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Solving Systems of Linear Equations under Conditions of Uncertainty on the Example of the Leontief Model[2]

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Abstract: The paper presents various methods of solving systems of linear equations under conditions of uncertainty. In a situation when the parameters of such systems cannot be precisely determined with real numbers, they can be represented by interval numbers, fuzzy numbers or ordered fuzzy numbers. Solutions of systems of linear equations with such representations of parameters are shown in the example of Leontief input-output model. It has also been shown that when ordered fuzzy numbers are applied, their additional feature – orientation – can broaden and deepen economic analysis.

Keywords: Systems of linear equations, interval arithmetic, fuzzy numbers, ordered fuzzy numbers, Leontief model.

JEL Codes: C02, C30, D57

1 Introduction

Our surrounding economic reality is characterized by a very high degree of complexity. This makes it difficult to describe it with accurate, precise concepts and quantities. The inverse relationship between the precision of the description of the problems that occur in the world around us, and their complexity is best reflected by the principle of inconsistency formulated by the creator of fuzzy logic Lotfi Zadeh. Namely (Zadeh, 1973): 'as the complexity of the system increases, our ability to make precise and yet significant statements about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics'. As a result, to describe the phenomena occurring in economic reality, vague or broad concepts are often applied, which can be mathematically described using interval numbers or fuzzy numbers.

In many disciplines, including economics, modelling specific processes or phenomena often lead to systems of linear equations. There are many methods for solving such systems if their parameters are described with real numbers. However, describing reality with numerical values may not be sufficient in the modelling of many phenomena. The main reasons for this stem from the fact that measurements, evaluations or ratings have a degree of uncertainty and imprecision. The uncertainty of measurement results, among other things, from the so-called 'human error' or from the imperfection of measuring instruments. Also, estimations or evaluations made by experts are often subjective and can be perceived by others in various ways. This necessitates a return to the mathematical tools that allow to perform operations on imprecise and unreliable data. Such tools include, for instance, interval mathematics (arithmetic), which allows to present data when only their range is known, and fuzzy numbers, used to present subjective information or information expressed by means of natural language (linguistic variables).

In the literature, many papers can be found which deal with solving systems of linear equations with parameters represented by fuzzy numbers. Many papers include a fuzzy constant term vector (e.g., Friedman, Ma and Kandel, 1998; Ma, Friedman and Kandel, 1999), but also the main matrix of the system is fuzzified (e.g., Sevastjanov and Dymova, 2009). However, there are no papers in the literature dealing with solving systems of linear equations with parameters in the form of ordered fuzzy numbers. The present paper attempts to begin to fill this gap.

The purpose of the paper is to present selected methods of solving systems of linear equations whose parameters are imprecise or uncertain and expressed using interval numbers, fuzzy numbers or ordered fuzzy numbers. In addition, using the partition into strong and weak solutions for fuzzy numbers (Friedman, Ma and Kandel, 1998), we will give the conditions under which the solution of the system is strong in the case of interval numbers and fuzzy numbers.

The paper consists of five sections. Section 2 shows systems of linear equations and methods to solve them under conditions of uncertainty. Section 3 discusses the Leontief model, which in Section 4 forms the basis for numerical examples illustrating the methods presented in Section 2. The paper ends with a summary.

2 Systems of linear equations and methods of their solutions in conditions of uncertainty

2.1 Systems of linear equations with real parameters

A system of m linear equations with n unknowns $x_1, x_2, \ldots, x_n$ (an $m \times n$ system), where $m, n \in \mathbb{N}$, is a system of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

with the parameters $a_{ij}, b_i \in \mathbb{R}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. A solution of the system (1) is every sequence $\{x_1, x_2, \ldots, x_n\}$ of real numbers, which satisfies this system. In matrix notation, system (1) can be written as:

$$AX = B \tag{2}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{3}$$

Matrix A is called the coefficient matrix of system (1), matrix X – the matrix (column) of variables, and matrix B – the matrix (column) of constant terms.

Linear algebra provides many methods of solving systems of equations of the form (1), when their parameters a_{ij}, b_i, are real numbers. However, descriptions of the surrounding economic reality can lead to systems with unreliable or imprecise parameters, represented using interval numbers or fuzzy numbers. Such systems can be solved using interval mathematics (arithmetic) or fuzzy number arithmetic. In this paper we will consider systems of linear equations (1) whose parameters are interval numbers or fuzzy numbers.

2.2 Interval arithmetic

A closed (sharp) interval (interval number) is defined as follows (Moore, Kearfott and Cloud, 2009, 7–9):

$$A = \left[\underline{a}, \overline{a}\right] = \left\{x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a}\right\}. \tag{4}$$

In what follows, the term 'interval' will always denote a closed interval of the form (4). We say that two intervals A and B are equal when $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$. The midpoint of interval (4) is determined as follows:

$$\text{mid}(A) = \frac{\underline{a} + \overline{a}}{2}, \tag{5}$$

and its width:

$$w(A) = \overline{a} - \underline{a}. \tag{6}$$

In the present paper, the most common approach, the so-called 'naive' approach to the construction of interval arithmetic is used. Let A and B be intervals and let $\circledcirc$ be an arbitrary operation, then (Moore, Kearfott and Cloud, 2009, 10):

$$A \circledcirc B = [a \circledcirc b, a \in A \wedge b \in B]. \tag{7}$$

From that it follows that an arbitrary operation on intervals can be completed by performing the corresponding operations on their endpoints:

$$\begin{aligned} A \circledcirc B = [&\min\{\underline{a} \circledcirc \underline{b}, \underline{a} \circledcirc \overline{b}, \overline{a} \circledcirc \underline{b}, \overline{a} \circledcirc \overline{b}\}, \\ &\max\{\underline{a} \circledcirc \underline{b}, \underline{a} \circledcirc \overline{b}, \overline{a} \circledcirc \underline{b}, \overline{a} \circledcirc \overline{b}\}]. \end{aligned} \tag{8}$$

Consider one of the four basic operations $\{+,-,\cdot,/\}$. Then (Moore, Kearfott and Cloud, 2009, 11–13):

$$A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{9}$$

$$A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \tag{10}$$

$$\begin{aligned} A \cdot B = [&\min\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}, \\ &\max\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}], \end{aligned} \tag{11}$$

$$\begin{aligned} A/B = [&\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \\ &\max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}] \text{ if } 0 \in B. \end{aligned} \tag{12}$$

If interval B contains zero ($0 \in B$), division is more complicated. In interval arithmetic, the roles of zero and one are played by the real numbers 0 and 1, which can be expressed as degenerate intervals [0, 0] and [1, 1], respectively.

Economic quantities and variables such as production level, demand, supply, price and so on are positive. In this paper, interval numbers will be used in an economic model, the so-called Leontief model; so, it is appropriate to assume that the intervals considered are subsets of the positive half-axis, that is, interval $A = \left[\underline{a}, \overline{a}\right] \subset \mathbb{R}_+$ where $\underline{a} \in \mathbb{R}_+$ and $\overline{a} \in \mathbb{R}_+$. If interval $B \subset \mathbb{R}_+$, the operations of multiplication (11) and division (12) take the form:

$$A \cdot B = [\underline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}], \tag{13}$$

$$A/B = [\underline{a}/\overline{b}, \overline{a}/\underline{b}]. \tag{14}$$

2.3 Systems of linear equations with interval parameters

Consider a system of linear equations (1) with parameters a_{ij}, b_i expressed by means of intervals and with $m = n$. The system is then of the form:

$$\begin{cases} [\underline{a}_{11}, \overline{a}_{11}][\underline{x}_1, \overline{x}_1] + [\underline{a}_{12}, \overline{a}_{12}][\underline{x}_2, \overline{x}_2] + \cdots + [\underline{a}_{1n}, \overline{a}_{1n}][\underline{x}_n, \overline{x}_n] = [\underline{b}_1, \overline{b}_1] \\ [\underline{a}_{21}, \overline{a}_{21}][\underline{x}_1, \overline{x}_1] + [\underline{a}_{22}, \overline{a}_{22}][\underline{x}_2, \overline{x}_2] + \cdots + [\underline{a}_{2n}, \overline{a}_{2n}][\underline{x}_n, \overline{x}_n] = [\underline{b}_2, \overline{b}_2] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [\underline{a}_{n1}, \overline{a}_{n1}][\underline{x}_1, \overline{x}_1] + [\underline{a}_{n2}, \overline{a}_{n2}][\underline{x}_2, \overline{x}_2] + \cdots + [\underline{a}_{nn}, \overline{a}_{nn}][\underline{x}_n, \overline{x}_n] = [\underline{b}_n, \overline{b}_n] \end{cases}. \tag{15}$$

