward. Moreover, as we have seen from the results, assessments of capital growth, depreciation, productivity growth etc., are interrelated. Therefore, industry-specific rates of technical change, consistent with capital stock estimates, can be used to formulate a more consistent baseline projection than otherwise, when assumptions are based on diffuse external sources.

More broadly, the above example illustrates a possibility of acquiring more empirical grounding in applied general equilibrium modeling by using the Bayesian methodology. It seems particularly well suited to that case, characterized by limited, low frequency, short series data, and a strong theoretical orientation, allowing to specify informative priors.

A number of further research routes and potential model generalizations emerge. First, the proposed models could be extended by introducing time-varying depreciation rates. One option is to model the depreciation coefficient ρ_t as independent beta latent variables, that is $\rho_t \sim ii\mathrm{Beta}\,(\alpha_\rho, \beta_\rho)$. Alternatively, a Gaussian AR(1) latent stochastic process can be applied for the logit of the depreciation coefficient, that is $\delta_t = \phi_0 + \phi_1 \cdot \delta_{t-1} + \zeta_t, \zeta_t \sim ii\mathrm{N}\,(0, \chi)$, where $\delta_t = \log[(1-\rho_t)/\rho_t]$. Second, as indicated earlier, variable capital utilization would allow to account for crises-related shocks. One option is to treat utilization rate as a latent variable, an alternative – to use survey data on capital utilization. Third, the capital income equation from the extended model considered in the paper could be generalized to an AR model. Furthermore, since capital rental rates in different sectors are likely interrelated (e.g., subject to common trends), panel estimation based on data for multiple sectors would likely improve inference. Fourth, a more general stochastic specification of TFP changes could be applied, along with more general production function forms. Fifth, the model should address the problem that investment is highly pro-cyclical, implying that large investment are often actually followed by a slowdown in output growth. This undermines identification of capital increments in the model which attempts to link output (capacity) changes to previous year's investment. Accordingly, one could consider enhancing model dynamics or distinguishing between short- and long-run effects or accounting for business cycle through capital utilization modeling – such that the model could fit the data more flexibly.

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Appendix A Sensitivity analysis

We consider three sensitivity analysis variants for the basic model (Model 1):

i) **TFP**: doubling of prior standard deviation of μ, and increasing prior standard deviation of σ roughly ten times.

ii) **Capital**: increasing prior standard deviation of ρ and K_0 by 50%. Since K_0 is exponentially distributed, increase in standard deviation also entails a corresponding increase in prior mean.

iii) **Elasticities**: doubling prior standard deviations of α and β, as well as truncating their (bi-variate normal) distribution at 0.05 to avoid non-identifiability of capital when $\alpha = 0$. Truncation results in standard deviations being actually only around 50% larger than in the base case, as well as prior means of the elasticities are shifted slightly, from 0.5 to 0.509.

Put otherwise, sensitivity scenarios assume an increase in prior uncertainty regarding technical change, initial capital stock and depreciation, and production function elasticities, respectively.
The magnitudes of changes in prior standard deviations are parameter-specific. They were chosen to conform, on the one hand, with economic plausibility - increasing standard deviations still further would essentially extend prior distributions over economically unreasonable values. On the other hand, our testing shows that allowing for even more prior uncertainty might, at least in some cases, lead to convergence problems, likely related to limited informativeness of data and weak identifiability of parameters (especially given the model structure with latent variables).
Tables A.1–A.4 show the results of sensitivity analysis for total economy, as well as industry, information and communication, and real estate sectors. The base variant represents the assumptions considered in the main part of the paper. Wherever prior assumptions differ compared to the base variant, this is highlighted with bold font.

TFP. Allowing for more prior uncertainty with respect to technical change characteristics – in particular, increasing prior standard deviation of σ, the dispersion of annual technical change rate, by a factor of nearly 11 − directly increases the posterior standard deviation of σ 4–9 times, depending on the sector. This further translates to posterior standard deviation of the initial capital stock, K_0, increased by $30 - 80\%$, except in the case of the industrial sector, in which posterior uncertainty was high already in the base case. Otherwise, the results of the TFP sensitivity analysis variant are mixed. For the total economy, the remaining results are hardly affected, with typically only slight increases in posterior standard deviations. In the case of the sectors considered, similarly most of the results are affected only in a rather limited way, incidentally however we observe a more substantial impact. For example, in the case of the information and communication sector the change in prior assumptions resulted in the increase of posterior mean of labour elasticity, β, from around 0.3 to around 0.4, and posterior standard deviation for that parameter doubled. A similar effect is found for the real estate sector. In this sensitivity variant, posterior mean depreciation coefficient is also lower by 1–2 p.p. than in the base case (except for the total economy). Overall, with limited data informativeness, prior technical change characteristics seem an important identifying factor for productive capital stock estimation, and less so for the other model parameters, although with some exceptions).

