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YABLO'S PARADOXES IN NON-ARITHMETICAL SETTING

Abstract

Proving a paradox from very weak assumptions helps us to reveal what the source of the paradox is. We introduce a weak non-arithmetical theory in a language of predicate logic and give proofs for various versions of Yablo's paradox in this weak system. We prove *Always*, *Sometimes*, *Almost Always*, and *Infinitely Often* versions of Yablo's paradox in the presented weak axiom system, which is much weaker than the arithmetical setting.

Keywords: non-self-referential paradox, Yablo's paradox, weak systems

1. YABLO'S PARADOX

To counter a general belief that all the paradoxes have self-referential (circular) nature, Stephen Yablo designed a paradox that seemingly avoids self-reference (Yablo 1985, 1993). Since then much debate has been sparked in the philosophical community as to whether Yablo's Paradox is really circularity-free or involves some circularity, at least hidden or implicit (cf. Beall 2001, Bringsjord, van Heuveln 2003, Bueno, Colyvan 2003a, b, Ketland 2004, 2005, Priest 1997, Sorensen 1998, Yablo 2004). In this section, we focus on Yablo's paradox and show the existence of Yablo formulas using diagonal techniques.

Yablo considers the following sequence of sentences $\{\mathcal{Y}_i\}$:

$\mathcal{Y}_1 : \forall k > 1;\ \mathcal{Y}_k$ is untrue,

$\mathcal{Y}_2 : \forall k > 2;\ \mathcal{Y}_k$ is untrue,

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$\mathcal{Y}_3 : \forall k > 3; \mathcal{Y}_k$ is untrue,

$\vdots$

There is no consistent way to assign truth values to all the statements, although no statement directly refers to itself. The paradox follows from the following deductions. Suppose $\mathcal{Y}_1$ is true. Then for any $k > 1$, $\mathcal{Y}_k$ is not true. In particular, $\mathcal{Y}_2$ is not true. Also, $\mathcal{Y}_k$ is not true for any $k > 2$. But this is exactly what $\mathcal{Y}_2$ says, hence y_2 is true after all. Contradiction! Suppose then that $\mathcal{Y}_1$ is false. This means that there is a $k > 1$ such that $\mathcal{Y}_k$ is true. But we can repeat the reasoning, this time with respect to $\mathcal{Y}_k$ and reach a contradiction again. No matter whether we assume $\mathcal{Y}_1$ to be true or false, we reach a contradiction. Hence the paradox.

Yablo's paradox can be viewed as a non-self-referential liar's paradox; it has been used to give alternative proof for Gödel's first incompleteness theorem (Cieśliński, Urbaniak 2013, Leach-Krouse 2014). In (Karimi 2019, Karimi, Salehi 2014, 2017), formalization of Yablo's paradox and its different versions in Linear Temporal Logic (LTL) yields genuine theorems in this logic. Recently, Yablo's strategy has been applied to present a non-self-referential version of Brandenburger–Keisler paradox (Brandenburger, Keisler 2006) in epistemic game theory (Karimi 2017).

Let S be a theory formulated in L_T, a language of first-order arithmetic extended with a one place predicate $T(x)$. By $\forall x T(\ulcorner \varphi(\dot{x}) \urcorner)$ we mean: for all natural numbers x, the result of substituting a numeral denoting $\dot{x}$ for a variable free in φ is true.