Assume that $[\underline{a}_{ij}, \overline{a}_{ij}] \subset \mathbb{R}_+$ and $[\underline{b}_{ij}, \overline{b}_{ij}] \subset \mathbb{R}_+$ $(i,j = 1, \dots, n)$. This allows to write system (15) as a $(2n \times 2n)$ system with real parameters:

$$\begin{cases} \underline{a}_{11}\underline{x}_1 + \underline{a}_{12}\underline{x}_2 + \cdots + \underline{a}_{1n}\underline{x}_n = \underline{b}_1 \\ \overline{a}_{11}\overline{x}_1 + \overline{a}_{12}\overline{x}_2 + \cdots + \overline{a}_{1n}\overline{x}_n = \overline{b}_1 \\ \underline{a}_{21}\underline{x}_1 + \underline{a}_{22}\underline{x}_2 + \cdots + \underline{a}_{2n}\underline{x}_n = \underline{b}_2 \\ \overline{a}_{21}\overline{x}_1 + \overline{a}_{22}\overline{x}_2 + \cdots + \overline{a}_{2n}\overline{x}_n = \overline{b}_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \\ \underline{a}_{n1}\underline{x}_1 + \underline{a}_{n2}\underline{x}_2 + \cdots + \underline{a}_{nn}\underline{x}_n = \underline{b}_n \\ \overline{a}_{n1}\overline{x}_1 + \overline{a}_{n2}\overline{x}_2 + \cdots + \overline{a}_{nn}\overline{x}_n = \overline{b}_n \end{cases}, \tag{16}$$

which can be partitioned into two subsystems:

$$\begin{cases} \underline{a}_{11}\underline{x}_1 + \underline{a}_{12}\underline{x}_2 + \cdots + \underline{a}_{1n}\underline{x}_n = \underline{b}_1 \\ \underline{a}_{21}\underline{x}_1 + \underline{a}_{22}\underline{x}_2 + \cdots + \underline{a}_{2n}\underline{x}_n = \underline{b}_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \\ \underline{a}_{n1}\underline{x}_1 + \underline{a}_{n2}\underline{x}_2 + \cdots + \underline{a}_{nn}\underline{x}_n = \underline{b}_n \end{cases}$$

and

$$\begin{cases} \overline{a}_{11}\overline{x}_1 + \overline{a}_{12}\overline{x}_2 + \cdots + \overline{a}_{1n}\overline{x}_n = \overline{b}_1 \\ \overline{a}_{21}\overline{x}_1 + \overline{a}_{22}\overline{x}_2 + \cdots + \overline{a}_{2n}\overline{x}_n = \overline{b}_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \\ \overline{a}_{n1}\overline{x}_1 + \overline{a}_{n2}\overline{x}_2 + \cdots + \overline{a}_{nn}\overline{x}_n = \overline{b}_n \end{cases}. \tag{17}$$

The solutions of subsystems (17) determine the lower and upper bounds $\underline{x}_i$ and $\overline{x}_i$, $(i = 1, \dots, n)$, of the intervals of the solutions of system (15). Assuming the partition introduced for the solutions of systems of linear equations with fuzzy parameters (Friedman, Ma and Kandel, 1998), the solutions of systems of the form (15) can be partitioned into two kinds. A solution is called a *strong solution*, if for every i the interval $[\underline{x}_i, \overline{x}_i]$ is of the form (4), that is, $\underline{x}_i \leq \overline{x}_i$. Otherwise, we call the solution a *weak solution*. System (15) has a strong solution if $\underline{A}^{-1}\underline{B} \leq \overline{A}^{-1}\overline{B}$, where $\underline{A}$, $\overline{A}$ are coefficient matrices, and $\underline{B}$ and $\overline{B}$ are columns of constant terms of subsystems (17).

Another method of presentation of imprecise data uses fuzzy numbers. Fuzzy numbers, as opposed to intervals (to which all elements belong in the same degree), allow for a variety of degrees of belonging of their individual elements, by means of the membership function.

2.4 Fuzzy numbers

The notions of a fuzzy set was introduced by Lotfi A. Zadeh in 1965 (Zadeh, 1965). A fuzzy set A in universe X is the set of pairs:

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A : X \to [0,1]\}, \tag{18}$$

where μ_A is the membership function of fuzzy set A, which assigns to each element $x \in X$ its degree of membership to fuzzy set A.

A number of characteristics, such as support, kernel, or α-cut, are related to a fuzzy set. The support of fuzzy set A is the set supp$A = \{x \in X: \mu_A(x) > 0\}$, the kernel of fuzzy set A is the set ker$A = \{x \in X: \mu_A(x) = 1\}$, and the α-cut of fuzzy set A is the set $A^\alpha = \{x \in X: \mu_A(x) \geq \alpha\}$, for $\alpha \in [0, 1]$. The following implication holds for α-cuts: if $\alpha < \beta$ then $A^\beta \subset A^\alpha$ and the following decomposition theorem holds (Łachwa, 2001, 32–33).

Theorem 1. Every fuzzy set A can be presented in the form:

$$A = \bigcup_{\alpha \in [0,1]} \alpha A^\alpha \tag{19}$$

where αA^α is the fuzzy set with the membership function:

$$\mu_{\alpha A^\alpha}(x) = \begin{cases} \alpha & \text{for} \quad x \in A^\alpha \\ 0 & \text{for} \quad x \notin A^\alpha \end{cases}. \tag{20}$$

A convex fuzzy number (CFN) A, or a fuzzy number for short, is a fuzzy set defined in the universe of real numbers ($X = \mathbb{R}$), with the following properties (Łachwa, 2001, 91):

- A is normal, that is, $\exists x \in \mathbb{R}: \mu_A(x) = 1$.
- A is convex, that is, $\forall x_1, x_2 \in \mathbb{R}$, $\forall \lambda \in [0, 1]$: $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$.
- The membership function of A is piecewise continuous.

Practical applications of fuzzy numbers show that the most commonly used are triangular fuzzy numbers (which are also used in this paper). Such numbers can be uniquely described using three real numbers a, b, and c ($a \leq b \leq c$), that is:

$$A = (a, b, c) \tag{21}$$

and their membership function is of the form (Fig. 1):

$$\mu_A(x) = \begin{cases} \dfrac{x - a}{b - a} & \text{for} \quad a \leq x \leq b \\ \dfrac{c - x}{c - b} & \text{for} \quad b \leq x \leq c \end{cases}. \tag{22}$$

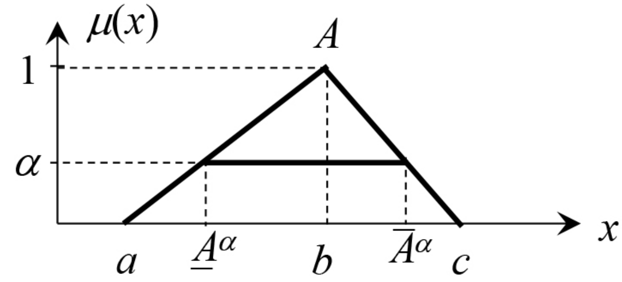

Fig. 1. Membership function of the triangular fuzzy number A = (a, b, c) and α-cut

Source: Author's own elaboration.

Fuzzy number $A = (a, b, c)$ for which $0 \le a \le b \le c$ is called a positive fuzzy number.

