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# The minimal-time growth problem and turnpike effect in the stationary Gale economy[1]

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**Abstract**: In mathematical economics there is a number of so called "turnpike theorems" proved mainly on the basis of multiproduct models of economic dynamics. According to these theorems, all optimal paths of economic growth over a long period of time converge to a certain path (turnpike) in which the economy achieves the highest growth rate while remaining in a specific dynamic (von Neumann) equilibrium. The article refers to this trend and presents some properties of optimal growth processes in the Gale-type model of the stationary economy when the quality criterion of growth processes is not the utility of production—which is normally postulated in the turnpike theory—but the time needed by the economy to achieve the desired final state, e.g. the level of production or production value. According to the author's knowledge, the idea of using time as a criterion for growth in turnpike theory (especially in Gale-type economy) is innovative.

It has been proven that changing the growth criterion does not deprive the Gale economy of its asymptotic / turnpike properties.

**Keywords**: stationary Gale economy, minimum-time growth problem, von Neumann equilibrium, production turnpike—von Neumann ray.

**JEL codes**: C62, C67, O41, O49.

## Introduction

In the papers devoted to the turnpike properties of the optimal processes in the Neumann-Gale-Leontief economies the role of the main growth criterion is usually held by the production utility (in particular the value of the production measured in the von Neumann prices) that has been produced in

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the last period of the fixed time horizon T = $\{0, 1, \ldots, t_1\}$[3] or (less frequently) total production utility produced in the economy in all periods of the time horizon $T$[4].

In this paper the assessment of the growth processes with the use of utility function is departed from. The time that an economy starting from a certain initial state (initial production level) needs to achieve the desired final state (target production level) plays here the role of the criterion by which the quality of growth processes is assessed. Therefore the interest lies in the minimal-time problem of growth. It will be formulated for the stationary Gale economy with a single production turnpike. The idea of such a minimal-time approach to the assessment of growth processes comes from the optimal control theory[5]. Three theorems are presented in which it is proved that despite the change in the growth criterion, the economy does not lose its, so-called, turnpike properties.

The choice of the simplest version of the input-output Gale model is not accidental. Due to its simplicity, the arguments contained in this article become more transparent.

The structure of this paper is as follows. In Section 1 a model of the stationary Gale economy with a single production turnpike is presented and the necessary concepts used in this paper, including production space, technological and economic efficiency of production, von Neumann equilibrium, production turnpike (von Neumann ray) are defined. In Section 2 the minimal-time growth problem is formulated and "weak" and "very strong" turnpike theorems in the Gale economy, in which the set of target states is a hyperplane in the space of production vectors are proved . Section 3 of this paper provides the proof of a "weak" turnpike theorem in a case of the minimal-time growth problem with a different, yet naturally interpreted, form of the set of target states. The final remarks indicate possible and interesting directions for further research.

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[3] Cf. Makarov and Rubinov (1977), McKenzie (1976), Nikaido (1968, chapter 4), Panek (2003, chapter 5), Takayama (1985, chapter 7); see also e.g. Babaei (2020), Babaei, Evstigneev, Schenk-Hoppé (2020), Dai and Shen (2013), Giorgi and Zuccotti (2016), Zhitlukhin (2019). McKenzie had a special contribution to the turnpike theory. An extensive selection of his works is contained in the Mitra and Nishimura monograph (2009).

[4] Examples of such an approach are presented, among others, in Panek (2014, 2015, 2018).

[5] In the last two decades turnpike theory has moved beyond the traditional boundaries of mathematical economics and especially results obtained in the optimal control theory are very interesting, see Zaslawski (2006, 2015), as well as e.g. Gurman and Gusieva (2013), Mammedov (2014), Trelat, Zhang and Zuazua (2018).

## 1. Model of the stationary Gale economy with a single production turnpike[6]

In the economy there are finite number n of consumed and/or produced commodities. Let $x = (x_1, x_2, \ldots, x_n)$ denote the non-negative vector of consumed commodities and let $y = (y_1, y_2, \ldots, y_n)$ denote non-negative of commodities produced in a certain unit of time, e.g., during a year[7]. If the available technology in the economy allows production from the input vector $x$, the output $y$, then the pair $(x, y)$ describes a technologically admissible production process. Let $Z \subset R^{2n}_+$ denote the set of all such processes. This is called it the production space here (technological set). It is assumed that the production space $Z$ is a non-empty set that meets the following conditions:

**(G1)** $\forall (x^1, y^1) \in \mathrm{Z}\ \forall (x^2, y^2) \in Z\ \forall\ \lambda_1, \lambda_2 \geq 0\ (\lambda_1(x^1, y^1) + \lambda_2\ (x^2, y^2) \in \mathrm{Z})$

(inputs/outputs proportionality condition and the additivity of production processes),

**(G2)** $$\forall (x, y) \in Z\ (x = 0 \Rightarrow y = 0)$$

(„no cornucopia" condition),

**(G3)** $$\forall (x, y) \in Z\ \forall x' \geqq x\ \forall\ 0 \leqq y' \leqq y\ ((x', y') \in Z)^8$$

(possibility of wasting the inputs / outputs),

**(G4)** a production space $Z$ is a closed subset of $R^{2n}_+$.

The set $Z$ satisfying conditions **(G1)-(G4)** is a convex and closed cone in $R^{2n}_+$ with a vertex at with the following property: if $(x, y) \in Z$ and $(x, y) \neq 0$, then We call it Gale's production space. The interest here is in nontrivial (non-zero) production processes $(x, y) \in Z\backslash\{0\}$.