DEFINITION 1. *Let S be a theory in L_T. Formula $Y(x)$ is called a Yablo formula in S if it satisfies the Yablo condition, i.e., if* $S \vdash \forall x[Y(x) \equiv \forall z > x \neg T(\ulcorner Y(\dot{z}) \urcorner)]$. *Yablo sentences are obtained by substituting numerals for x in $Y(x)$.* (Cieśliński 2013)

It is easy to prove the existence of Yablo formulas for all theories extending Robinson's arithmetic (Ketland 2004, Priest 1997). For this, we apply generalized diagonal lemma:

THEOREM 1. (*Generalized Diagonal Lemma*) *Let S be a theory in L_T extending Robinson's arithmetic. Then for any formula $\mathcal{G}(x, y)$ in L_T, there is a formula $\varphi(x)$ such that*

(1) $$S \vdash \varphi(x) \equiv \mathcal{G}(x, \ulcorner \varphi(x) \urcorner).$$

To construct a Yablo formula, consider the following arithmetized formula of Yablo's sequence:

$$\mathcal{G}(x, y) := \forall z > x \neg T\,(Sub(y, \dot{z})),$$

where $Sub(y, \dot{z})$ is the substitution function. Applying diagonal lemma for $\mathcal{G}(x, y)$, there exists Yablo formula $Y(x)$ for which:

(2) $$S \vdash Y(x) \equiv \forall z > x \neg T\,(Sub(\ulcorner Y(x) \urcorner, \dot{z})).$$

Yablo's paradox appears in several varieties (Yablo 2004):

Always Yablo's paradox: $\forall x\,[Y(x) \equiv \forall z > x \neg T\,(\ulcorner Y(\dot{z}) \urcorner)]$

Sometimes Yablo's paradox: $\forall x\,[Y(x) \equiv \exists z > x \neg T\,(\ulcorner Y(\dot{z}) \urcorner)]$

Almost Always Yablo's paradox: $\forall x\,[Y(x) \equiv \exists z > x\ \ \forall t \geq z \neg T\,(\ulcorner Y(\dot{t}) \urcorner)]$

Infinitely Often Yablo's paradox: $\forall x\,[Y(x) \equiv \forall z > x\ \ \exists t \geq z \neg T\,(\ulcorner Y(\dot{t}) \urcorner)]$

Sometimes Yablo's paradox is the dual version of the *Always* one. Indeed, using $\neg Y(x)$ instead of $Y(x)$ in *Always* Yablo's paradox, one can easily derive *Sometimes* Yablo's paradox. Note that *Infinitely Often* Yablo's paradox is also the dual version of *Almost Always* Yablo's paradox.

The dual of Yablo's paradox, i.e., *Sometimes* Yablo's paradox, was first given by Roy T. Cook (2004). *Almost Always* Yablo's paradox and its dual variant, i.e., *Infinitely Often* Yablo's paradox, were put forward by Yablo (2004). Yablo's paradox and the above two variants were generalized by Philippe Schlenker (2007). The notion of unwinding was formulated by Cook (2004, 2014), who presented a uniform framework in which many variations of Yablo's paradox proceed by varying the quantifier used at the beginning of each of the sentences. He considers an infinite sequence of sentences, each of which says that infinitely many (but not necessarily all) of the sentences below it are false. The main problem here is what patterns of sentential reference generate semantic paradoxes.

2. YABLO'S PARADOX AND ω-INCONSISTENT THEORIES OF TRUTH

Jeffrey Ketland (2005) shows that Yablo's sentences have a non-standard model. He argues that the list of Yablo sentences is ω-paradoxical, in the sense that it is unsatisfiable on the standard model $\mathbb{N}$ of arithmetic. He has translated Yablo's paradox into first-order logic called Uniform Homogeneous Yablo Scheme (UHYS):

(UHYS): $$\forall x(\varphi(x) \leftrightarrow \forall y[xRy \rightarrow \neg\varphi(y)]),$$

where R is a binary relation symbol, with the auxiliary axioms stating that R is serial and transitive:

(SER): $\forall x \exists y (xRy)$,

(TRANS): $\forall x, y, z (xRyRz \rightarrow xRz)$.

A Yablo-like argument can show that the formula $\neg$(UHYS $\wedge$ SER $\wedge$ TRANS) is a first-order tautology (Karimi, Salehi 2014), i.e., UHYS is inconsistent, together with SER and TRANS. Note that the inconsistency of UHYS arises irrespective of what φ means, provided that R is serial and transitive. Ketland (2005) shows that the associated set of numerical instances of UHYS is consistent as it has a non-standard model.