Consider a triangular positive fuzzy number $A = (a, b, c)$. The endpoints of the α-cut, for some $\alpha \in [0, 1]$, are determined from the following equations (Fig. 1):

$$\alpha = \frac{\underline{A}^\alpha - a}{b - a} \text{ and } \alpha = \frac{c - \overline{A}^\alpha}{c - b}. \tag{23}$$

We obtain the interval:

$$A^\alpha = [\underline{A}^\alpha, \overline{A}^\alpha] = [a + \alpha(b - a), c - \alpha(c - b)]. \tag{24}$$

Operations on fuzzy numbers of the form (21) can be reduced to operations on intervals, using Theorem 1 on decomposition.

Commonly used is also another notation of fuzzy numbers, the so-called parametric representation, similar to α-cuts (24). In this representation, a triangular fuzzy number has the form of an ordered pair (Goetschel and Voxman, 1986):

$$\left(\underline{A}(\alpha), \overline{A}(\alpha)\right) = \left(a + \alpha(b - a), c - \alpha(c - b)\right), \alpha \epsilon [0,1]. \tag{25}$$

Equality, sum and product of fuzzy numbers A and B in parametric form, as well as product of a positive real number $r \in \mathbb{R}_+$ and a fuzzy number A are of the form (Friedman, Ma and Kandel, 1998; Ma, Friedman and Kandel, 1999):

- $A = B \Leftrightarrow \underline{A}(\alpha) = \underline{B}(\alpha) \wedge \overline{A}(\alpha) = \overline{B}(\alpha)$,
- $A + B = \left(\underline{A}(\alpha) + \underline{B}(\alpha), \overline{A}(\alpha) + \overline{B}(\alpha)\right)$,
- $AB = \left(\underline{AB}, \overline{AB}\right)$,
  where
  $\underline{AB} = \min\{\underline{A}(\alpha)\underline{B}(\alpha), \underline{A}(\alpha)\overline{B}(\alpha), \overline{A}(\alpha)\underline{B}(\alpha),$
  $\overline{A}(\alpha)\overline{B}(\alpha)\}$,
  $\overline{AB} = \max\{\underline{A}(\alpha)\underline{B}(\alpha), \underline{A}(\alpha)\overline{B}(\alpha), \overline{A}(\alpha)\underline{B}(\alpha),$
  $\overline{A}(\alpha)\overline{B}(\alpha)\}$,
- $r \cdot A = \left(r\underline{A}(\alpha), r\overline{A}(\alpha)\right)$.

2.5 Systems of linear equations with fuzzy parameters

To solve system (1), with parameters expressed by triangular positive fuzzy numbers we decompose them into α-cuts (24). Then for every $\alpha \in [0, 1]$, we obtain a system with interval parameters:

$$\begin{cases} [\underline{a}_{11}^\alpha, \overline{a}_{11}^\alpha][\underline{x}_1^\alpha, \overline{x}_1^\alpha] + [\underline{a}_{12}^\alpha, \overline{a}_{12}^\alpha][\underline{x}_2^\alpha, \overline{x}_2^\alpha] + \cdots \\ [\underline{a}_{21}^\alpha, \overline{a}_{21}^\alpha][\underline{x}_1^\alpha, \overline{x}_1^\alpha] + [\underline{a}_{22}^\alpha, \overline{a}_{22}^\alpha][\underline{x}_2^\alpha, \overline{x}_2^\alpha] + \cdots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [\underline{a}_{n1}^\alpha, \overline{a}_{n1}^\alpha][\underline{x}_1^\alpha, \overline{x}_1^\alpha] + [\underline{a}_{n2}^\alpha, \overline{a}_{n2}^\alpha][\underline{x}_2^\alpha, \overline{x}_2^\alpha] + \cdots \end{cases}$$

$$\begin{matrix} + [\underline{a}_{1n}^\alpha, \overline{a}_{1n}^\alpha][\underline{x}_n^\alpha, \overline{x}_n^\alpha] = [\underline{b}_1^\alpha, \overline{b}_1^\alpha] \\ + [\underline{a}_{2n}^\alpha, \overline{a}_{2n}^\alpha][\underline{x}_n^\alpha, \overline{x}_n^\alpha] = [\underline{b}_2^\alpha, \overline{b}_2^\alpha] \\ \dots\dots\dots\dots\dots\dots\dots \\ + [\underline{a}_{nn}^\alpha, \overline{a}_{nn}^\alpha][\underline{x}_n^\alpha, \overline{x}_n^\alpha] = [\underline{b}_n^\alpha, \overline{b}_n^\alpha] \end{matrix}. \tag{26}$$

Analogously as for intervals, system (26) can be written as two subsystems with real parameters whose solutions are represented by lower and upper endpoints $\underline{x}_i^\alpha$ and $\overline{x}_i^\alpha$ $(i = 1, \dots, n)$, respectively, for each α-cut x_i^α. Theorem 1 on decomposition allows to determine the solution of system (1) with parameters expressed by triangular positive fuzzy numbers.

In the literature, systems of linear equations of the form (1), with real numbers a_{ij} and fuzzy numbers b_i, are often considered. Many authors use the so-called parametric form of fuzzy numbers (Friedman, Ma and Kandel, 1998; Ma, Friedman and Kandel, 1999).

Moreover, Friedman, Ma and Kandel (1998) define the solution of such systems as a vector of fuzzy numbers

$$u = \left\{\left(\underline{u}_i(\alpha), \overline{u}_i(\alpha)\right), i = 1, \dots, n\right\},$$

where $\underline{u}_i(\alpha) = \min\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}$,

$\overline{u}_i(\alpha) = \max\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}$.

The constant function $\underline{x}_i$ (1) is used to eliminate triangular fuzzy numbers with angles greater than 90 degrees. Furthermore, the authors define two types of solutions of these systems. They call a solution u a *strong fuzzy solution* if $\underline{u}_i(\alpha) = \underline{x}_i(\alpha)$ and $\overline{u}_i(\alpha) = \overline{x}_i(\alpha)$ for $i = 1, \dots, n$. Otherwise, they call it a *weak fuzzy solution*.

In the α-cut notation, system (1) with parameters expressed as triangular positive fuzzy numbers has a strong solution if it satisfies the following conditions (*) and (**):

(*) $\forall \alpha \in [0, 1]$

$$\left(\underline{A}^{\alpha}\right)^{-1}\underline{B}^{\alpha} \leq \left(\overline{A}^{\alpha}\right)^{-1}\overline{B}^{\alpha}$$

(**) $\forall \alpha, \beta \in [0, 1]$ such that $\alpha \leq \beta$

$$\left(\underline{A}^{\alpha}\right)^{-1}\underline{B}^{\alpha} \leq \left(\underline{A}^{\beta}\right)^{-1}\underline{B}^{\beta} \leq \left(\overline{A}^{\beta}\right)^{-1}\overline{B}^{\beta} \leq \left(\overline{A}^{\alpha}\right)^{-1}\overline{B}^{\alpha},$$

where $\underline{A}^{\alpha}$, $\overline{A}^{\alpha}$ are the coefficient matrices, and $\underline{B}^{\alpha}$ and $\overline{B}^{\alpha}$ are the columns of constant terms of the subsystems of system (26).

2.6 Imperfections of interval arithmetic and fuzzy numbers

Because of the specific character of operations on interval numbers and on fuzzy numbers, subtraction is not an operation opposite to addition ($A - A = A + (-A) \neq 0$), nor is division inverse to multiplication

$(A \cdot A^{-1} = A \cdot \dfrac{1}{A} \neq 0)$,

where A is an interval number or a fuzzy number. As a result, one cannot use the elimination method to solve equations or systems of equations, which restricts the use of interval numbers or fuzzy numbers in modelling.

An additional flaw of interval arithmetic and fuzzy number arithmetic is that the width of the interval numbers and the fuzziness (support) of fuzzy numbers increase rapidly as the number of arithmetic operations grows. It can therefore happen that after a number of operations, the resulting interval number or fuzzy number support will be so wide that the information represented by it is not very useful.

Because of such imperfections, among other things, various modifications of the presented methods emerge whose main goal is to overcome these limitations. One of the tools created with this in mind is the model with ordered fuzzy numbers.

