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IGOR HANZEL*

THE METHODS OF CONSTRUCTION IN SCHRÖDINGER'S *MITTEILUNGEN***

Abstract

The paper reconstructs the methods used by Schrödinger in the construction of wave mechanics as given in his four *Mitteilungen*. These methods are described from the point of view of modern philosophy of science, with a focus on the relationships between scientific theories and on the reconstruction of the structure of scientific laws and the relations between scientific laws. After reconstructing the derivation of the first equation in *Mitteilung 1*, it analyzes the methodology of the optical-mechanical analogy in *Mitteilung 2* and reconstructs the two heuristic pathways that led to the construction of wave equations as the basis of wave mechanics in the first two *Mitteilungen*. Finally, it addresses the methods of generalization, application, and explanation by which the second, third, and fourth *Mitteilungen* are constructed.

Keywords: Erwin Schrödinger, *Mitteilungen*, scientific laws, explanation, application, generalization

> The transition from Hamilton's partial differential equation to the wave equation . . . signifies for mechanics exactly the same as for optics the transition from ray optics . . . to wave optics proper. One can speak of an undulatory theory of mechanics. (Schrödinger 1926b: [14])

In the last fifty years or so, the formation of Erwin Schrödinger's wave mechanics has been the subject of detailed analyses by historians of science.[1] This paper takes a different approach to Schrödinger's wave mechanics. Instead of providing a historical analysis of its formation, it aims to reconstruct

* Department of Logic and Methodology of Science, Philosophical Faculty, Comenius University, Gondova 2, 814 99 Bratislava, Slovakia, e-mail: hanzel@fphil.uniba.sk, ORCID: 0000-0001-9378-7292.

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[1] For a more recent historical analysis, see, e.g., (Darrigol 2009, 2014).

the methods used by Schrödinger, as given in his four *Mitteilungen*, to arrive at wave mechanics. These methods are described from the perspective of contemporary philosophy of science and, more specifically, the idealizational conception of science proposed by Leszek Nowak (1971, 1972, 1977, Nowak, Nowakowa 2000), whose focus on the structure of scientific laws and their interrelations will enable us to understand the methods used by Schrödinger to build the inner structure of *Mitteilungen 1-4* and see how these methods shed light on the relations that obtain between these *Mitteilungen*.[2]

In Section 1, I reconstruct the derivation of the first wave equation in *Mitteilung 1* and show how Schrödinger, by drawing on inter-theoretic relations, determined the value of the constant *K*, which appears at the beginning of the derivation. I then analyze the optical-mechanical analogy in *Mitteilung 2* and reconstruct the two heuristic pathways that, related via the principle of correspondence, led Schrödinger in the first two *Mitteilungen* to the formulation of the wave equations, which form the basis of wave mechanics. Section 2 addresses the methods of generalization, application, and explanation by which the second, third, and fourth *Mitteilungen* are constructed. The conclusion summarizes the methodological moral of Schrödinger's construction and the relations between the four *Mitteilungen*.

1. THE METHODS OF CONSTRUCTIONS OF THE WAVE EQUATIONS IN *MITTEILUNG 1* AND *MITTEILUNG 2*

Schrödinger begins by specifying the type of entity to which his new quantum postulate should initially be applied: it is the non-relativistic and unperturbed hydrogen atom. But although he starts with the simplest atom, he claims that his "new conception is capable of generalization" (1926a: 361).[3]

His next step is to employ Hamilton's partial differential equation:[4]

$$H\left(q, \frac{\partial S}{\partial q}\right) = E. \qquad \text{(1-M1)}$$

[2] For a more recent development of the views of Nowak, see (Hanzel 2015); for their application to physics, see, e.g., (Hanzel 2008, 2013).

[3] All the quotes from Schrödinger's *Mitteilungen* cited in this paper are in my translation.

[4] The numbers of formulas, supplemented by the symbols "-M1," "-M2," "M3," and "-M4," correspond to the numbers from Schrödinger's first, second, third, and fourth *Mitteilung*, respectively. Starred numbers placed into square brackets indicate formulas not used by Schrödinger.

S stands for *action*, for which Schrödinger substitutes

(2-M1) $S = K \ln \psi$.

K is a constant, while ψ is a function initially characterized mathematically, not physically, as a product of functions, each depending on only one coordinate. By substituting (2-M1) into (1-M1), Schrödinger obtains the following:

(1′-M1) $$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E.$$

Then, instead of searching for a solution to this equation, Schrödinger transforms (1′-M1) into a quadratic form, of ψ and of ψ's first derivatives, that is equal to zero. This means is that he looks for a function ψ such that it is real, finite, and differentiable to the second order so that, for its arbitrary variation, the integral (over the whole space of coordinates) is stationary. In other words, Schrödinger arrives at a variation problem, which he specifies for the case of a Keplerian motion. For this case, the energy of an electron with charge *e* and mass *m* moving in a Coulomb field is:

([1*]) $$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_y^2) - \frac{e^2}{r}.$$

Then, by taking into account ([1*]) together with (2-M1) and the equations: $p_x = \partial S/\partial x$, $p_y = \partial S/\partial y$, $p_z = \partial S/\partial z$, one obtains:

(1″-M1) $$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0.$$

By subjecting (1″-M1) to the variation procedure, Schrödinger obtains two conditions under which the variation yields zero; they are as follows:[5]

(5-M1) $$\Delta\psi + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0.$$

(6-M1) $$\int df\, \delta\psi \frac{\partial \psi}{\partial n} = 0.$$

Next, Schrödinger subjects (5-M1) to a series of transformations and arrives at the following formula:[6]

[5] The expression d*f* stands for the element of closed surface in infinity; δ*n* stands for the differential of a line element perpendicular to this surface.