However, for getting a contradiction from UHYS, the formula R need not be transitive (i.e., satisfy TRANS) — a weaker condition such as the following will do:

(W-TRANS): $\forall x \exists y (xRy \wedge \forall z [yRz \rightarrow xRz])$.

One can easily show that W-TRANS does not imply TRANS (but implies SER), and so W-TRANS is weaker than TRANS.[1]

Eduardo A. Barrio (2010), Barrio and Lavinia Picollo (2013), and Barrio and Bruno Da Ré (2018) focus on Yablo's paradox and ω-inconsistent theories of truth. Barrio (2010) argues that Yablo's sequences yield new boundaries to expressive capabilities of certain axiomatic theories of truth. He shows that Yablo's sentences have no model in second order PA, but even in this case the sequence is consistent. Barrio and Lavinia Picollo (2013) show that adopting ω-inconsistent truth theories for arithmetic in the second-order case leads to unsatisfiability.

Shunsuke Yatabe (2011) reviews Yablo's paradox to analyze the computational content of ω-inconsistent theories and explains the correspondence between co-induction and ω-inconsistent theories of truth. Yatabe argues in favor of ω-inconsistent first-order theories of truth as he believes that they are intrinsically equipped with a machinery, co-induction, which is useful to prove properties of infinite structures. Against Yatabe, Barrio and Da Ré (2018) present some undesirable philosophical features of ω-inconsistent theories and identify five conceptual problems as results of ω-inconsistency.

In the rest of the paper, we introduce a weak non-arithmetical setting and prove various versions of Yablo's paradox in this weak system.

[1] To see that $\neg$(UHYS $\wedge$ W-TRANS) is a first-order logical tautology, cf. (Karimi, Salehi 2014).

3. WEAK NON-ARITHMETICAL SETTING

In order to prove Yablo's paradox in a weak setting, Volker Halbach and Shuoying Zhang introduce a weak theory in a language of predicate logic (Halbach, Zhang 2017). By "proving a paradox" we mean that in a given theory, with strictly specified assumptions, we can derive a contradiction and conclude that this theory with such and such inference rules captures or formalizes the paradoxical reasoning. We assume the reader is familiar with the general notations Halbach and Zhang (2017) used to proceed, but we list the main notations, definitions, and theorems that will be employed below. The language contains identity symbol, a binary predicate symbol <, and a ternary predicate symbol $\mathsf{Sat}(x, y, z)$. Adding countably many new constants c_1, c_2, ... to the language, for each formula φ in the language one can define a closed term $\overline{\varphi}$ in that language. Applying the same reasoning as Yablo's, we simply prove that the theory Y given by T-schema (TS) and two axioms in inconsistent:

(TS): $\forall x \forall y\, (\mathsf{Sat}(\overline{\varphi(x,y)}, x, y) \leftrightarrow \varphi(x, y))$

(SER): $\forall x \exists y\, x < y$

(W-TRANS) : $\forall x \exists y\, x < y \wedge \forall z\, (y < z \rightarrow x < z))$,

where TS is a variant of the uniform T-schema (Halbach, Zhang 2017). "Sat" stands for satisfaction and it is to be read: For all x and y, the formula $\overline{\varphi(x,y)}$ is satisfied by x and y if and only if $\varphi(x, y)$.

Although the sentences in Yablo's paradox are indexed by numbers, Ketland showed, for an arithmetical setting, that only a serial and transitive relation is needed to obtain the paradox (2005). Halbach and Zhang (2017) introduce a weak non-arithmetical theory including only TS, SER, and TRANS axioms in a language of predicate logic and prove Yablo's paradox in this weak setting.