2.7 Ordered fuzzy numbers

In 2002, Professor Witold Kosiński together with two collaborators, Piotr Prokopowicz and Dominik Ślęzak, in a series of papers (Kosiński, Prokopowicz and Ślęzak, 2002; Kosiński, Prokopowicz and Ślęzak, 2003; Kosiński and Prokopowicz, 2004; Kosiński, 2006), suggested a model of fuzzy numbers called *Ordered Fuzzy Numbers* (OFN),[3] which is an extension of the model of fuzzy numbers.

An ordered fuzzy number A is an ordered pair (f_A, g_A) of continuous functions such that $f_A, g_A\colon[0, 1]\to\mathbb{R}$. The elements of ordered fuzzy number A are called: f_A – *UP* and g_A – *DOWN*. To preserve agreement with the CFN model, we denote by y the independent variable of functions f_A and g_A, and by x, their values (dependent variable) (Fig. 2a). From the continuity of f_A and g_A, it follows that their images are bounded intervals, denoted by UP_A and $DOWN_A$, respectively (Fig. 2a), whose endpoints are: $UP_A = [f_A(0), f_A(1)]$ and $DOWN_A = [g_A(1), g_A(0)]$. Taking into account the condition of normality of fuzzy numbers on the set $[f_A(1), g_A(1)] = CONST_A$, we include the constant function equal to 1. The interval $[f_A(0), g_A(0)]$ is then the support of the ordered fuzzy number A, while the set $[f_A(1), g_A(1)]$ is its kernel.

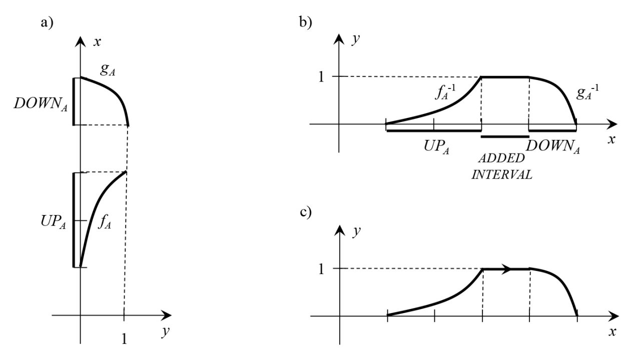

Fig. 2. a) Ordered fuzzy number , b) OFN presented in relation to CFN, c) Arrow showing the order of the reverse functions and the orientation of

Source: (Kosiński and Prokopowicz, 2004).

In the definition of an ordered fuzzy number A, there is no requirement for functions f_A and g_A to be invertible, only to be continuous. But if we assume, additionally, that f_A is increasing and g_A, decreasing, and that $f_A(y) \leq g_A(y)$ for $\forall y \in [0, 1]$, we can define a membership

3 After the death of Professor Kosiński, to commemorate him and his contribution to the development of the OFN model, instead of the term "*Ordered Fuzzy Numbers*" the term "*Kosiński Fuzzy Numbers*" (KFN) is used (Prokopowicz and Pedrycz, 2015).

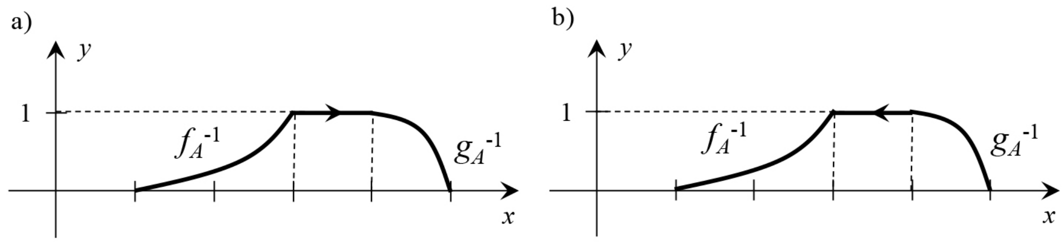

Fig. 3. a) OFN with positive orientation, b) OFN with negative orientation
Source: (Kacprzak, 2010).

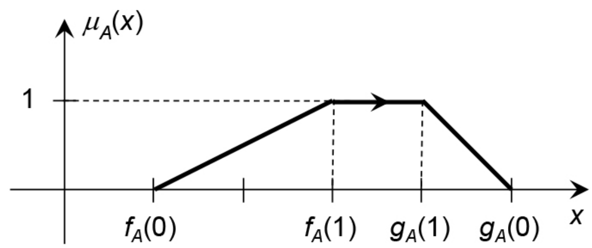

Fig. 4. An example of an OFN together with its characteristic points
Source: (Kacprzak, 2010).

function μ_A of the ordered fuzzy number A as follows (Fig. 2b) (Kosiński, 2006):

$$\mu_A(x) = \begin{cases} f_A^{-1}(x) & \text{for} \quad x \in UP_A \\ 1 & \text{for} \quad x \in CONST_A \\ g_A^{-1}(x) & \text{for} \quad x \in DOWN_A \end{cases}. \tag{27}$$

Fig. 2c shows the ordered fuzzy number A with an arrow – its orientation – which shows the order of the reverse functions f_A and g_A, that is, the orientation of the ordered fuzzy number A. The pair of continuous functions (g_A, f_A) determines a fuzzy number different from that determined by the pair (f_A, g_A). This is shown graphically by their orientation (Fig. 3). This allows to partition the set of ordered fuzzy numbers into two subsets: ordered fuzzy numbers with positive orientations, if their orientation is the same as that of the $0x$ axis (Fig. 3a) and ordered fuzzy numbers with negative orientation otherwise (Fig. 3b).

Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ be OFNs. The arithmetic operation $\circledcirc$ on ordered fuzzy numbers $C = A \circledcirc B$ is defined as follows:

$$\forall y \in [0, 1]\ [f_C(y) = f_A(y) \circledcirc f_B(y) \text{ and } g_c(y) = g_A(y) \circledcirc g_B(y)] \tag{28}$$

where $\circledcirc \in \{+, -, \cdot, /\}$, A/B is defined when $\forall y \in [0, 1]\ |f_B(y)| > 0$ and $|g_B(y)| > 0$.

Real numbers are special case of ordered fuzzy numbers and are identified with pairs of constant functions. Let $r \in \mathbb{R}$ and let r' be a constant function, that is, $r'(y) = r$ for $y \in [0, 1]$. Then $r^* = (r', r')$ is an ordered fuzzy number representing the real number r. This allows to define the multiplication of the real number r by the ordered fuzzy number $A = (f_A, g_A)$ as follows:

$$\forall y \in [0, 1]\ [r \cdot A = (r \cdot f_A(y), r \cdot g_A(y))]. \tag{29}$$

Four real numbers occur in the definition of the membership function $\mu_A(x)$ (27) of the ordered fuzzy number A, namely $f_A(0)$, $f_A(1)$, $g_A(1)$ and $g_A(0)$. If functions f_A and g_A are linear, then these four numbers uniquely describe the ordered fuzzy number A, which can be written as follows (Fig. 4):

$$A = (f_A(0), f_A(1), g_A(1), g_A(0)). \tag{30}$$

In the case when $f_A(1) < g_A(1)$, this OFN will be called a trapezoidal ordered fuzzy number; if $f_A(1) = g_A(1)$, we have a triangular ordered fuzzy number. Representation (30) allows to quickly perform basic operations on OFNs. Let $A = (f_A(0), f_A(1), g_A(1), g_A(0))$ and $B = (f_B(0), f_B(1), g_B(1), g_B(0))$ be ordered fuzzy numbers, and let $r \in \mathbb{R}$. Addition and subtraction of A and B, and multiplication of A by r are determined as follows:

$$A \pm B = (f_A(0) \pm f_B(0), f_A(1) \pm f_B(1), g_A(1) \pm g_B(1), g_A(0) \pm g_B(0)), \tag{31}$$

$$r \cdot A = (r \cdot f_A(0), r \cdot f_A(1), r \cdot g_A(1), r \cdot g_A(0). \tag{32}$$

2.8 Systems of linear equations with OFN parameters

As in the case of CFNs, system (1) with parameters expressed by positive OFNs can be solved using α-cuts, by means of an arithmetic analogous to interval arithmetic, partitioning the system into two subsystems with real parameters. This arithmetic based on α-cuts is more intuitive and similar to operations on real numbers. Additionally, in the case of OFNs, α-cuts can be improper intervals (that is, intervals $[a, b]$ with $a > b$). Using (28), we can define arithmetic operations on α-cuts as follows:

$$\forall\alpha\in[0,1]\ [f_A(\alpha), g_A(\alpha)] \circledcirc [f_B(\alpha), g_B(\alpha)] = [f_A(\alpha) \circledcirc f_B(\alpha), g_A(\alpha) \circledcirc g_B(\alpha)] \tag{33}$$

where $\circledcirc \in \{+,-,\cdot,/\}$.