[6] For details on these transformations, see (Mawhin, Ronveaux 2010).

(19-M1) $$-E_l = \frac{me^4}{2K^2 l^2}.$$

He interprets the formula physically as follows (1926a: 371):

> The well-known energy-levels of Bohr are thus obtained corresponding to the Balmer--terms, if the constant K, which we had to introduce in (2[-M1]) for reasons of dimension, is assigned the value
>
> (20-M1) $$k = \frac{h}{2\pi}.$$
>
> Then, one obtains
>
> (19′-M1) $$-E_l = \frac{2\pi^2 me^4}{h^2 l^2}.$$

It is worth noting that in order to derive the value of K, Schrödinger had had to compare formula (19-M1) with Bohr's formula for the energy radiated by the formation of one of the stationary states of a hydrogen atom as derived in his paper "On the Constitution of Atoms and Molecules." This formula is as follows (1913: 8):

([2*]) $$W_\tau = \frac{2\pi^2 me^4}{h^2 \tau^2}.$$

Thus, the derivation of the value of the constant K in the first *Mitteilung* is tied to the derivation of the formula for W_τ in Bohr's 1913 theory of the hydrogen atom.

The dependence of the determination of the constant K on Bohr's theory of 1913 means, in turn, that the interpretation of equation (5-M1) as a *quantum* equation is also tied to Bohr's theory. Only by substituting into this equation the value $h/2\pi$ for K does one obtain:

([3*]) $$\Delta\psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r}\right)\psi = 0,$$

where the presence of Planck's constant indicates that it is already a quantum equation.

Let me now turn to *Mitteilung 2*. Here, Schrödinger's construction of wave mechanics follows a different route than it did in *Mitteilung 1*. As he sees it, it aims at the clarification of the "*general* relation that exists between the Hamiltonian partial differential equation (H.P.) of a mechanical problem and the 'corresponding' wave equation" (1926c: 489).

Schrödinger reflects on the analogy, going back to William R. Hamilton, between classical mechanics and ray optics, and where the former, in its Hamiltonian form, was derived from ray optics.[7] The starting point here is the partial differential equation:

(1-M2) $$\frac{\partial W}{\partial t} + T(q_k, \frac{\partial W}{\partial q_k}) + V(q_k) = 0,$$

where W is Hamilton's principal function, T is the kinetic energy, and V is potential energy. In order to solve this equation, Schrödinger chooses the Ansatz

(2-M2) $$W = -Et + S(q_k).$$

By substituting (2-M2) into (1-M2), one obtains

(1′-M2) $$2T(q_k, \frac{\partial W}{\partial q_k}) = 2(E - V).$$

Schrödinger then transforms this equation, by drawing on a non-Euclidean metric, into the following two equivalent equations:

(1″-M2) $$(\text{grad } W)^2 = 2(E - V).$$

(1‴-M2) $$|\text{grad } W| = \sqrt{2(E - V)}.$$

The function W is viewed by Schrödinger as a family of surfaces with W = const, which then enables him to consider a system of waves, where W = const refers to the surface of these waves. Next, Schrödinger derives the formula for the normal velocity u with which these surfaces are propagated:

(6-M2) $$u = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{E}{\sqrt{2(E - V)}}.$$

However, despite the success of physical descriptions based on the analogy between optics and mechanics, Schrödinger makes a fundamental turn in his approach. He declares:

> Although the preceding deliberations refer to wave surfaces, velocity of propagation, and the Huygens principle, the analogy we are drawing is not between mechanics and *wave* optics, but between mechanics and *geometrical* optics. This is because the concept of rays, which is of chief concern for mechanics, belongs to *geometrical* optics; it is only there that it has a clearly defined meaning. (1926c: 495)

The conclusion Schrödinger draws from this analogy between mechanics and optics is that:

[7] See Hamilton's works in (Conway, Singe 1931).

> The analogy *obtains precisely* with a *geometrical*, or, indeed, with a primitive wave optics and not with a fully developed wave optics. ... In order to *preserve* the analogy in the course of successive wave-theoretical development of the optics of q-space, one has to make sure not to depart markedly from the limiting case of geometrical optics — say, by choosing the wavelength to be small compared with all the dimensions of the trajectories. In which case, however, the addition does not teach us anything new. (1926c: 496)

In order to make the move to a fully developed wave theory, Schrödinger brings in two more pieces of information. The first is "*that our classical mechanics fails for very small trajectories and very great curvatures of trajectories*" (1926c: 496) and the second is that geometrical optics, as an "'optics with an infinitely small wavelength'" (1926c: 497), fails "as soon as the 'obstacles' and 'openings' are not large any more compared with the real finite wavelength" (1926c: 497).