Note that W-TRANS not only is weaker than TRANS but also implies SER. Therefore, in this paper, we weaken the non-arithmetical system Y to include only TS and W-TRANS and prove various versions of Yablo's paradox in this non necessarily arithmetical setting. The presented system is much weaker than the one applied by Halbach and Zhang (2017) to prove Yablo's paradox. In fact, we try to obtain various versions of Yablo's paradox from as weak assumptions as we can get. By proving the paradox in minimal settings, we come closer to a better understanding of the paradox structure and hope to extend our knowledge of what the source of the paradox is.

3.1. *ALWAYS* YABLO'S PARADOX

An abbreviation of *Always* Yablo's paradox is:

$$\forall n(y_n \Leftrightarrow \forall i > n\ (y_i \text{ is not true})).$$

We use the notation used by Halbach and Zhang (2017) and prove Yablo's paradox in our weak system. We write $\forall z > y\ \varphi$ for $\forall z\ (z < y \to \varphi)$ and $\overline{\psi_\forall}$ for $\overline{\forall z > y \neg \mathsf{Sat}(x,x,z)}$. Instantiating $\varphi(x, y)$ with $\psi_\forall$ in TS, we have:

(3) $$Y \vdash \forall x \forall y(\mathsf{Sat}(\overline{\psi_\forall}, x, y) \leftrightarrow \forall z > y\ \neg\mathsf{Sat}(x, x, z))$$

(4) $$Y \vdash \forall y(\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, y) \leftrightarrow \forall z > \mathrm{y}\ \neg\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, z))$$

The second equation is obtained by universal instantiation. To prove Yablo's paradox in our weak system Y, we assume that $\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, a)$ for some a and reason in Y as follows: by $\mathsf{W\text{-}TRANS}$, there is some b such that $a < b$ and $\forall z\ (b < z \to a < z)$. Since $a < b$ and $\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, a)$ hold, then by (4), $\neg\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, b)$ should hold. Again by (4), there exists some c for which $b < c$ and $\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, c)$. Since $a < b$, and $b < c$, then $a < c$ by $\mathsf{W\text{-}TRANS}$, therefore by (4), we have $\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, c)$, which is a contradiction! Thus, $\neg\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, a)$ holds for all a. For arbitrary a, by $\mathsf{W\text{-}TRANS}$, there exists some b such that $a < b$ and by (4) we have $\mathsf{Sat}(\overline{\psi_\forall}, \overline{\psi_\forall}, b)$. Again, repeating the above argument yields contradiction.

3.2. *SOMETIMES* YABLO'S PARADOX

As mentioned before, Yablo's paradox comes in several versions (Yablo 2004). A dual version of *Always* Yablo's paradox is *Sometimes* Yablo's paradox: Consider infinite sequence of sentences $\{\mathcal{Y}_i\}_{i=0}^{\infty}$ each of which says "there exists a forthcoming sentence that is not true":

$$\mathcal{Y}_0 : \exists i > 0\ (\mathcal{Y}_i \text{ is not true})$$

$$\mathcal{Y}_1 : \exists i > 1\ (\mathcal{Y}_i \text{ is not true})$$

$$\mathcal{Y}_2 : \exists i > 2\ (\mathcal{Y}_i \text{ is not true})$$

$$\vdots \qquad\qquad \vdots$$

In short,

$$\forall n(\mathcal{Y}_n \Leftrightarrow \exists i > n\ (\mathcal{Y}_i \text{ is not true})).$$

Again we write $\forall z > y\ \varphi$ for $\forall z\ (z < y \to \varphi)$ and $\overline{\psi_\exists}$ for $\overline{\exists z > y\, \neg\mathsf{Sat}(x,x,z)}$. Instantiating $\varphi(x, y)$ with $\psi_\exists$ in TS and then universal instantiation $\overline{\psi_\exists}$ yield:

$$(5) \qquad Y \vdash \forall y(\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, y) \leftrightarrow \exists z > y\, \neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, z))$$

