HEDGING OF EQUITY-LINKED CONTRACT WITH A MAXIMAL SUCCESS FACTOR*

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Abstract: The author considers an equity-linked contract whose payoff depends on the lifetime of the policy holder and the stock price, and assumes the limited capital for hedging and provides the best strategy for an insurance company in the meaning of the so-called success factor $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}} + \mathbf{1}_{\{V_T < D\}} \frac{V_T}{D}]$, where V_T denotes the end value of the strategy and D is the payoff of the contract. The study is a generalisation of the work by Föllmer and Schied (2004), and Klusik and Palmowski (2011), but it considers the much more general 'incompleteness' of the market, among others, midterm nonmarket information signals and infinite nonmarket scenarios.

Keywords: quantile hedging, equity-linked contract.

1. Introduction

This study considers an incomplete financial market. On such a market the seller of the contract can superhedge it. This is a very expensive, but worst-case-scenario resistant strategy. A very good example here is the case of unit-linked insurance products, where the payoff, a function of price of some financial index, is paid only if the insured is alive at some specified date. The safest strategy is to be prepared for the case where everyone insured survives, but it creates costs. Competition forces insurance companies to assume some risk, as it is very likely (especially in large cohorts) that some of the insured die before maturity.

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One way of controlling the risk of the implemented strategy is the so-called quantile hedging. Classically one assumes some allowed capital and implements the strategy maximizing the probability of satisfying all the claims $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}}]$, where D denotes the contingent claim and V_T denotes the final value of the hedging portfolio. Equivalently, one can fix the probability of successful hedging and look for the cheapest strategy. However, this approach can be criticised because it leaves the unsuccessful scenarios completely out of control. That is why the expected success ratio criterion $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}} + \mathbf{1}_{\{V_T < D\}} \frac{V_T}{D}]$ is also being used.

Föllmer and Leukert (1999) investigated the general semi-martingale setting, and they pointed out the optimal strategy for a complete market with maximal $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}}]$ and proved the existence of such a strategy maximizing the expected successful ratio in an incomplete market. The proof was based on various versions of the Neymann-Pearson lemma.

Spivak and Cvitanic (1999) studied a complete market framework of assets modelled with Ito processes. They constructed a strategy with maximal $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}}]$. They also implemented this technique for a market with partial observations. Finally, they considered the case where the drift of the wealth process is a nonlinear (concave) function of the investment strategy of the agent.

Sekine (2000) considered defaultable securities in an incomplete market, where the security-holder can default at some random time and receives a payoff modelled by the martingale process. The study (2000) shows a strategy maximizing the probability of a successful hedge.

Klusik, Palmowski, Zwierz (2010) solved the problem of quantile hedging from the point of view of a better informed agent acting on the market. The additional knowledge of the agent was modelled by a filtration initially enlarged by some random variable.

The above approaches concentrated on complete market frameworks, which could not be used for equity- linked contracts. The payoff of equity-linked policies is a function of two random factors: the price of the stock or the financial index (hence the term equity-linked), and some insurance-type (i.e. nonmarket) event in the life of the owner of the contract (death, retirement, survival to a certain date, etc.). As such, the payoff depends on both financial and insurance risk elements, which have to be priced so that the resulting premium is fair to both the seller and the buyer of the contract. There are very few methods providing an appropriate risk management in connection with such contracts which exploit some imperfect hedging forms, as the mortality risk makes the market incomplete.

The maximum success factor was proposed by Klusik and Palmowski (2011). They considered an equity-linked product where the insurance event can take a finite number of states and is independent of the financial asset modelled with the geometric Brownian motion. They constructed an optimal strategy for both maximal probability and maximal expected success ratio. In their framework, the knowledge about the insurance event was not revealed before the maturity.