Based on these two pieces of information, he draws the conclusion about the possibility of an analogy between these two failures in the following sense:

> Perhaps, our mechanics is a *complete* analogy of geometrical optics and as such [the analogy is] false . . . it fails as soon as the radii of curvature and the dimensions of the trajectory are not large anymore compared to a certain wavelength. (1926c: 497)

As a way out of this failure, he suggests "to search for an 'undulatory mechanics,' and the most obvious way to accomplish this is by developing Hamilton's picture wave-theoretically" (1926c: 497). He does this by using the *quantum* formula for the frequency ν:

(11-M2) $$\nu = \frac{E}{h}.$$

Because, in such a case, the energy is $E = h\nu$, by substituting for E in (6-M2), one obtains:

(6′-M2) $$u = \frac{h\nu}{\sqrt{2(h\nu - V)}}.$$

Schrödinger's final steps in the introduction of the wave equation in the second *Mitteilung* are as follows (1926c: 509). He presupposes the validity of the following equation for the wave function ψ:

(18-M2) $$\operatorname{div}\operatorname{grad}\psi - \frac{1}{u^2}\ddot{\psi} = 0,$$

which he views as valid for processes that depend on time only via the factor $e^{2\pi i\nu t}$; that is, $\psi(q, t) = \psi(q)\, e^{2\pi i\nu t}$. By substituting this formula and (6′-M2) into (18-M2), he obtains:

$$(18'\text{-M2}) \qquad \operatorname{div} \operatorname{grad} \psi + \frac{8\pi^2}{h^2}(h\upsilon - V)\psi = 0,$$

or

$$(18''\text{-M2}) \qquad \operatorname{div} \operatorname{grad} \psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0.$$

What are the methods Schrödinger employs in the inferences involved in his reasoning about the analogies between the formulas of mechanics and optics, and about the analogies of the failures of these formulas in certain physical conditions, that ultimately lead him to equation (18′-M2)?

In order to reconstruct these methods it is necessary to understand the methodological nature of a special form of the correspondence principle that underlies Schrödinger's reasoning about those analogies.[8] Its nature is that of a relation between two already existing theories: one historically prior and one historically later, and where *from the point of view of the historically later theory*, one can, under certain conditions, derive formulas which, even though still part of the historically later theory, bear a similarity to formulas already derived in the historically prior theory. The difference between the formulas derived in the historically prior theory and the formulas derived in the framework of the historically later theory is accounted for by the fact that the former formulas are embedded in a network of concepts that is different from the network of concepts in which the latter formulas are embedded.[9]

This holds, for example, for Sommerfield's and Runge's (1911) wave-theoretical derivation of the equation for the eikonal, referred to by Schrödinger in the second *Mitteilung*, where the eikonal and the equation for the eikonal had already been introduced in the framework of ray optics (Bruns 1895).[10] The wave-theoretical derivation is based on the use of the differential equation:

$$([4^*]) \qquad \Delta u + k^2 u = 0,$$

[8] The idea that a form of the correspondence principle is at work in the second *Mitteilung* was suggested to me by (Joas, Lehner 2009). As to the correspondence principle, I found (Post 1971, Radder 1991) very helpful. It remains an open question that I do not address here whether the correspondence principle that is at the basis of Schrödinger's reasoning about analogies is the same as Bohr's understanding of the correspondence principle.

[9] This holds also, e.g., for the relation between relativistic mechanics and pre-relativistic mechanics. The derivation of formulas in the framework of relativistic mechanics that are similar to formulas in pre-relativistic mechanics is possible only *after* relativistic mechanics has assigned a prominent role to light and its velocity c. Only then can one employ the condition under which the derivation can be performed — namely, that velocity v of objects is much smaller than c; $v << c$. In the framework of pre-relativistic mechanics, neither light nor its velocity has a prominent role.

[10] On this function and equation, see also (Klein 1901).

where u stands for the excitation of light (separated from the factor e^{ivt}) and k is the wave number related to the wavelength λ by means of $k = 2\pi/\lambda$. The very derivation is then performed, after a sequence of computations, by introducing the condition that $1/k$ is extremely small.

The difference between the original ray-optical equation for the eikonal and the wave-theoretical derivation of the equation for the eikonal is that the latter derivation is embedded in a theory that involves concepts that are absent from ray optics. So, for example, the differential equation ([4*]) is explicitly characterized as a "*differential equation of wave optics*" (1911: 290) and the function u employed in it is characterized as being measured by one of the "electric or magnetic components" (1911: 290-291) of light's excitation — that is, this equation and the whole derivation of the equation for the eikonal, is already embedded in the *electro-magnetic wave theory of light* whose central concepts are absent from geometrical optics.

The absence of central concepts of wave optics in ray optics is emphasized by Sommerfeld and Runge, who — when moving temporarily from a wave-optical to an exclusively ray-optical treatment of certain phenomena — declare:

> We thus return to the point of view of pure geometric optics, which proceeds from the rectilinearity of rays and the law of refraction as empirical data and *knows nothing about wave surfaces*. (Sommerfeld, Runge 1911: 287; emphasis added)

The heuristic ends to which the correspondence principle is employed in the second *Mitteilung* derives from the features of the correspondence principle just described. Once it is possible, *under certain conditions* in the framework of a historically later theory, to derive formulas that are similar to formulas already derived in a historically prior theory, then it is also possible to *infer from the knowledge about those conditions the conditions under which the historically prior theory fails*. In the case of the Sommerfeld-Runge derivation, it is the realization that ray optics fails once $1/k$ acquires large values; under this condition, "the laws of geometrical optics are no longer a sufficient approximation" (1911: 496). Thus, so as equation $\lambda = 2\pi/k$ holds, ray optics fails when the wavelength of the rays has such large values that it is comparable to the dimension of the trajectory along which the rays propagate.[11]

[11] The same methodological principle holds for pre-relativistic mechanics. Once it is possible to derive the formulas of relativistic mechanics from analogous formulas of pre-relativistic mechanics by applying the condition $v << c$, then it is possible to infer that pre-relativistic mechanics fails when the velocity of the objects becomes comparable with the velocity of light, $v \approx c$.