As in the case of fuzzy numbers, for the solution of system (1) with OFN parameters to be a strong solution, the following conditions (*) and (**) have to be satisfied:

(*) $\forall\alpha\in[0, 1]$

$(f_A(\alpha))^{-1} f_B(\alpha) \le (g_A(\alpha))^{-1} g_B(\alpha)$

(**) $\forall\alpha, \beta\in[0, 1]$ such that $\alpha \le \beta$

$(f_A(\alpha))^{-1} f_B(\alpha) \le (f_A(\beta))^{-1} f_B(\beta) \le (g_A(\beta))^{-1} g_B(\beta) \le (g_A(\alpha))^{-1} g_B(\alpha)$

or

(*) $\forall\alpha\in[0, 1]$

$(f_A(\alpha))^{-1} f_B(\alpha) \ge (g_A(\alpha))^{-1} g_B(\alpha)$

(**) $\forall\alpha, \beta\in[0, 1]$ such that $\alpha \le \beta$

$(f_A(\alpha))^{-1} f_B(\alpha) \le (f_A(\beta))^{-1} f_B(\beta) \le (g_A(\beta))^{-1} g_B(\beta) \le (g_A(\alpha))^{-1} g_B(\alpha)$

where $f_A(\alpha)$, $g_A(\alpha)$ are the coefficient matrices, and $f_B(\alpha)$ and $g_B(\alpha)$ are the columns of constant terms of the subsystems of system (1) with parameters expressed by positive ordered fuzzy numbers.

Note that in the OFN model, for any ordered fuzzy number, there is an opposite number with respect to addition and an inverse number with respect to multiplication (if the divisor of the support does not contain zero); so in this model, systems of equations can be solved by methods of linear algebra.

The presented methods for solving systems of equations with imprecise parameters will be discussed using the Leontief model.

3 The Leontief model

The Leontief model, also known in the literature as the input-output model, was created by American scholar Wassily Leontief, a laureate of the Nobel Prize in Economics in 1973 'for the development of the input-output method and for its application to important economic problems'.

This model allows to describe and analyse complex economic systems. It is based on the observation that an economy consists of many production sectors whose activities are mutually related. These relationships result from the fact that the output of some sectors is used as an input to other sectors. In addition, part of the output is used to meet the needs of end users (i.e., for consumption, investments or inventory stocking).

The Leontief model provides an answer to the question: what should be the output of each sector of the economy, to satisfy the demand from the sectors themselves and from the household sector? It also allows to analyse changes in the structure of the production, resulted from changes in the demand of the household sector or from the output volume of one of the sectors. As such analysis usually includes many sectors and has a fairly complicated structure, for simplicity, we make certain assumptions (Chiang, 1994, 127; Gawinecki, 2000, 98):

- each sector produces only one homogeneous product,
- each sector uses in its production the inputs of one factor (or of the combination of factors in a fixed ratio),
- the output of each sector is characterized by constant returns to scale, so a k-fold change in all the inputs will cause an exactly k-fold change of the output.

Suppose that an economy consists of n production sectors (we are considering an open model that contains an 'open' sector such as households). We introduce the following notation (expressed in monetary units):

- X_i $(i = 1, \dots, n)$ is the size of total (global) output of the i-th sector,
- x_{ij} $(i,j = 1, \dots, n)$ is that part of the output of the i-th sector that is used for the needs of the production in the j-th sector,
- d_i $(i = 1, \dots, n)$ is the final output of the j-th sector (the difference between the total output of the i-th sector and its inputs to all the sectors).

Tab. 1. Input-output table

Sector		inputs-outputs				Final output	Total output
		j					
		1	2	...	*n*		
i	1	x_{11}	x_{12}	...	x_{1n}	d_1	X_1
	2	x_{21}	x_{22}	...	x_{2n}	d_2	X_2
	⋮	⋮	⋮	⋱	⋮	⋮	⋮
	n	x_{n1}	x_{n2}	...	x_{nn}	d_n	X_n

Source: Author's own elaboration based on (Gruszczyński and Podgórska, 2004, 371).

The starting point of the Leontief model is an economic balance in the form of an input-output table (Tab. 1) prepared so as to allow quantification of the interconnections between the individual parts of the system. This table contains numerical data characterizing business operations in a certain period. Since the total output of the i-th sector ($i = 1, ..., n$) is the sum of the inputs-outputs and the final output, we obtain the following system of balance equations:

$$\begin{cases} X_1 = x_{11} + x_{12} + \cdots + x_{1n} + d_1 \\ X_2 = x_{21} + x_{22} + \cdots + x_{2n} + d_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ X_n = x_{n1} + x_{n2} + \cdots + x_{nn} + d_n \end{cases}. \tag{34}$$

The demand for outputs of the i-th sector needed for the production of the j-th sector is determined by the coefficients:

$$a_{ij} = \frac{x_{ij}}{X_j} \; (i,j = 1, \dots, n) \tag{35}$$

called cost coefficients. Coefficients a_{ij} admit values from the interval [0, 1] and are interpreted as follows: to obtain, in the j-th sector, a total output worth one monetary unit, it is necessary to use the output of the i-th sector worth a_{ij} monetary units. From (35), we obtain:

$$x_{ij} = a_{ij} X_j \; (i,j = 1, \dots, n) \tag{36}$$

which allows to write system (34) in the following form:

$$\begin{cases} X_1 = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + d_1 \\ X_2 = a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n + d_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ X_n = a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + d_n \end{cases}. \tag{37}$$

This, in turn, allows to write the system of balance equations (37) in matrix form:

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \tag{38}$$

or, briefly, as:

$$X = AX + d \tag{39}$$

where X denotes the matrix (vector) of the total (global) output, A the matrix of cost coefficients, and d the matrix (vector) of the final output.

Equation (39) is written in the form of a Leontief model (I is the $n \times n$ identity matrix):

$$X - AX = d \leftrightarrow (I - A)X = d. \tag{40}$$

Matrix $(I - A)$ is called the Leontief matrix; it transforms the vector X of total output into vector d of final output.

Immediately, the question arises: can we reverse the situation and, knowing the vector d of final output, determine the vector X of total output? To answer this question, we introduce the notion of a productive matrix. Matrix A of cost coefficients is called productive if there exists a non-negative vector X of total output such that $X > AX$. From an economic point of view, this means that there must exist at least one vector of total output for which the total output exceeds the production consumption (inputs-outputs). If such a vector did not exist, this would mean that the economy is not able to produce, in each sector, more than it consumes for the current production. For this reason, in real economy, we can assume that matrix A is productive. We will apply now the following two theorems (Czerwiński, 1980, 312–314).

Theorem 2. If matrix A is productive, then the Leontief matrix $(I - A)$ is a non-singular matrix.

Theorem 3. If matrix A is productive, then all the elements of matrix $(I - A)^{-1}$ are non-negative.

From Theorem 2, it follows directly that in an actual economy, the final output d uniquely determines the total output X according to the rule:

$$X = (I - A)^{-1} d. \tag{41}$$

Additionally, from Theorem 3 we obtain that for any non-negative final output vector d, the obtained total output vector X is also non-negative.

The Leontief model can be applied for short-term prediction of the future value of the final output vector or the total output vector, assuming that the elements of the matrix A of cost coefficients are constant in time. In practice, it is assumed that this matrix is identical to the one obtained from the balance of the previous period. Forecasts calculated using the Leontief model can be divided into three types:

- Forecast of type 1 – determination of the final output for a specific future total output vector based on model (40);
- Forecast of type 2 – determination of the total output, which allows to achieve the desired final output on the basis of model (41);
- Mixed forecast – determination of selected elements of the total output and the final output vectors, when the other elements of those vectors are known.