In order to prove *Sometimes* Yablo's paradox in a weak setting, assume $\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, a)$ for some a. By (5), there is some $b > a$ for which $\neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, b)$, thus, for all $c > b$ we have $\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, c)$. Choose an arbitrary fixed c. Since $\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, c)$ holds, there exists $d > c > b$ for which $\neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, d)$. On the other hand, since $\neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, b)$ holds, then $\neg\mathsf{Sat}(\overline{\psi_\exists},\overline{\psi_\exists}, b)$ must hold. Contradiction! Therefore, $\neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, a)$ for all a. Thus, for all a, $\forall z > a\ \neg\mathsf{Sat}(\overline{\psi_\exists}, \overline{\psi_\exists}, z)$. Choose a specific and fixed z and apply above reasoning to reach a contradiction.

3.3. *ALMOST ALWAYS* YABLO'S PARADOX

Let us focus now on *Almost Always* Yablo's paradox. Let $\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be a sequence of sentences in which each sentence, roughly speaking, says "all sentences, except finitely many, after this sentence are false." Mathematically, this sequence is as follows:

$$\begin{array}{l} \mathcal{Y}_0 : \exists i > 0\ \forall j \geq i\ (\mathcal{Y}_j \text{ is not true}) \\ \mathcal{Y}_1 : \exists i > 1\ \forall j \geq i\ (\mathcal{Y}_j \text{ is not true}) \\ \mathcal{Y}_2 : \exists i > 2\ \forall j \geq i\ (\mathcal{Y}_j \text{ is not true}) \\ \vdots \qquad\qquad\qquad\qquad \vdots \end{array}$$

In short,

$$\forall n(\mathcal{Y}_n \Leftrightarrow \exists i > n\ \forall j \geq i\ (\mathcal{Y}_j \text{ is not true})).$$

The paradox arises when we try to assign truth values in a consistent way to all sentences $\mathcal{Y}_i$'s. Assume for a moment that there is a sentence (say) $\mathcal{Y}_n$ which is true; so there exists $i > n$ for which all $\mathcal{Y}_j$ with $j \geq i$ are untrue. In particular, $\mathcal{Y}_i$ is untrue. Since all the sentences $\mathcal{Y}_{i+1}, \mathcal{Y}_{i+2}, \ldots$ are untrue, so $\mathcal{Y}_i$ has to be true. Therefore, $\mathcal{Y}_i$ is true and false at the same time, which is a contradiction. Hence, all $\mathcal{Y}_n$'s are untrue, so $\mathcal{Y}_0$ is true; again a contradiction.

We write $\overline{\psi_{\exists\forall}}$ for $\overline{\exists z > y\, \forall w \geq z\, \neg\mathsf{Sat}(x,x,w)}$. Instantiating $\varphi(x, y)$ with $\psi_{\exists\forall}$ in TS and then universal instantiation $\overline{\psi_{\exists\forall}}$ yield:

$$(6) \qquad Y \vdash \forall y(\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, y) \leftrightarrow \exists z > y\ \forall w \geq z\, \neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w))$$

To prove the paradox, assume that $\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, a)$ for some a. By (6), there exists $z > a$ for which $\forall w \geq z \, \neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w)$. In particular, for $w = z$, we have $\neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, z)$. Since for all $w > z$, $\neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w)$ hold, $\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, z)$ must hold, which is a contradiction! Thus, $\neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, a)$ for all a. Consider a fixed a. There exists $z > a$ (by W-TRANS) such that

$$\forall w \geq z \, \neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w)$$

holds (indeed, $\neg\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w)$ holds for all w). By (6), $\mathsf{Sat}(\overline{\psi_{\exists\forall}}, \overline{\psi_{\exists\forall}}, w)$ holds, which is a contradiction.