This explains why Schrödinger refers to the Sommerfeld-Runge article in a footnote of the second *Mitteilung* (1926c: 496); he employs this realization as a heuristic principle in his movement toward a wave mechanics he is constructing.

From the methodological point of view, it is also worth noting that Schrödinger constructs his wave mechanics (in the form of wave equations (5-M1) and (18′-M2)) by moving along two different paths — one, in the first *Mitteilung*, originating from Hamilton's mechanics, and another, in the second *Mitteilung*, originating from wave optics. *Schrödinger's wave mechanics thus lies at the intersection of two different paths*.

Once this is taken into account, it is possible to understand a specific feature of the relation between these two paths and thus also between the first and second *Mitteilungen*. Namely, that the *justification* of the movement from Hamiltonian mechanics to the wave equation (5-M1) in the first *Mitteilung* is given only *post festum* — i.e., in the framework of the second *Mitteilung*, where the necessity of that movement is justified by stating "*that our classical mechanics fails for very small trajectories and very great curvatures of trajectories*" (1926c: 496).

Figure 1 represents the heuristics leading to the first two *Mitteilungen* set against a historical background.[12]

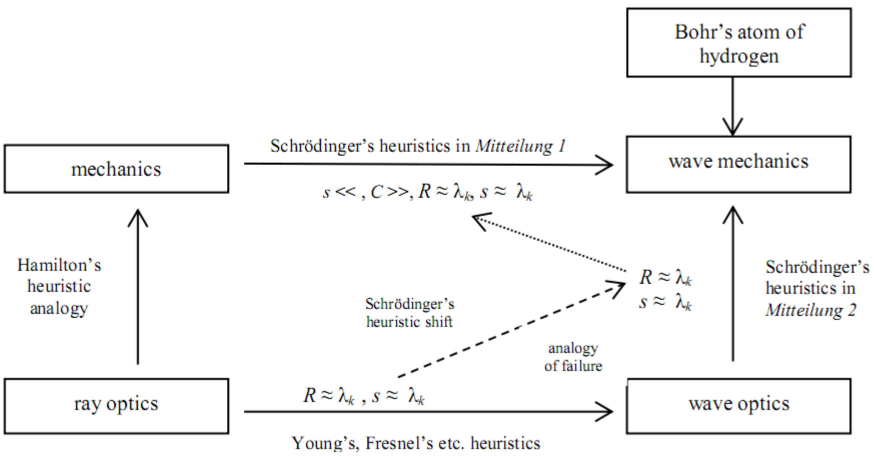

Fig. 1. Schrödinger's heuristics leading to the first and second *Mitteilungen* and its historical background

The downward-pointing arrow indicates that Schrödinger had to draw on Bohr's formula ([2*]) to determine the value of the constant K introduced in

[12] For a different figure, see (Joas, Lehner 2009: 346).

the first *Mitteilung*. The expressions $s \approx \lambda_k$ and $R \approx \lambda_k$ state that the dimensions s of the trajectories and the radii R of their curvature are compatible to a certain wavelength λ_k. The dashed arrow means that Schrödinger acquired the knowledge, initially stated in optics, that, under these conditions, ray optics fails and it is necessary to switch to wave optics, and he then employed this knowledge from optics in a heuristic manner and applied it in the second *Mitteilung*. In the second *Mitteilung*, he used this knowledge to justify the move from wave optics to wave mechanics. The expressions $s <<<$ and $C >>>$ stand for Schrödinger's claim (1926c: 496) that Hamiltonian mechanics fails when the dimensions of the trajectories are very small and their curvature is very large.

The dotted arrow indicates that the information about the conditions under which this failure takes place is, even if it justifies the move from mechanics to wave mechanics in the first *Mitteilung*, stated only post festum in the second *Mitteilung*. The heuristic movement from ray to wave optics is indicated at the bottom of the figure. Young and Fresnel are mentioned as examples of physicists who participated in the movement from ray to wave optics.[13]

Let me now turn to the methods used in the second *Mitteilung* after Schrödinger proposed wave equations (18′-M2) and (18″-M2). Both of them had been derived from wave equation (18-M2), whose introduction had been accompanied by the following words of caution (1926c: 509):

> The wave equation has not been used explicitly in this communication and has not been stated yet. The only datum for its construction is the *wave velocity*, given by (6[-M2]) and (6′[-M2]) as the function of the parameter of mechanical energy or the frequency; and, needless to say, the wave equation is not uniquely defined by this datum. It is not at all certain that it must definitely be of the second order; only considerations of simplicity lead us to try this.

Thus, at this point, wave equation (18-M2) has the status of a *tentative conjecture*, which is inherited by the quantum wave equations (18′-M2) and (18″-M2) even after equation (18-M2) has been combined with the "datum" (6-M2) and (6′-M2). In order to give some credence and validity to his conjecture, Schrödinger uses equation (18″-M2) as the starting point for a quantum-mechanical treatment of four physical systems, all of which are restricted to the nonrelativistic case and with no magnetic field being at work. These four systems are: the Planck oscillator, a rotator with a fixed axis, a rigid rotator with a free axis, and a nonrigid rotator (diatomic molecule).

From the point of view of this paper, it is worth noting that Schrödinger reflects on the specific properties of each system and, based on those proper-

[13] For details on this movement, see (Darrigol 2012).

ties, uses equation (18′-M2) to derive a wave equation for that system. Here, I only deal with Planck's oscillator and the rotator with a fixed axis.