The Leontief model is homogeneous and additive. From its additivity we obtain:

$$(I - A)(X + \Delta X) = (I - A)X + (I - A)\Delta X = d + \Delta d \qquad (42)$$

that is,

$$(I - A)\Delta X = \Delta d \qquad (43)$$

and

$$(I - A)^{-1}\Delta d = \Delta X. \qquad (44)$$

Relationships (43) and (44) allow to determine the increment of the final output vector based on a given increment of the total output vector and conversely, without taking into account the original value of the total or final output.

The application of the Leontief model to forecasting of the total output or the final output is based on the assumption that these quantities are measurable and can be written as real numbers. But in an actual economy, some economic quantities are difficult to measure. Precise determination of the demand from the household sector for the output of a particular sector may be impossible, because it is saddled with uncertainty and subject to constant fluctuations. Also, the level of total output is prone to all kinds of interference and signals coming from the economy, which makes it difficult to precisely describe its value. In this situation, we can turn to interval numbers and fuzzy numbers, which allow for a mathematical description of unreliable and imprecise quantities.

4 Examples of a Leontief model with uncertain parameters

In this section, we present numerical examples of methods for solving systems of linear equations with uncertain or imprecise parameters presented in the second section. We will base our discussion on the Leontief model of a system consisting of two sectors, for which we will make predictions of the second type.

Example 1: A Leontief model with interval parameters.

Consider a system in which the cost coefficient and the final output matrices are of the form:

$$\tilde{A} = \begin{pmatrix} [0.1388, 0.1396] & [0.0321, 0.0327] \\ [0.0214, 0.0216] & [0.0400, 0.0402] \end{pmatrix}$$

and

$$\tilde{d} = \begin{pmatrix} [23305, 24325] \\ [22125, 23105] \end{pmatrix}.$$

Substituting matrices $\tilde{A}$ and $\tilde{d}$ into equation (40), we obtain a matrix equation of the form:

$$\begin{pmatrix} [0.8604, 0.8612] & [-0.0327, -0.0321] \\ [-0.0216, -0.0214] & [0.9598, 0.9600] \end{pmatrix} \begin{pmatrix} [\underline{x}, \overline{x}] \\ [\underline{y}, \overline{y}] \end{pmatrix} =$$

$$= \begin{pmatrix} [23305, 24325] \\ [22125, 23105] \end{pmatrix},$$

that gives a system of equations with interval coefficients, which according to (17) can be written as two subsystems with real coefficients, of the form:

$$\begin{cases} 0.8604\underline{x} - 0.0327\overline{y} = 23305 \\ -0.0214\underline{x} + 0.9600\overline{y} = 23105 \end{cases}$$

and

$$\begin{cases} 0.8612\overline{x} - 0.0321\underline{y} = 24325 \\ -0.0216\overline{x} + 0.9598\underline{y} = 22125 \end{cases}.$$

After having solved the subsystems, we obtain the endpoints of the total output vector:

$$\tilde{X} = \begin{pmatrix} [\underline{x}, \overline{x}] \\ [\underline{y}, \overline{y}] \end{pmatrix} = \begin{pmatrix} [28024.7, 29129.1] \\ [23707.2, 24692.4] \end{pmatrix}.$$

Note that the solution obtained is a strong solution.

Example 2: A Leontief model with interval parameters reduced to a point.

Consider the data from Example 1, in which the elements of matrices $\tilde{A}$ and $\tilde{d}$ are reduced to midpoints of the interval as in (5), that is, they are of the form:

$$A = \begin{pmatrix} 0.1392 & 0.0324 \\ 0.0215 & 0.0401 \end{pmatrix} \text{ and } d = \begin{pmatrix} 23815 \\ 22615 \end{pmatrix}$$

We want to check whether the solution is also reduced to the midpoint. Substituting matrices A and d into equation (40), we obtain a matrix equation of the form:

$$\begin{pmatrix} [0.8608,0.8608] & [-0.0324,-0.0324] \\ [-0.0215,-0.0215] & [0.9599,0.9599] \end{pmatrix} \begin{pmatrix} [\underline{x},\overline{x}] \\ [\underline{y},\overline{y}] \end{pmatrix} =$$

$$= \begin{pmatrix} [23815,23815] \\ [22615,22615] \end{pmatrix}$$

which generates two subsystems with real coefficients:

$$\begin{cases} 0.8608\underline{x} - 0.0324\overline{y} = 23815 \\ -0.0215\underline{x} + 0.9599\overline{y} = 22615 \end{cases}$$

and

$$\begin{cases} 0.8608\overline{x} - 0.0324\underline{y} = 23815 \\ -0.0215\overline{x} + 0.9599\underline{y} = 22615 \end{cases}.$$

Solving them, we obtain a solution reduced to the midpoints of the interval from Example 1:

$$\tilde{X} = \begin{pmatrix} [\underline{x},\overline{x}] \\ [\underline{y},\overline{y}] \end{pmatrix} = \begin{pmatrix} [28577,28577] \\ [24199.8,24199.8] \end{pmatrix}.$$

Example 3: A Leontief model with parameters expressed as fuzzy numbers.

Consider again the data from Example 1, in which the elements of matrices $\tilde{A}$ and $\tilde{d}$ are written as triangular fuzzy numbers, with the interval numbers from Example 1 being the supports, and the midpoint of the interval numbers, as in (5), being the kernels, that is:

$$\tilde{A} = \begin{pmatrix} (0.1388,0.1392,0.1396) & (0.0321,0.0324,0.0327) \\ (0.0214,0.0215,0.0216) & (0.0400,0.0401,0.0402) \end{pmatrix}$$

and

$$\tilde{d} = \begin{pmatrix} (23305,23815,24325) \\ (22125,22615,23105) \end{pmatrix}.$$

Using α-cuts we obtain, according to (23), for each :

$$\tilde{A}^\alpha = \begin{pmatrix} [0.1388 + 0.0004\alpha, 0.1396 - 0.0004\alpha] & [0.0321 + 0.0003\alpha, 0.0327 - 0.0003\alpha] \\ [0.0214 + 0.0001\alpha, 0.0216 - 0.0001\alpha] & [0.0400 + 0.0001\alpha, 0.0402 - 0.0001\alpha] \end{pmatrix}$$

and

$$\tilde{d}^\alpha = \begin{pmatrix} [23305 + 510\alpha, 24325 - 510\alpha] \\ [22125 + 490\alpha, 23105 - 490\alpha] \end{pmatrix},$$

which allows us to write the Leontief model for any $\alpha \in [0, 1]$ in the form

$$(I - \tilde{A}^\alpha)\tilde{X}^\alpha = \tilde{d}^\alpha$$

where

$$\tilde{X}^\alpha = \begin{pmatrix} [\underline{x}^\alpha, \overline{x}^\alpha] \\ [\underline{y}^\alpha, \overline{y}^\alpha] \end{pmatrix}.$$

The solution of α-cut $\tilde{X}^\alpha$ is as follows:

$$\tilde{X}^\alpha = \begin{pmatrix} \left[\frac{50000(192\alpha^2 + 928630\alpha + 46256667)}{-\alpha^2 + 30111\alpha + 82528422}, \frac{50000(192\alpha^2 - 929398\alpha + 48114695)}{-\alpha^2 - 30107\alpha + 82588640}\right] \\ \left[\frac{50000(290\alpha^2 - 799379\alpha - 39158940)}{\alpha^2 + 30107\alpha - 82588640}, \frac{50000(290\alpha^2 + 798219\alpha - 40756538)}{\alpha^2 - 30111\alpha - 82528422}\right] \end{pmatrix},$$

and in the parametric form (25):

$$\tilde{X}^\alpha = \begin{pmatrix} \left(\frac{50000(192\alpha^2 + 928630\alpha + 46256667)}{-\alpha^2 + 30111\alpha + 82528422}, \frac{50000(192\alpha^2 - 929398\alpha + 48114695)}{-\alpha^2 - 30107\alpha + 82588640}\right) \\ \left(\frac{50000(290\alpha^2 - 799379\alpha - 39158940)}{\alpha^2 + 30107\alpha - 82588640}, \frac{50000(290\alpha^2 + 798219\alpha - 40756538)}{\alpha^2 - 30111\alpha - 82528422}\right) \end{pmatrix},$$

$\alpha \in [0, 1]$.