Capital. With the exception of the information and communication sector, the increase in prior standard deviation for the initial capital stock, K_0, translates to a roughly proportional (or even more than proportional – in the case of the real estate sector) in posterior standard deviation for that model parameter. At the same time, given that under exponential distribution standard deviation equals the expected value, the change in prior assumptions also tends to increase posterior mean of the initial capital stocks (although not as much as it increases the posterior dispersion). Accordingly, the obtained posterior mean depreciation coefficients tend to be higher than in the base case. At the same time, higher prior uncertainty of depreciation coefficient is hardly reflected in posterior results. To sum up, prior assumptions regarding initial capital stock are significant for its posterior evaluation. As mentioned above, the one exception is the information and communication sector, where – among the examples considered – precision of posterior capital stock was the highest (which we attributed to a regime shift in the data – a change in investment-to-output ratio which helped distinguish the impact of capital from the impact of technical change and employment on production.

Elasticities. The most striking result of the increase in prior uncertainty of the production function elasticities is the sharp drop in posterior mean of capital elasticity, α – in all but the information and communication sector case. We conclude that, unless data is informative enough to avoid this, stretching the distribution of α towards zero, allows technical change to take over a larger part of production growth explanation (mean posterior average technical change rates become higher). This effect is particularly strong in the real estate sector, and it hinders estimation of capital stocks. In the other sectors, as well as for total economy, the impact on the posterior distributions of the initial capital stock and depreciation coefficient are not substantial.

We conclude from sensitivity analysis, that the use of prior information rooted in economic theory is an essential ingredient of capital stock and depreciation rate estimation in the proposed framework. However, this comes as no surprise given little informativeness of data. We could not identify an universal pattern in which sensitivity to prior assumptions exhibits itself – rather, depending on a specific case (sector), an increase in prior uncertainty could affect one or another parameter significantly, leaving other outcomes relatively stable. A promising insight comes from the information and communication sector example – it suggest that more variability of (*inter alia*) investment in the data should lead to more robust results. Even in the worst case, though, we still see our prior economic assumptions revised when confronted with data, and we obtain a consistent set of estimates to be used in applied economic analysis based on simulation. Otherwise a common option is to use just the prior assumptions, based on estimates for possibly unrelated sources.

Table A.1: Sensitivity analysis: Total economy

	μ	σ	α	β	$\alpha+\beta$	ρ	K_0
	Prior mean						
Base	0.000	0.010	0.500	0.500	1.000	0.920	1910
TFP	0.000	0.010	0.500	0.500	1.000	0.920	1910
Capital	0.000	0.010	0.500	0.500	1.000	0.920	**2865**
Elasticities	0.000	0.010	**0.509**	**0.509**	**1.018**	0.920	1910
	Prior standard deviation						
Base	0.020	0.0005	0.150	0.150	0.095	0.031	1910
TFP	**0.040**	**0.0053**	0.150	0.150	0.095	0.031	1910
Capital	0.020	0.0005	0.150	0.150	0.095	**0.045**	**2865**
Elasticities	0.020	0.0005	**0.238**	**0.238**	**0.185**	0.031	1910
	Posterior mean						
Base	0.012	0.010	0.391	0.545	0.937	0.949	1052
TFP	0.012	0.013	0.400	0.548	0.948	0.944	1095
Capital	0.010	0.010	0.416	0.533	0.949	0.964	1239
Elasticities	0.017	0.010	0.288	0.520	0.809	0.942	903
	Posterior standard deviation						
Base	0.006	0.0005	0.098	0.081	0.090	0.021	303
TFP	0.007	0.0021	0.108	0.093	0.093	0.022	449
Capital	0.006	0.0005	0.101	0.082	0.090	0.021	496
Elasticities	0.007	0.0005	0.123	0.093	0.147	0.024	363