For the oscillator, Schrödinger first considers the case of a one-dimensional oscillator, and derives the wave equation

$$\text{(22-M2)} \qquad \frac{d^2\psi}{dq^2} + \frac{8\pi^2}{h^2}(E - 2\pi^2 v_0 q^2)\psi = 0.$$

He then moves from the case of a one-dimensional oscillator to the case with several dimensions (space-oscillator, rigid body) and reflects on the possible modifications of the formula for the energy spectrum initially derived for the one-dimensional case.

In the case of the rotator with a fixed axis, for A as the moment of inertia and φ as the angle of rotation, the wave equation is

$$\text{(29-M2)} \qquad \frac{1}{A}\frac{d^2\psi}{d\varphi^2} + \frac{8\pi^2}{h^2}\psi = 0.$$

2. THE METHODS OF THE *MITTEILUNGEN*

2.1. METHODOLOGICAL PRELIMINARIES

If one looks from the point of view of philosophy of science, at the way Schrödinger formulates the wave equations in the first and second *Mitteilungen*, one can discern a recurrent methodological pattern.

When Schrödinger states or derives a wave equation, he always does so by considering the *type of entities* to which the wave equation should apply as well as the ideal conditions, *idealizations*, that the entities of this type should satisfy. So, for example, the first wave equation, (5-M1), is stated for the hydrogen atom under the idealization that no external forces act on it and the velocity of its electron is much smaller than the velocity of light. The same method is applied by Schrödinger in the formulation of the wave equation for the oscillator, (22-M2). Equation (22-M2) is stated for an oscillator under the following three idealizations: from all the possible N dimensions, $N - 1$ is set to zero; the oscillator is a non-relativistic one, and it is not subject to the action of a magnetic force.

According to Nowak (1971, 1972, 1977, Nowak, Nowakowa 2000), the triplet consisting of *equation*, *type of entities*, and *idealizations* is constitutive of the structure of scientific laws. Following Nowak, we can express this structure like this:

(1) $L^{(k)}$: $\forall x[Cmod_1 x = 0 \;\&\; Cmod_2 x = 0 \;\&\; \ldots \;\&\; Cmod_k x = 0 \rightarrow Fx = f_k(H_1 x, H_2 x,\ldots, H_r x)]$,

where "x" refers to the individual variable ranging over a universe of discourse that specifies the type of entities for which the law is stated. "$F = f_k (H_1, H_2,\ldots, H_r)$" stands for an equation expressing the functional relation between quantity F and quantities $H_1, H_2,\ldots, H_r$ ($r \geq 1$). "$Cmod_i = 0$" expresses the idealizing assumption that a quantity, expressed as "$Cmod_i$" ($i \geq 1$), is equal to zero. The idealization assumptions as expressed in (1) enable us to understand the relation between the equation with the structure of "$F = f_k (H_1, H_2,\ldots, H_r)$" and the type of entities for which the scientific law with the structure of (1) is stated — namely, that the equation holds for idealized types of entities.[14]

As I will show, once this quantity is at work (i.e., different from zero), it modifies F; hence, I label it a *modification condition*. The symbol "k" expresses the number of idealizing assumptions, so that "$L^{(k)}$" stands for the scientific law L under k idealizing assumptions.

Based on the structure (1), it is possible to reconstruct the structure of the scientific laws in which the wave equations stated by Schrödinger are embeded; I start with structures for the wave equations (5-M1), (18″-M2), (22-M2), and (29-M2). As I will show, based on structure (1), it is also possible to reconstruct the methods by which the laws with their respective wave equations are derived in the four *Mitteilungen*.

The law for (5-M1) is stated for hydrogen atoms (the range of x) under the following three idealizations: $v/c = 0$, where v is the velocity of the electron and c is the velocity of light; no external force is acting on the atom, $F_E = 0$; and the potential $V = -e^2/r$ is not an explicit function of time, $\partial V/\partial t = 0$.[15] The law is expressed as

(2) $$L^{(3)} : \forall x[\frac{vx}{c} = 0 \;\&\; F_E x = 0 \;\&\; \frac{\partial Vx}{\partial tx} = 0 \rightarrow \Delta\psi x + \frac{2mx}{K^2}\left(E + \frac{e^2 x}{rx}\right)\psi x = 0].$$

The law in which (18″-M2) is embedded is stated for entities (the range of x) whose trajectories have dimensions and radii of curvature that are compara-

[14] Paradigmatic examples of such idealized entities in physics include the ideal (mathematical) pendulum in classical mechanics and the ideal gas in thermodynamics.

[15] Here and in the following reconstructions of scientific laws, the symbol "x" stands for the individual variable and should not be confused with the physical coordinate.

ble to a certain wavelength. Its three idealizations are the same as those in (2). We thus obtain the following:

(3) $L^{(3)} : \forall x[\frac{vx}{c} = 0 \;\&\; F_E x = 0 \;\&\; \frac{\partial Vx}{\partial tx} = 0 \rightarrow \text{divgrad}\,\psi x + \frac{8\pi^2}{h^2}(Ex - Vx)\,\psi x = 0]$.

It is readily seen that two elements of (2) underwent a transformation when we moved to (3). Compared to the type of entities for which (2) is stated (hydrogen atoms), the type of entities invoked in (3) is specified in such a *new manner* that it covers other types of entities besides hydrogen atoms. So the transformation of the type of entities that took place in the movement from (2) to (3) is that of a generalization.