Equity-linked policies have been studied since the 1970s (see Boyle and Schwartz, 1977; Brennan and Schwartz, 1976; Delbaen, 1986). Some later works were by Bacinello (Bacinello and Ortu, 1993), Aase and Persson (1994), Ekern and Persson (1996), Boyle and Hardy (1997), Bacinello (2001), Moeller (2001), Melnikov (2004a, 2004b), Melnikov and Romanyuk (2008).

In this paper the author states a general problem of optimizing success factor $E^{\mathbb{P}}[\mathbf{1}_{\{V_T\geq D\}} + \mathbf{1}_{\{V_T<D\}}\frac{V_T}{D}]$ in an incomplete market, as in Klusik and Palmowski (2011), but allowing for a very general flow of information outside the market. The study found an optimal strategy using a geometric approach.

The paper is organized as follows. Section 2 introduces a model of the financial market and the structure of the considered insurance product. The author also formulates and gives a solution of both problems of hedging. Section 3 provides the proof of the main result. To make this section clearer, the author moved one technical proof (proof of Lemma 3.3) outside the text, to the Appendix.

2. Mathematical model and investment problem

Let us consider a discounted price process $\mathrm{X} = (\mathrm{X}_{\mathrm{t}})_{\mathrm{t}\in[0,\mathrm{T}]}$ being a semi-martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}^{\mathrm{X}} = \left(\mathcal{F}_{\mathrm{t}}^{\mathrm{X}}\right)_{\mathrm{t}\in[0,\mathrm{T}]}$ and assume that there is unique equivalent martingale measure on $\mathcal{F}_T^X$ denoted by $\mathbb{R}$.

Let us consider a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and a fixed sequence $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and assume that $t \in (t_i, t_{i+1})$ holds the following equality $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_{t_i}$. The interpretation is as follows: the knowledge modelled by $\mathcal{F}$ could be augmented by information outside the market only at times $t_0, t_1, \dots, t_n$ assuming that $\mathcal{F} = \mathcal{F}_{\mathrm{T}}$.

The augmentation of filtration here could be interpreted as the information signal about non-market variables important to the value of the contract. An example here could be the ‘life’ part of information about the equity-linked contract.

With the slight abuse of notation one can extend measure $\mathbb{R}$ to σ-field $\mathcal{F}$:

$$\mathbb{R}(A) := \int_A \frac{d\mathbb{R}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^X} d\mathbb{P}$$

for $A \in \mathcal{F}$.

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The author considered the contingent claim D being an $\mathcal{F}_T$-measurable non-negative random variable and the replicating investment strategies, which are expressed in terms of the integrals with respect to X. This deals with the self-financing admissible trading strategies (V_0, ξ) where V_0 is constant and ξ is $\mathcal{F}$-predictable process on $[0, T]$ for which the value process

$$V_t = V_0 + \int_0^t \xi_u \, \mathrm{d}X_u, \qquad t \in [0, T]$$

is well-defined and generates non-negative wealth:

$$V_t \geq 0, \mathbb{P} - \text{a. s.}$$

for all $t \in [0, \mathrm{T}]$.

Denote the set of all equivalent martingale measures by $\mathcal{P}$. This means that process X follows a martingale in respect to all measures from $\mathcal{P}$.

Problem 2.1. Fix an initial capital $\tilde{V}_0$. Among all admissible strategies satisfying $V_0 \leq \tilde{V}_0$ find one that maximizes *expected success ratio*:

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T \geq D)} + \mathbf{1}_{\{V_T < D\}} \frac{V_T}{D}\right]. \tag{2.1}$$

Define:

$$f(k) = \text{ess sup}_{s \in L^1(\Omega, \mathcal{F}_T^X)} \left\{ s \colon E^{\mathbb{R}}\left[\frac{d\mathbb{P}}{d\mathbb{R}} \frac{1_{\{s \leq D\}}}{D} \middle| \mathcal{F}_T^X\right] \geq k \right\}. \tag{2.2}$$

Assume that constant k is defined by:

$$\mathbb{E}^{\mathbb{R}}[f(k)] = \tilde{V}_0 \,. \tag{2.3}$$

Theorem 2.1. A solution of Problem 2.1 is a super replicating strategy of $min(D, f(k))$. The maximal success ratio is equal to $\mathbb{E}^{\mathbb{P}}\left[\min\left(1, \frac{f(k)}{D}\right)\right]$.