Note that the solution obtained is strong.

In this example, as the starting point for the construction of fuzzy numbers in matrices $\tilde{A}$ and $\tilde{d}$, we used the interval numbers from Example 1. We used the same ranges of the parameters, but we assigned various degrees of membership to the individual elements, using the membership function. Because of that, when $\alpha = 0$, that is, when we use the supports of the fuzzy numbers of the solution, they are at the same time the solution from Example 1, while when $\alpha = 1$, that is, when we use the kernels of the solution, we obtain the situation discussed in Example 2.

Example 4: A Leontief model with parameters expressed as ordered fuzzy numbers.

We continue to use the data from Example 1, where the elements of matrices $\tilde{A}$ and $\tilde{d}$ are represented as triangular ordered fuzzy numbers, with the interval numbers from Example 1 as the supports, and the midpoint of the interval numbers as the kernels, according to (5). Consider a situation in which vector $\tilde{d}$ consists of OFNs with positive orientation. If the elements of matrix $\tilde{A}$ also have positive orientation, then we obtain the following solution of α-cut $\tilde{X}^\alpha$:

$$\tilde{X}^\alpha = \left(\begin{array}{l} \left[\frac{50000(192\alpha^2 + 1019272\alpha + 46166025)}{\alpha^2 - 47975\alpha + 82606506}, \frac{50000(192\alpha^2 - 1020040\alpha + 48205337)}{\alpha^2 + 47971\alpha + 82510560}\right] \\ \left[\frac{50000(-290\alpha^2 + 852765\alpha + 39105554)}{\alpha^2 - 47975\alpha + 82606506}, \frac{50000(-290\alpha^2 - 851605\alpha + 40809924)}{\alpha^2 + 47971\alpha + 82510560}\right] \end{array} \right),$$

while when the elements of matrix $\tilde{A}$ have negative orientation, the solution of α-cut $\tilde{X}^\alpha$ is:

$$\tilde{X}^\alpha = \left(\begin{array}{l} \left[\frac{50000(-192\alpha^2 + 1002428\alpha + 46183253)}{\alpha^2 + 47971\alpha + 82510560}, \frac{50000(-192\alpha^2 - 1001660\alpha + 48187341)}{\alpha^2 - 47975\alpha + 82606506}\right] \\ \left[\frac{50000(290\alpha^2 + 878263\alpha + 39079476)}{\alpha^2 + 47971\alpha + 82510560}, \frac{50000(290\alpha^2 - 879423\alpha + 40837162)}{\alpha^2 - 47975\alpha + 82606506}\right] \end{array} \right).$$

The corresponding solutions in the form of ordered fuzzy numbers are, respectively:

$$\tilde{X} = \left(\begin{array}{l} \left(\frac{50000(192\alpha^2 + 1019272\alpha + 46166025)}{\alpha^2 - 47975\alpha + 82606506}, \frac{50000(192\alpha^2 - 1020040\alpha + 48205337)}{\alpha^2 + 47971\alpha + 82510560}\right) \\ \left(\frac{50000(-290\alpha^2 + 852765\alpha + 39105554)}{\alpha^2 - 47975\alpha + 82606506}, \frac{50000(-290\alpha^2 - 851605\alpha + 40809924)}{\alpha^2 + 47971\alpha + 82510560}\right) \end{array} \right),$$

$\alpha \in [0, 1]$.

and

$$\tilde{X} = \left(\begin{array}{l} \left(\frac{50000(-192\alpha^2 + 1002428\alpha + 46183253)}{\alpha^2 + 47971\alpha + 82510560}, \frac{50000(-192\alpha^2 - 1001660\alpha + 48187341)}{\alpha^2 - 47975\alpha + 82606506}\right) \\ \left(\frac{50000(290\alpha^2 + 878263\alpha + 39079476)}{\alpha^2 + 47971\alpha + 82510560}, \frac{50000(290\alpha^2 - 879423\alpha + 40837162)}{\alpha^2 - 47975\alpha + 82606506}\right) \end{array} \right),$$

$\alpha \in [0, 1]$.

Fig. 5 shows the fuzzy numbers (CFNs and OFNs) of the total output vector obtained in Examples 3 and 4. The smallest fuzziness (the most accurate result measured using the support) has been obtained when fuzzy numbers and interval arithmetic were used. The application of ordered fuzzy numbers results in slightly wider intervals of the support.

Example 5: Application of the orientation of OFNs in the Leontief model.

The additional property of OFNs, orientation, allows to take into account additional information. The elements of the matrix of parameters can be interpreted as follows: $f(0)$ is a value from the current period, while $g(0)$ is the forecast value. Then the orientation provides information on the predicted direction of the change (an increase when $f(0) < g(0)$ or a decrease when $f(0) > g(0)$).

Using OFNs from Example 4, we can analyse the level and the direction of changes in total output caused by a given level and direction of changes in the final output.[4] Assume that $f_d(0)$ describes the level of final output in the current period, while $g_d(0)$, the forecast value in the upcoming period. Then, the sign of the difference $f_d(0) - g_d(0)$ (corresponding to the orientation of d) determines an increase (+) or a decrease (–) of the final output, while the value $|f_d(0) - g_d(0)|$ determines the level of the change. For simplicity, we assume that matrix A consists of real numbers representing the midpoints of the interval numbers from Example 1, that is:

$$A = \begin{pmatrix} 0.1392 & 0.0324 \\ 0.0215 & 0.0401 \end{pmatrix} \Rightarrow (I - A)^{-1} =$$
$$= \begin{pmatrix} 1.1627 & 0.0392 \\ 0.0260 & 1.0426 \end{pmatrix},$$