A generalization can also be discerned in the transformation of the equation in (3) into the equation in (2). The potential V in law (3) covers not only the potential — e^2/r given in law (2) but also other types of potentials. Thus, the derivation of law (3) from law (2) is produced by the method of *generalization*.

For Planck's oscillator with equation (22-M2), the structure of the scientific law is as follows:

(4) $L^{(3)} : \forall x[\frac{vx}{c} = 0 \;\&\; Mx = 0 \;\&\; Dx = 0 \rightarrow \frac{d^2\psi x}{dq^2 x} + \frac{8\pi^2}{h^2}(Ex - 2\pi^2 v_0 x q^2 x)\,\psi x = 0]$.

x ranges over the set of oscillating objects whose radii of curvature of their trajectories and the dimensions of their trajectories are comparable to a certain wavelength. Three idealizations involved here are: $v/c = 0$; the external magnetic field is equal to zero, $M = 0$; and the 2nd, .., Nth dimension of the oscillator is equal to zero, $D = 0$.

Finally, the scientific law for the rotator with a fixed axis is as follows:

(5) $L^{(3)} : \forall x[\frac{vx}{c} = 0 \;\&\; Mx = 0 \;\&\; CDx = 0 \rightarrow \frac{1}{Ax}\frac{d^2\psi x}{d\varphi^2 x} + \frac{8\pi^2}{h^2}\psi x = 0]$.

x ranges over the set of rotating objects whose radii of curvature of their trajectories and the dimensions of their trajectories are comparable to a certain wavelength. The idealizations are: $v/c = 0$; the external magnetic field is equal to zero, $M = 0$; and the change of direction of its axis is equal to zero, $CD = 0$.

By comparing the structure of laws (4) and (5) with that of (3), one can discover yet another method of derivation of scientific laws being at work in the second *Mitteilung* — namely, the method of *applying* a scientific law to particular types of entities.[16] What underlies this method is a *bidirectional* movement. One movement is the *thought-transformation of the types of entities for which the derived laws are to be stated so that they can be subsumed under* the type of entities for which the applied law is fulfilled. In the case of the oscillator and the rotator, both these types of entities are transformed in the mind into entities that already satisfy the condition represented above as $s \approx \lambda_k$ and $R \approx \lambda_k$. The other movement, going in the opposite direction, is the *adjustment* of the equation from the applied law to the specific features of the entities for which the respective derived laws are to be stated. In the case of entities for which laws (4) and (5) are stated, it is the transformation of equation (18″-M2) embedded in law (3).[17]

A reconstruction of the structure of scientific laws in terms of (1) reveals yet another method that is used in the *Mitteilungen*: the method of *explanation*. This method consists in gradually abolishing the idealizations — a process called *gradual concretization* — and modifying of the functional dependence of quantity *F*.

The modification conditions play a central role in the process of explanation by gradual concretization. Once these conditions are at work, the idealizations are cancelled — symbolized as "$Cmod_i \neq 0$" — they gradually modify the functional dependence between the quantities $H_1, H_2,\ldots, H_r$ and the quantity F as it is initially stated in the scientific law with the structure of $L^{(k)}$. One derives a sequence of different functional dependencies, $f_{k-1},\ldots, f_0$, between the quantities $H_1, H_2,\ldots, H_r$ and the quantity F thereby yielding different computations of F. The result is a derivation of a sequence of scientific laws that can be expressed as follows:

$$(6)\quad \begin{array}{l} L^{(k-1)}: (x)[Cmod_1x \neq 0 \;\&\; Cmod_2x = 0 \;\&\; \ldots \;\&\; Cmod_kx = 0 \rightarrow Fx = \\ \quad f_{k-1}(H_1x,\ldots, H_rx, Cmod_1x)] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L^{(0)}: (x)[Cmod_1x \neq 0 \;\&\; \ldots \;\&\; Cmod_kx \neq 0 \rightarrow Fx = f_0(H_1x,\ldots, H_rx, \\ \quad Cmod_1x, Cmod_kx)]. \end{array}$$

A specific feature of explanation by gradual concretization is that, in the course of it, the type of entities for which these laws are stated is constant; that is, the range of x for the laws in the sequence $L^{(k)}, L^{(k-1)},\ldots, L^{(0)}$ remains the same. This, as I will show now, holds for the derivation of the law in which the wave equation from the third *Mitteilung* is embedded.

[16] On the crucial role of applications in the development of quantum mechanics into a finished canonical theory, see (James, Joas 2015).

[17] On subsumption in Nowak's approach to scientific explanation, see, e.g., (Hanzel 2007; 2015: 11-13).

2.2. THE DERIVATION OF SCIENTIFIC LAWS IN *MITTEILUNG 3* AND *MITTEILUNG 4*

In the third *Mitteilung*, which is central from the point of view of the structure of scientific laws and the method of explanation described above, Schrödinger takes up the Stark-effect for the hydrogen atom — that is, the shifting and splitting of its spectral lines due to the impact of an external electric field with the force F_{EE} acting on the electron with charge e in the positive direction of axis z. The starting point is equation (5-M1) for the unperturbed hydrogen atom with its corresponding scientific law with the structure of $L^{(3)}$, stated above as (2). The wave equation that Schrödinger states for the Stark-effect is as follows:[18]

$$(32\text{-M3}) \qquad \Delta\psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r} - eF^*\right)\psi = 0.$$