3. Solution of Problem 2.1

Let us begin with an auxiliary problem:

Problem 3.1. Find a $\mathcal{F}_{\mathrm{T}}^{X}$-measurable random variable Γ that maximizes $\mathbb{E}^{\mathbb{P}}\left[\min\left(1, \frac{\Gamma}{D}\right)\right]$ subject to condition $\mathbb{E}^{\mathbb{R}}\Gamma \leq \tilde{V}_0$.

Lemma 3.1. *A solution of Problem 3.1 is given by* $\Gamma = f(k)$ *where* k *is defined by 2.3.*

Proof. Consider a function

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$$F(s) := \mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{P}}{d\mathbb{R}}\min\left(1,\frac{s}{D}\right)\Big|\mathcal{F}_T^X\right] \tag{3.1}$$

it is almost everywhere increasing and concave in respect of s, so

$$F(s) \le F(f(k)) + k(s - f(k)). \tag{3.2}$$

Put $s = \Gamma$ and take the expectation:

$$\mathbb{E}^{\mathbb{R}}[F(\Gamma)] \le \mathbb{E}^{\mathbb{R}}[F(f(k))] + k\left(\mathbb{E}^{\mathbb{R}}[\Gamma] - \mathbb{E}^{\mathbb{R}}[f(k)]\right). \tag{3.3}$$

Finally as $\mathbb{E}^{\mathbb{R}}[\Gamma] \le \tilde{V}_0 = E^{\mathbb{R}}[f(k)]$ we get $\mathbb{E}^{\mathbb{R}}[F(\Gamma)] \le \mathbb{E}^{\mathbb{R}}[F(f(k))]$, i.e.

$$\mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{\Gamma}{D}\right)\right] \le \mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{f(k)}{D}\right)\right]. \tag{3.4}$$

For random variable M, let us introduce the set $\mathcal{K}^M$ of $\mathcal{F}_T^{\mathcal{X}}$ – measurable random variables K such that $\mathbb{R}(M \ge K|\mathcal{F}_T^{\mathcal{X}}) > 0$ almost surely.

Using convention $\overline{M} = \text{ess sup}\,\mathcal{K}^M$.

Now one can solve another auxiliary problem:

Problem 3.2. Find a $\mathcal{F}$ -measurable random variable ϕ such that $0 \le \phi \le 1$ and that maximizes $\mathbb{E}^{\mathbb{P}}[\phi]$ subject to condition $\mathbb{E}^{\mathbb{R}}[\overline{\phi D}] \le \tilde{V}_0$.

Lemma 3.2. *A solution of Problem 3.2 is* $\phi^* := \min\left(1,\frac{f(k)}{D}\right)$.

Proof. First, check that ϕ^* is in the domain of Problem 3.2:

$$\mathbb{E}^{\mathbb{R}}[\overline{\phi^* D}] = \mathbb{E}^{\mathbb{R}}\left[\overline{\min\left(1,\frac{f(k)}{D}\right)D}\right] \le \mathbb{E}^{\mathbb{R}}[f(k)] = \tilde{V}_0.$$

Second, check that ϕ^* maximizes $\mathbb{E}^{\mathbb{P}}[\phi]$:

Assume that $\tilde{\phi}$ is some other solution. As $\mathbb{E}^{\mathbb{R}}[\tilde{\phi}D] \le \mathbb{E}^{\mathbb{R}}\left[\overline{\tilde{\phi}D}\right] \le \tilde{V}_0$ from Lemma 3.1 one gets: $\mathbb{E}^{\mathbb{P}}\phi^* = \mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{f(k)}{D}\right)\right] \ge \mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{\overline{\tilde{\phi}D}}{D}\right)\right]$. Further $\mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{\overline{\tilde{\phi}D}}{D}\right)\right] \ge \mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{\tilde{\phi}D}{D}\right)\right] = \mathbb{E}^{\mathbb{P}}[\min(1,\tilde{\phi})] = \mathbb{E}^{\mathbb{P}}\tilde{\phi}$.