Table A.2: Sensitivity analysis: Industry

	μ	σ	α	β	$\alpha + \beta$	ρ	K_0
				Prior mean			
Base	0.000	0.010	0.500	0.500	1.000	0.920	351
TFP	0.000	0.010	0.500	0.500	1.000	0.920	351
Capital	0.000	0.010	0.500	0.500	1.000	0.920	**527**
Elasticities	0.000	0.010	**0.509**	**0.509**	**1.018**	0.920	351
				Prior standard deviation			
Base	0.020	0.0005	0.150	0.150	0.095	0.031	351
TFP	**0.040**	**0.0053**	0.150	0.150	0.095	0.031	351
Capital	0.020	0.0005	0.150	0.150	0.095	**0.045**	**527**
Elasticities	0.020	0.0005	**0.238**	**0.238**	**0.185**	0.031	351
				Posterior mean			
Base	0.029	0.015	0.455	0.449	0.904	0.936	775
TFP	0.026	0.035	0.482	0.491	0.973	0.924	629
Capital	0.029	0.015	0.481	0.441	0.922	0.953	1056
Elasticities	0.033	0.014	0.297	0.424	0.722	0.917	712
				Posterior standard deviation			
Base	0.006	0.0007	0.106	0.072	0.091	0.028	305
TFP	0.011	0.0055	0.130	0.119	0.093	0.029	309
Capital	0.007	0.0007	0.110	0.072	0.094	0.033	521
Elasticities	0.006	0.0007	0.140	0.078	0.142	0.039	298

Table A.3: Sensitivity analysis: Information and Communication

	μ	σ	α	β	$\alpha+\beta$	ρ	K_0
	Prior mean						
Base	0.000	0.010	0.500	0.500	1.000	0.920	39
TFP	0.000	0.010	0.500	0.500	1.000	0.920	39
Capital	0.000	0.010	0.500	0.500	1.000	0.920	**59**
Elasticities	0.000	0.010	**0.509**	**0.509**	**1.018**	0.920	39
	Prior standard deviation						
Base	0.020	0.0005	0.150	0.150	0.095	0.031	39
TFP	**0.040**	**0.0053**	0.150	0.150	0.095	0.031	39
Capital	0.020	0.0005	0.150	0.150	0.095	**0.045**	**59**
Elasticities	0.020	0.0005	**0.238**	**0.238**	**0.185**	0.031	39
	Posterior mean						
Base	0.032	0.015	0.638	0.295	0.933	0.943	46
TFP	0.033	0.040	0.582	0.389	0.971	0.928	44
Capital	0.030	0.015	0.655	0.287	0.943	0.955	50
Elasticities	0.034	0.015	0.593	0.262	0.856	0.941	44
	Posterior standard deviation						
Base	0.006	0.0007	0.091	0.055	0.090	0.020	10
TFP	0.012	0.0062	0.116	0.104	0.093	0.027	18
Capital	0.007	0.0007	0.093	0.056	0.090	0.022	11
Elasticities	0.007	0.0007	0.147	0.060	0.160	0.022	12

Table A.4: Sensitivity analysis: Real Estate

	μ	σ	α	β	$\alpha+\beta$	ρ	K_0
				Prior mean			
Base	0.000	0.010	0.500	0.500	1.000	0.920	200
TFP	0.000	0.010	0.500	0.500	1.000	0.920	200
Capital	0.000	0.010	0.500	0.500	1.000	0.920	**300**
Elasticities	0.000	0.010	**0.509**	**0.509**	**1.018**	0.920	200
				Prior standard deviation			
Base	0.020	0.0005	0.150	0.150	0.095	0.031	200
TFP	**0.040**	**0.0053**	0.150	0.150	0.095	0.031	200
Capital	0.020	0.0005	0.150	0.150	0.095	**0.045**	**300**
Elasticities	0.020	0.0005	**0.238**	**0.238**	**0.185**	0.031	200
				Posterior mean			
Base	−0.014	0.015	0.596	0.084	0.680	0.963	197
TFP	−0.010	0.044	0.659	0.242	0.901	0.938	223
Capital	−0.012	0.015	0.674	0.070	0.745	0.980	384
Elasticities	0.008	0.014	0.114	0.082	0.196	0.927	21
				Posterior standard deviation			
Base	0.008	0.0008	0.129	0.047	0.114	0.017	120
TFP	0.016	0.0073	0.123	0.105	0.094	0.024	154
Capital	0.012	0.0008	0.125	0.046	0.110	0.013	393
Elasticities	0.005	0.0007	0.039	0.025	0.046	0.028	11