Thus, Schrödinger cancels the idealization stated in $L^{(2)}$ that no external forces act on the atom, $F_E \neq 0$. Yet, at the same time, he introduces another idealization, namely — that no external magnetic force F_{EM} is acting on the hydrogen atom. This equation should hold under the idealizations: $v/c = 0$ and $F_{EM} = 0$, while already taking into account the impact of the modification condition that an external electric force is present, $F_{EE} \neq 0$. The scientific law in which (32-M3) is embedded is as follows:

$$(7) \qquad L^{(3)} : \forall x[\frac{vx}{c} = 0 \; \& \; F^*x = 0 \; \& \; \frac{\partial Vx}{\partial tx} = 0 \; \& \; F_{EE}x \neq 0 \rightarrow \Delta\psi x + \frac{8\pi^2 mx}{h^2}\left(Ex + \frac{e^2 x}{rx} - exF^*x\right)\psi x] = 0.$$

By comparing the structure of law (7) with that of (2), it becomes apparent that the former is derived from the latter by the method of gradual concretization sketched above, but at the same time transcends the abstract scheme (5) because, in the course of the derivation, *while some idealizations are being cancelled, new idealizations are also being introduced.*

The derivation of scientific laws by the method of application and the method of generalization can also be identified in the fourth *Mitteilung*. Here, Schrödinger restates equation (18″-M2) as:

$$(1\text{-M4}) \qquad \Delta\psi - \frac{2(E-V)}{E^2}\frac{\partial^2\psi}{\partial t^2} = 0.$$

[18] F^*z stands for the action of an external electric force exerted on the electron along axis z; my notation for the external electric force differs slightly from that of Schrödinger.

But even if he regards this equation as part of the new foundation of mechanics, he voices two objections against it. The first objection is that it does not state the law of change (*Veränderungsgesetz*) for ψ generally enough. The problem here is that (1-M4) is valid only for processes that depend on time solely via a certain periodic parameter. This then determines the type of the function ψ for which this equation is (for a definite value of E) valid — it should be $\psi \approx e^{\pm 2\pi i E/h}$.

The second objection to (1-M4) pertains to the supposition on which it was based — namely, that the potential V should not be an explicit function of time, or, in other words, that $\partial V/\partial t = 0$ should hold. This idealization has to be given up because, according to Schrödinger (1926e: 110),

> There exists, however, an urgent need to extend the theory to *nonconservative* systems, because only in this way it is possible to study the behavior of the system under the impact of pregiven external forces, for example of a light wave or of another atom flying past.

Schrödinger tries to remedy this by deriving the following equation:[19]

(4″-M4) $$\Delta\psi - \frac{8\pi^2}{h^2}V\psi \mp \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0.$$

The derivation of this wave equation enables Schrödinger to consider a special type of a time-dependent potential — namely, that a time-independent potential V_0 is disturbed by a small function of time (and the coordinates). He introduces the formula:

(5-M4) $$V = V_0(x) + r(x, t).$$

The disturbance $r(x, t)$ he considers is an oscillating variable field; the following holds for the case of linearly polarized monochromatic light with frequency ν:

(5′-M4) $$V = V_0(x) + A(x)\cos 2\pi\nu t.$$

Schrödinger then substitutes this formula into the equation (4″-M4) and derives the equation:

(7-M4) $$\Delta\psi - \frac{8\pi^2}{h^2}(V_0 + A(x)\cos 2\pi\nu t)\psi \mp \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0.$$

[19] Like in the second *Mitteilung*, the operator Δ has to be understood in the context of a non-Euclidean metric.

From the methodological point of view, by applying the reconstruction of the structure of scientific laws and of the method of application, it is possible to understand the derivation of equation (7-M4) from (4″-M4) as part of the derivation of a scientific law from another scientific law. The latter has the structure:

$$(8)\quad L^{(1)}: \forall x[\frac{vx}{c} = 0 \;\&\; F_{E}x \neq 0 \;\&\; \frac{\partial Vx}{\partial tx} \neq 0 \rightarrow \Delta\psi x - \frac{8\pi^2}{h^2}Vx\psi x \mp \frac{4\pi i}{h}\frac{\partial \psi x}{\partial tx} = 0],$$

where x ranges over a set of objects whose radii of curvature of their trajectories and the dimensions of their trajectories are comparable to a certain wavelength.

From it one derives, as an *application*, the law with the embedded equation (7-M4):[20]

$$(9)\quad L^{(1)}: \forall x[\frac{vx}{c} = 0 \;\&\; F_{E}x \neq 0 \;\&\; \frac{\partial Vx}{\partial tx} \neq 0 \rightarrow$$
$$\Delta\psi x - \frac{8\pi^2}{h^2}(V_0 x + Ax\cos 2\pi vxtx)\psi x \mp \frac{4\pi i}{h}\frac{\partial \psi x}{\partial tx} = 0].$$

Law (8) is a two-fold *generalization* of law (2). The first generalization concerns to the type of entities involved — (2) applies to hydrogen atoms whereas (8) applies to a set of objects whose radii of curvature of their trajectories and the dimensions of their trajectories are comparable to a certain wavelength. The second generalization concerns the type of the ψ function — while, in (2), the ψ function depends on time just via factor $e^{2\pi i v t}$ while, in (8), it is already inherently dependent on time.

Finally, let me address another generalization, which Schrödinger initially labels as the "relativistic-magnetic generalization of the basic equation (4″[-M4])" (1926e: 132). After deriving this generalization, he adds that it is "the conjectured relativistic generalization of (4″[-M4]) for the case of *a single electron*" (1926e: 133-134; emphasis added), which he views as directly applicable to the hydrogen atom.