3.4. *INFINITELY OFTEN* YABLO'S PARADOX

For *Infinitely Often* Yablo's paradox, let $\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be a sequence of sentences in which each sentence says "infinitely many sentences after this sentence are false." Mathematically, this sequence is as follows:

$$\begin{array}{l} \mathcal{Y}_0 : \forall i > 0 \, \exists j \geq i \, (\mathcal{Y}_j \text{ is not true}) \\ \mathcal{Y}_1 : \forall i > 1 \, \exists j \geq i \, (\mathcal{Y}_j \text{ is not true}) \\ \mathcal{Y}_2 : \forall i > 2 \, \exists j \geq i \, (\mathcal{Y}_j \text{ is not true}) \\ \vdots \qquad\qquad\qquad \vdots \end{array}$$

In short,

$$\forall n (\mathcal{Y}_n \Leftrightarrow \forall i > n \, \exists j \geq i \, (\mathcal{Y}_j \text{ is not true})).$$

We write $\overline{\psi_{\forall\exists}}$ for $\overline{\forall z > y \, \exists w \geq z \, \neg\mathsf{Sat}(x,x,w)}$. Instantiating $\varphi(x, y)$ with $\psi_{\exists\forall}$ in TS and then universal instantiation $\overline{\psi_{\exists\forall}}$ yield:

$$Y \vdash \forall y (\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, y) \leftrightarrow \forall z > y \, \exists w \geq z \, \neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, w)) \tag{7}$$

In order to prove the paradox, assume that $\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$ for some a. By (7), we have:

$$\forall i > a \, \exists j \geq i \, \neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, j).$$

Choose fixed $b > a$ and $c \geq b$ for which $\neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, c)$. On the other hand, by W-TRANS, for all $i > c$ we have $i > a$, therefore, $\forall i > c \, \exists j \geq i \, \neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, j)$ holds, which means $\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, c)$. Contradiction! Thus,

$\neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$ for all a.

Let $\neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$ for an arbitrary and fixed a. By W-TRANS, for every $i > a$ there exists $j \geq i$ for which $\neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$.[2] This means that $\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$, which is a contradiction.

The main problem here is that the schema TS by itself is inconsistent. Halbach and Zhang (2017) reformulate Visser's paradox (Visser 1989) to give a solution to this problem. They weaken TS into a consistent principle that can be shown to be inconsistent with W-TRANS. Visser (1989) considered the set of T-sentences $T_n\overline{\varphi} \leftrightarrow \varphi$. His main observation was that φ is a sentence containing only truth predicates with index $k > n$. He showed that this set of T-sentences is ω-inconsistent over arithmetic. Reformulating Visser's paradox, we use the satisfaction predicate $\mathsf{Sat}_x(y, z, w)$ (x is a quantifiable variable) for formulas where all quantifiers over indices are restricted to objects v with $v > y$ (Halbach, Zhang 2017). Instead of TS we use the following axiom:

(VS): $$\forall x\, \forall y\, (\mathsf{Sat}_y(\overline{\varphi(x,y)}, x, y) \leftrightarrow \varphi(x, y)),$$

where all occurrences of variables in index position $\varphi(x, y)$ must be bound. In contrast to TS, it is shown that the schema VS by itself is consistent (Halbach, Zhang 2017). If $\overline{\psi_{\forall}}$ is defined as $\overline{\forall z > y\, \neg\mathsf{Sat}_z(x,x,z)}$ in Section 3.1, the proof still works and shows that W-TRANS together with VS are inconsistent.

CONCLUSIONS

The above proofs for various versions of Yablo's paradox in a weak system can help to characterize the nature of these paradoxes. Usually, Yablo's paradox is presented in the context of the theory of arithmetic, but we have shown that one can derive a contradiction in a very weak theory of first-order logic, including axioms TS and W-TRANS.

[2] Actually, $\neg\mathsf{Sat}(\overline{\psi_{\forall\exists}}, \overline{\psi_{\forall\exists}}, a)$ holds for all j.

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