4 An application of the OFN model to the Leontief model can be found in Kacprzak (2008; 2010; 2017).

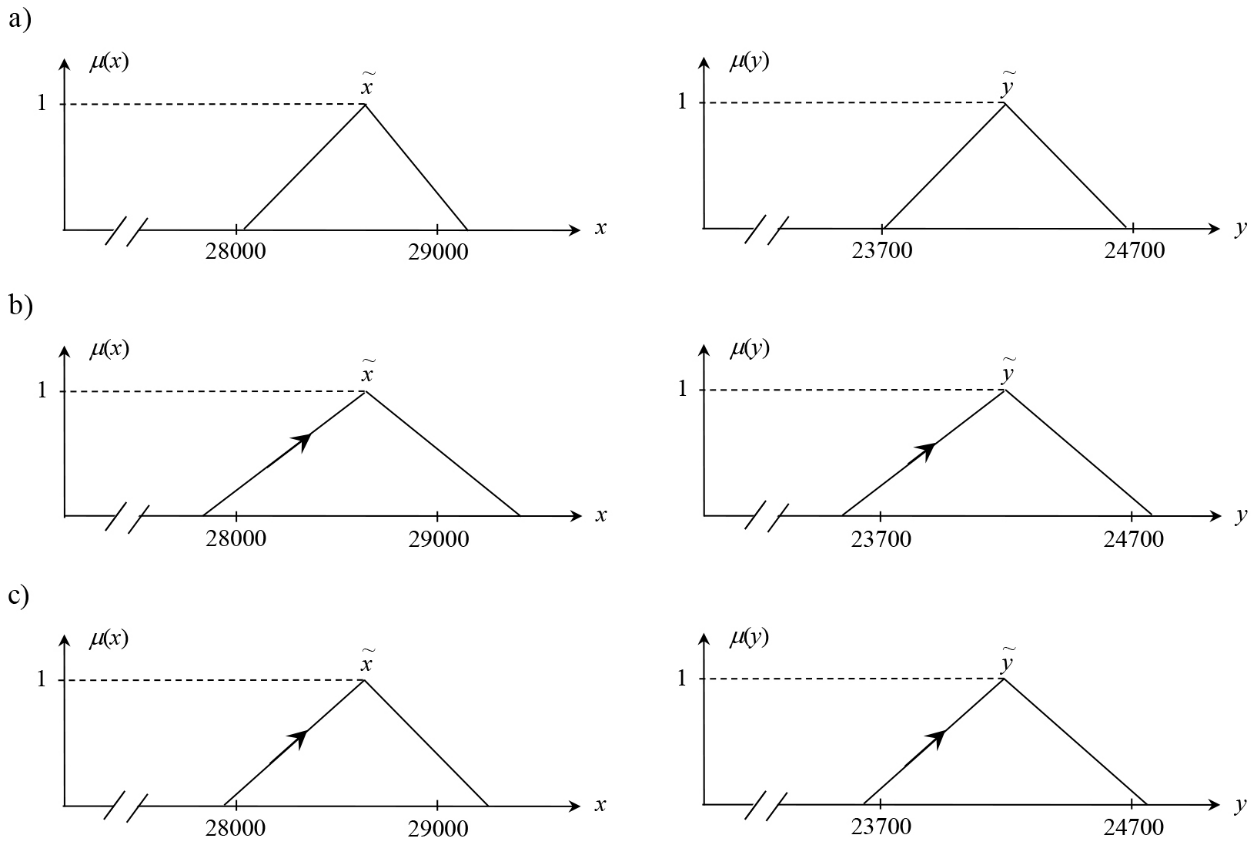

Fig. 5. Results obtained in Examples 3 and 4: a) constructed on a CFN, b) based on positively oriented OFNs, c) based on negatively oriented OFNs

Source: Author's own elaboration

while the elements of vector $\tilde{d}$ are represented by ordered fuzzy numbers:

$$\tilde{d} = \begin{pmatrix} (23305 + 510\alpha, 24325 - 510\alpha) \\ (22125 + 490\alpha, 23105 - 490\alpha) \end{pmatrix}.$$

Tab. 2 lists the results of the calculation of the total output levels $\tilde{X}$ corresponding to various directions of change of final output $\tilde{d}$ as well as the values of these changes $\Delta d = f_d(0) - g_d(0)$ and $\Delta X = f_X(0) - g_X(0)$.

Row 1 in Table 2 illustrates a situation in which the forecast of the final output level is identical to its current value. Therefore, there is no reason for the total output level to change. Row 2 shows that the final output level in the second sector will increase by about 980, and that in the first one, it will remain unchanged. This will result in an increase of the total output levels in both sectors. In the second sector, there will be an increase in the output by 1021.7, which exceeds the growth level of the final output by 41.7. This additional increase of the output is an essential component of the production growth of the first sector, which must meet the growing demand on the part of the second sector. Row 3 shows the effect of the change in the direction of the final output level (in this case, a decrease of demand) in the second sector. Rows 4 and 5 are analogous to rows 2 and 3 with the changes in the final output level occurring in the first sector. In row 6, the effect of the increase of the final output of the two sectors can be seen. This will result in an increase in the total output of both sectors, with the cumulation of three effects: an increase of the final output, an increase in demand for own production, and an increase in demand on the part of the other sector. In row 7, we have an increase of the final output level in the first sector and a decrease of this level in the second. In the first sector, it will result in an increase of the output level, but in a smaller degree than in row 6, reflecting a decrease in demand on the part of the other sector. In the second sector, however, there will be a decrease in the output level greater than that caused by the decrease in the final output. This is an effect of a larger decrease

Tab. 2. Total output level as dependent on the direction of the changes of the final output level

Lp.	$\tilde{d}$	$\tilde{X}$	Δd	ΔX
1.	$\begin{pmatrix}(23815,23815,23815)\\(22615,22615,22615)\end{pmatrix}$	$\begin{pmatrix}(28576.2,28576.2,28576.2)\\(24197.6,24197.6,24197.6)\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$
2.	$\begin{pmatrix}(23815,23815,23815)\\(22125,22615,23105)\end{pmatrix}$	$\begin{pmatrix}(28557,28576.2,28595.4)\\(23686.7,24197.6,24708.5)\end{pmatrix}$	$\begin{pmatrix}0\\980\end{pmatrix}$	$\begin{pmatrix}38.4\\1021.7\end{pmatrix}$
3.	$\begin{pmatrix}(23815,23815,23815)\\(23105,22615,22125)\end{pmatrix}$	$\begin{pmatrix}(28557,28576.2,28595.4)\\(23686.7,24197.6,24708.5)\end{pmatrix}$	$\begin{pmatrix}0\\-980\end{pmatrix}$	$\begin{pmatrix}-38.4\\-1021.7\end{pmatrix}$
4.	$\begin{pmatrix}(23305,23815,24325)\\(22615,22615,22615)\end{pmatrix}$	$\begin{pmatrix}(27983.2,28576.2,29169.2)\\(24184.3,24197.6,24210.9)\end{pmatrix}$	$\begin{pmatrix}1020\\0\end{pmatrix}$	$\begin{pmatrix}1186\\26.5\end{pmatrix}$
5.	$\begin{pmatrix}(24325,23815,23305)\\(22615,22615,22615)\end{pmatrix}$	$\begin{pmatrix}(27983.2,28576.2,29169.2)\\(24184.3,24197.6,24210.9)\end{pmatrix}$	$\begin{pmatrix}-1020\\0\end{pmatrix}$	$\begin{pmatrix}-1186\\-26.5\end{pmatrix}$
6.	$\begin{pmatrix}(23305,23815,24325)\\(22125,22615,23105)\end{pmatrix}$	$\begin{pmatrix}(27964,28576.2,29188.4)\\(23673.5,23197.6,24721.7)\end{pmatrix}$	$\begin{pmatrix}1020\\980\end{pmatrix}$	$\begin{pmatrix}1224.4\\1048.3\end{pmatrix}$
7.	$\begin{pmatrix}(23305,23815,24325)\\(23105,22615,22125)\end{pmatrix}$	$\begin{pmatrix}(28002.4,28576.2,29150)\\(24695.2,24197.6,23700)\end{pmatrix}$	$\begin{pmatrix}1020\\-980\end{pmatrix}$	$\begin{pmatrix}1147.5\\-995.2\end{pmatrix}$
8.	$\begin{pmatrix}(24325,23815,23305)\\(22125,22615,23105)\end{pmatrix}$	$\begin{pmatrix}(29150,28576.2,28002.4)\\(23700,24197.6,24695.2)\end{pmatrix}$	$\begin{pmatrix}-1020\\980\end{pmatrix}$	$\begin{pmatrix}-1147.5\\995.2\end{pmatrix}$
9.	$\begin{pmatrix}(24325,23815,23305)\\(23105,22615,22125)\end{pmatrix}$	$\begin{pmatrix}(29188.4,28576.2,27964)\\(24721.7,24197.6,23673.5)\end{pmatrix}$	$\begin{pmatrix}-1020\\-980\end{pmatrix}$	$\begin{pmatrix}-1224.4\\-1048.3\end{pmatrix}$

Source: Author's own elaboration.

in the demand for own production (a_{22} = 0.0401) than the increase requested by the first sector (a_{21} = 0.0215). Rows 8 and 9 show a situation reversed with respect to rows 7 and 6, respectively.

5 Summary

In this paper, methods of solving systems of linear equations with uncertain or imprecise parameters have been presented. Such parameters can be mathematically described using, for instance, interval numbers, fuzzy numbers or ordered fuzzy numbers. By dividing the solutions into two types, strong and weak, we gave conditions that should be satisfied for the solution to be of the specified type. Numerical examples of these methods have been shown on the example of the Leontief model.

From an economic point of view, of particular interest may be the model of ordered fuzzy numbers. These numbers allow not only to mathematically model and process (as in the case of real number arithmetic) imprecise or unreliable quantities, but can also enrich and simplify economic analysis. Using the orientation, the support and possibly also the kernel and α-cuts, we can easily draw conclusions about the results caused by specific changes of input values, as shown in Example 5, on the basis of the Leontief model.

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