Problem 3.3. Find a $\mathcal{F}$ -measurable random variable ϕ such that $0 \le \phi \le 1$ and that maximizes $\mathbb{E}^{\mathbb{P}}[\phi]$ subject to condition $\mathbb{E}^{\mathbb{Q}}[\phi D] \le \tilde{V}_0$ for all $\mathbb{Q} \in \mathcal{P}$.

To solve it let us use the following lemma:

Lemma 3.3. *Let* $M \in L(\Omega,\mathcal{F})$. *Then*

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$$\sup_{\mathbb{Q}\in\mathcal{P}} E^{\mathbb{Q}}M = \mathbb{E}^{\mathbb{R}}\overline{M}. \tag{3.5}$$

The proof of this is in the Appendix. Now one can state:

Lemma 3.4. A solution of Problem 3.3 is $\phi = \min\left(1, \frac{f(k)}{D}\right)$.

Proof. The proof is immediate from Lemmas 3.3 and 3.2.

Proof of Theorem 2.1. Take any admissible strategy (V_0, ξ) from the domain of Problem 2.1, i.e. such that $V_0 \leq \tilde{V}_0$. For all $\mathbb{Q} \in \mathcal{P}$ one has:

$$\begin{aligned} \tilde{V}_0 &\geq V_0 \\ &\geq \mathbb{E}^{\mathbb{Q}} V_T \\ &\geq \mathbb{E}^{\mathbb{Q}}\left[D\mathbf{1}_{\{V_T\geq D\}} + V_T\mathbf{1}_{\{V_T<D\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[D\phi^{(V_0,\xi)}\right]. \end{aligned}$$

where $\phi^{(V_0,\xi)} = 1_{\{V_T\geq D)} +_{\{V_T<D)} \frac{V_T}{D}$. Note that from Lemma 3.4 one gets $\mathbb{E}^{\mathbb{P}}[\phi^{(V_0,\xi)}] \leq \mathbb{E}^{\mathbb{P}}\left[min\left(1,\frac{f(k)}{D}\right)\right]$, as $\phi^{(V_0,\xi)}$ is in the domain of Pro-blem 3.3. It can be shown that one can choose strategy (V_0, ξ) so that the value $\mathbb{E}^{\mathbb{P}}\left[min\left(1,\frac{f(k)}{D}\right)\right]$ is attained.

Take the strategy (V_0*, ξ*) being the super replicating strategy of contingent claim min(D, f(k)), where $V_0^* = sup_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}[min(D, f(k))] \leq \tilde{V}_0$ (from Lemma 3.4). This shows that for this strategy the value $\mathbb{E}^{\mathbb{P}}\left[min\left(1,\frac{f(k)}{D}\right)\right]$ is attained:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[\min\left(1,\frac{f(k)}{D}\right)\right] &\leq \mathbb{E}^{\mathbb{P}}\left[1_{\{V_T^*\geq D\}} + \min\left(1,\frac{f(k)}{D}\right)1_{\{V_T^*<D\}}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[1_{\{V_T^*\geq D\}} + \frac{\min(D, f(k))}{D}1_{\{V_T^*<D\}}\right] \\ &\leq \mathbb{E}^{\mathbb{P}}\left[1_{\{V_T^*\geq D\}} + \frac{V_T^*}{D}1_{\{V_T^*<D\}}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\phi^{(V_0^*,\xi^*)}\right] \end{aligned} \tag{3.6}$$

So (V_0*, ξ*) is a solution of Problem 2.1.