Thus, the law in which that generalized equation is embedded is stated for the hydrogen atom (the range of x), where the velocity of its electron is $v \approx c$ and so it is acted upon by an external electromagnetic field; the conditions represented in (7) as F_{EE} and F_{EM} are now supposed to be both at work, $F_{EE} \neq 0$ and $F_{EM} \neq 0$. The scientific law is as follows:[21]

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[20] In order not to confuse the coordinate-variable with the individual variable, I eliminate the symbol for the former when using the symbol "A" for the amplitude.

[21] V and $\mathfrak{A}$ are the electromagnetic potentials of the external electromagnetic field at the

(10) $$L^{(0)}: \forall x[vx \approx c \ \&\ \mathrm{F}_{\mathrm{EM}}x \neq 0 \rightarrow \frac{1}{c^2}\frac{\partial^2 \psi x}{\partial t^2 x} \mp \frac{4\pi i e}{hc}(\frac{Vx}{c}\frac{\partial \psi x}{\partial tx} + \mathfrak{A}x \operatorname{grad} \psi x) + \frac{4\pi^2 e^2}{h^2 c^2}(V^2 x - \mathfrak{A}^2 x - \frac{m^2 x c^2 x}{e^2 x})\psi x = 0.$$

By comparing law (10) with law (8) we discover that condition $F_{\mathrm{E}} \neq 0$ in (8) is bifurcated into two conditions given in (10): $F_{\mathrm{EE}} \neq 0$ and $F_{\mathrm{EM}} \neq 0$.

The generalization involved in the movement from (8) to (10) pertains to the generalization of the equation, but not to the type of entities. While the former law is stated for x that ranges over a set of entities that fulfill the condition symbolized above as $s \approx \lambda_k$ and $R \approx \lambda_k$, the latter law is stated for x that ranges only over a subset of that set — namely, hydrogen atoms. Law (10) can also be viewed as a generalization of law (2), but here both these laws are stated for the same type of entities — hydrogen atoms.

CONCLUSION

Drawing on the reconstruction presented here of the structure of scientific laws combined with the reconstruction of the methods of generalization, application, and explanation based on scientific laws, Figure 2 indicates the methods that are foundational to Schrödinger's thought movements between the wave-equations and their corresponding laws in the framework of the respective *Mitteilung* and between *Mitteilungen*.[22]

location of the electron. The symbols of operators "Δ" and "grad" have a three-dimensional Euclidean meaning.

[22] The lower number refers to the respective scientific law as reconstructed above; the dotted arrows indicate the application of the law (3) to the rigid rotator with fixed axis and to the nonrigid rotator.

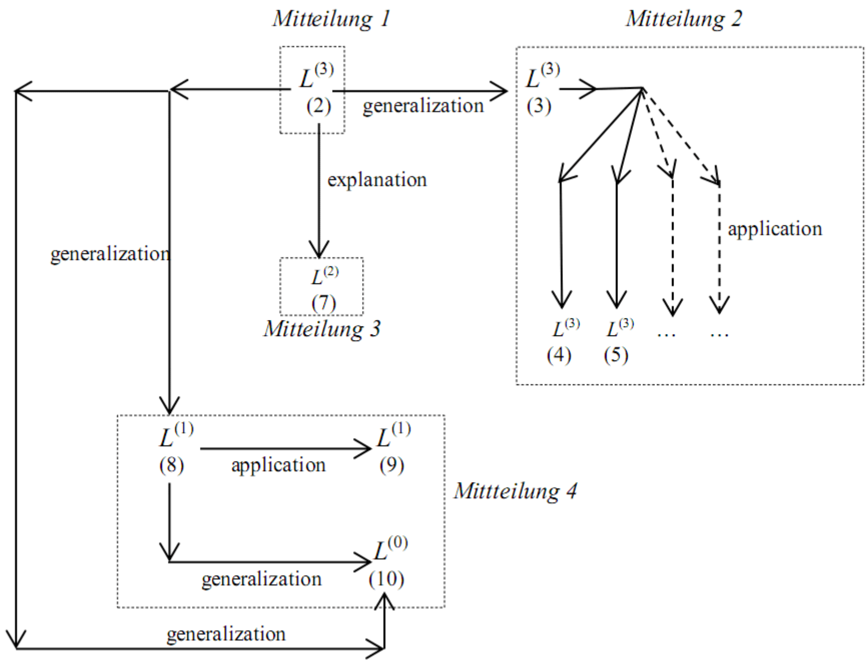

Fig. 2. Methods used by Schrödinger in *Mitteilungen 1-4*

Based on the presented analysis of the four *Mitteilungen*, it is possible to draw the following methodological conclusions.

First, the three elements regularly reappearing in the *Mitteilungen* — namely, the wave equation, the type of entities the wave equation should apply to, and the idealizations that this type of entities should fulfill are the basis for the methods of generalization, application, and explanation by which Schrödinger built these *Mitteilungen*. These methods serve to transform some or all of these elements.

Second, these three methods enabled Schrödinger to derive, as shown in Figure 2, a branching sequence of scientific laws, where the starting point of this sequence is law (2) for the hydrogen atom stated in the first *Mitteilung*. Thus, a reconstruction of these three methods can be viewed as a methodological explication of Schrödinger's term "generalization," which he employed at the beginning of the first *Mitteilung*, where he claimed that the "new conception is capable of generalization" (1926a: 361).

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