4. Appendix

Proof of Lemma 3.3. Fix a random variable M. For each $\mathbb{Q} \in \mathcal{P}$ there is an inequality $E^{\mathbb{Q}}M \leq E^{\mathbb{Q}}\overline{M} = E^{\mathbb{R}}\overline{M}$, so $sup_{Q\in\mathcal{P}} E^{\mathbb{Q}}M \leq E^{\mathbb{R}}\overline{M}$ and one only has to show that supremum is attainable.

For any $\alpha \in [0,1]$ and $K \in \mathcal{K}^M$ define a measure $\mathbb{R}^{\alpha,K}$

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$$\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}} = \prod_{i=1}^{n} A_i^{\alpha,K}, \tag{4.1}$$

where

$$A_i^{\alpha,K} = \alpha + \frac{\mathbb{R}(M \geq K|\mathcal{F}_{t_i})(1-\alpha)}{\mathbb{R}(M \geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}. \tag{4.2}$$

Note that for any $s < t_i$ there is

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}}[A_i^{\alpha,K}|\mathcal{F}_T^X \vee \mathcal{F}_s] &= \mathbb{E}^{\mathbb{R}}\left[\alpha + \frac{\mathbb{R}(M \geq K|\mathcal{F}_{t_i})(1-\alpha)}{\mathbb{R}(M \geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\middle|\mathcal{F}_T^X \vee \mathcal{F}_s\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[\alpha + \frac{\mathbb{R}(M \geq K|\mathcal{F}_{t_i})(1-\alpha)}{\mathbb{R}(M \geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\middle|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}}\right] \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}}[A_i^{\alpha,K}|\mathcal{F}_s] &= \mathbb{E}^{\mathbb{R}}\left[\alpha + \frac{\mathbb{R}(M \geq K|\mathcal{F}_{t_i})(1-\alpha)}{\mathbb{R}(M \geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[\mathbb{E}^{\mathbb{R}}\left[\alpha + \frac{\mathbb{R}(M \geq K|\mathcal{F}_{t_i})(1-\alpha)}{\mathbb{R}(M \geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\middle|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}}\right]\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}^{\mathbb{R}}[1|\mathcal{F}_s] = 1
\end{aligned}$$

So for $t \in (t_j, t_{j+1})$

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}\middle|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n} A_i^{\alpha,K}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{R}}\left[\mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n} A_i^{\alpha,K}\middle|\mathcal{F}_{t_{n-1}}\right]\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n-1} A_i^{\alpha,K}\mathbb{E}^{\mathbb{R}}\left[A_n^{\alpha,K}\middle|\mathcal{F}_{t_{n-1}}\right]\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n-1} A_i^{\alpha,K}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n-2} A_i^{\alpha,K}\middle|\mathcal{F}_t\right] = \ldots = \mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{j+1} A_i^{\alpha,K}\middle|\mathcal{F}_t\right] \\
&= \prod_{i=1}^{j} A_i^{\alpha,K}\mathbb{E}^{\mathbb{R}}\left[A_j^{\alpha,K}\middle|\mathcal{F}_t\right] = \prod_{i=1}^{j} A_i^{\alpha,K}
\end{aligned}$$

In the same way one gets

$$\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}\middle|\mathcal{F}_T^X \vee \mathcal{F}_t\right] = \prod_{i=1}^{j} A_i^{\alpha,K},$$

i.e. for every $t \in [0,T]$ there is

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$$\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}\Big|\mathcal{F}_T^X \vee \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}\Big|\mathcal{F}_t\right] \tag{4.4}$$

Note that $\mathsf{R}^{\alpha,K} \in \mathrm{P}$ for $\alpha \in (0,1]$, because for $s_1 < s_2$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}^{\alpha,K}}[X_{s_2}|\mathcal{F}_{s_1}] &= \frac{1}{\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}|_{\mathcal{F}_{s_1}}}\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}X_{s_2}\Big|\mathcal{F}_{s_1}\right] \\
&= \frac{1}{\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}|_{\mathcal{F}_{s_1}}}\mathbb{E}^{\mathbb{R}}\left[\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}X_{s_2}\Big|\mathcal{F}_T^X \vee \mathcal{F}_{s_1}\right]\Big|\mathcal{F}_{s_1}\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[X_{s_2}\frac{\mathbb{E}^{\mathbb{R}}\left[\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}\Big|\mathcal{F}_T^X \vee \mathcal{F}_{s_1}\right]}{\frac{d\mathbb{R}^{\alpha,K}}{d\mathbb{R}}|_{\mathcal{F}_{s_1}}}\Big|\mathcal{F}_{s_1}\right] \\
&= \mathbb{E}^{\mathbb{R}}[X_{s_2}|\mathcal{F}_{s_1}] = X_{s_1}
\end{aligned}$$

For every random variable $K \in \mathrm{K}^M$ there is an inequality:

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}}\overline{M} &\geq \sup_{\alpha\in(0,1]}\mathbb{E}^{\mathbb{R}^{\alpha,K}}M \geq \mathbb{E}^{\mathbb{R}^{0,K}}M = \mathbb{E}^{\mathbb{R}}\left[M\prod_{i=1}^{n}A_i^{0,K}\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[M\prod_{i=1}^{n-1}A_i^{0,K}\frac{1_{\{M\geq K\}}}{\mathbb{R}(M\geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\right] \\
&\geq \mathbb{E}^{\mathbb{R}}\left[K\prod_{i=1}^{n-1}A_i^{0,K}\frac{1_{\{M\geq K\}}}{\mathbb{R}(M\geq K|\mathcal{F}_{t_i}^X \vee \mathcal{F}_{t_{i-1}})}\right] \\
&= \mathbb{E}^{\mathbb{R}}\left[K\prod_{i=1}^{n}A_i^{0,K}\right] = \mathbb{E}^{\mathbb{R}}\left[K\mathbb{E}^{\mathbb{R}}\left[\prod_{i=1}^{n}A_i^{0,K}\Big|\mathcal{F}_T^X\right]\right] = \mathbb{E}^{\mathbb{R}}[K]
\end{aligned}$$

but from the other side $\sup_{K\in \mathrm{K}^M} \mathsf{E}^{\mathsf{R}}K = \mathsf{E}^{\mathsf{R}}M$.

Note that the second inequality above follows from the form of the Radon-Nikodym derivative in (4.2) and from continuity.

This implies that

$$\sup_{(\alpha,K)\in(0,1]\times\mathcal{K}^M}\mathbb{E}^{\mathbb{R}^{\alpha,K}}M = \mathbb{E}^{\mathbb{R}}\overline{M}. \tag{4.5}$$

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ZABEZPIECZENIE KONTRAKTU TYPU *EQUITY-LINKED* Z MAKSYMALNYM WSPÓŁCZYNNIKIEM SUKCESU

Streszczenie: W artykule opisano ubezpieczenie typu *equity-linked*, którego wypłata zależy od czasu życia ubezpieczonego oraz kursu akcji. Autor zakłada ograniczony kapitał zakładu ubezpieczeń na jego zabezpieczenie i pokazuje najlepszą strategię dla zakładu ubezpieczeń w sensie tzw. współczynnika sukcesu $E^{\mathbb{P}}[\mathbf{1}_{\{V_T \geq D\}} + \mathbf{1}_{\{V_T < D\}} \frac{V_T}{D}]$, gdzie V_T oznacza końcową wartość strategii, a D jest wypłatą z kontraktu. Publikacja jest uogólnieniem prac (Föllmer i Schied, 2004; Klusik i Palmowski, 2011), ale rozważa dużo bardziej ogólną 'niezupełność' rynku, między innymi dopuszcza bieżący dopływ informacji w trakcie życia instrumentu oraz nieskończoną liczbę scenariuszy w odniesieniu do świata „pozarynkowego".

Słowa kluczowe: hedging kwantylowy, kontrakty *equity-linked*.