Let $(x, y) \in Z\backslash\{0\}$. The number

$$\alpha(x, y) = \max\ \{\alpha \mid \alpha x \leqq y\}$$

[6] The model is presented in the most synthetic form. An interested reader is referred to the works and papers mentioned in the footnote 2. The notation used further in this paper refers to Panek (2003, Chapter 5).

[7] The coordinates of vectors $x$, $y$ are expressed in physical units (kilograms, litres, meters, etc.).

[8] If $a, b \in R^n$, then $a \geqq b$ indicates that $\forall i(a_i \geq b_i)$. Notation $a \geq b$ indicates that $a \geqq b$ and $a \neq b$. The condition $a \leqq b$ is defined similarly.

is called the index of the technological efficiency of the process $(x, y)$. If conditions **(G1)-(G4)** are met the function $\alpha(\cdot)$ is positively homogeneous of degree 0 on $Z\setminus\{0\}$ and there exists a number

$$\alpha_M = \max_{(x,y)\in Z\setminus\{0\}} \alpha(x,y) = \alpha(\bar{x},\bar{y}) \geq 0,$$

called the optimal indicator of the technological production efficiency, see for example Takayama (1985, th. 6.A.1).

Vector $(\bar{x}, \bar{y})$ is called the optimal production process in the stationary Gale economy. Since the function is positively homogeneous of degree 0, hence the optimal production process is determined with the accuracy of multiplication by a positive constant (with structure accuracy):

$$\forall\lambda > 0\big(\alpha(\bar{x},\bar{y}) = \alpha(\lambda\bar{x},\lambda\bar{y}) = \alpha_M \geq 0\big).$$

A particular case of the Gale economy will be taken into consideration in which

**(G5)** $$\exists(\bar{x},\bar{y}) \in Z\setminus\{0\}\big(\alpha(\bar{x},\bar{y}) = \alpha_M > 1 \;\&\; \bar{y} > 0\big).$$

The condition $\bar{y} > 0$ is called the regularity condition (all commodities are produced in the optimal production process). If an economy satisfies the condition $\alpha(\bar{x}, \bar{y}) = \alpha_M > 1$ then it is productive (there are more outputs than inputs).

If the condition **(G5)** is satisfied, then (considering **(G3)**):

$$\exists(\bar{x},\bar{y}) \in Z\setminus\{0\}\big(\alpha_M\bar{x} = \bar{y} > 0\big).$$

When the optimal production process is spoken about it always means every process $(\bar{x}, \bar{y})$ that satisfies this condition. The vector

$$\bar{s} = \frac{\bar{y}}{\|\bar{y}\|} > 0$$

characterizes the production structure in the optimal production process $(\bar{x}, \bar{y})$[9]. A ray:

$$N = \{\lambda\bar{s} \mid \lambda > 0\} \subset R^n_+$$

[9] From now on if $a \in R^n_+\setminus\{0\}$, then $\frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}, \ldots, \frac{a_n}{\|a\|}\right)$, $a = \sum_{i=1}^{n} a_i$. Since the optimal production process satisfies the condition $\alpha_M\bar{x} = \bar{y} > 0$, then $\bar{s} = \frac{\bar{y}}{\|\bar{y}\|} = \frac{\alpha_M\bar{x}}{\|\alpha_M\bar{x}\|} = \frac{\bar{x}}{\|\bar{x}\|}$.

is called the production turnpike (von Neumann ray) in the stationary Gale economy. In the optimal process $(\bar{x}, \bar{y})$ both the output vector $\bar{y}$ and the input vector $\bar{x}$ lie on the turnpike $N$.

By $p = (p_1, p_2, \ldots, p_n) \geq 0$ we denote the commodity price vector in the Gale economy. Let us consider any process $(x, y) \in Z\backslash\{0\}$. A number[10]

$$\beta(x, y, p) = \frac{\langle p, y \rangle}{\langle p, x \rangle}$$

$(\langle p, x \rangle \neq 0)$ is called the index of economic efficiency of the process $(x, y)$ with prices $p$. Let $(\bar{x}, \bar{y}) \in Z\backslash\{0\}$ be an optimal production process in Gale economy. Hence:

$$\alpha_M \bar{x} = \bar{y} > 0. \tag{1}$$

☐ **Theorem 1.** If conditions **(G1)-(G5)** are satisfied, then there exist prices $\bar{p} \geq 0$, such that

$$\forall (x, y) \in Z \left( \bar{p}, y \leq \alpha_M \bar{p}, x \right) \tag{2}$$

**Proof**. Cf. Panek (2003, Chapter 5, th. 5.4). ■

It follows from (1) that

$$\langle \bar{p}, \bar{y} \rangle > 0. \tag{3}$$

Conditions (1)-(3) lead to the conclusion that:

$$\beta(\bar{x}, \bar{y}, \bar{p}) = \frac{\langle \bar{p}, \bar{y} \rangle}{\langle \bar{p}, \bar{x} \rangle} = \max_{(x,y) \in Z\backslash\{0\}} \beta(x, y, \bar{p}) = \alpha_M$$

It is stated that the triple $\{\alpha_M, (\bar{x}, \bar{y}), \bar{p}\}$ represents the (optimal) von Neumann equilibrium state (in the stationary Gale economy). Prices $\bar{p}$ are called equilibrium prices or von Neumann prices. In the von Neumann equilibrium state, the technological production efficiency matches its economic efficiency at the maximum possible level $\alpha_M$ that can be achieved by the economy.

## 2. Dynamics. The first minimal-time growth problem

An economy is considered where time is discrete, $t \in T = \{0, 1, \ldots, t_1\}$, $t_1 < +\infty$. The set $T$ is called the horizon of the economy. By $x(t) = (x_1(t), x_2, (t), \ldots, x_n(t))$

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[10] A scalar product $\sum_{i=1}^{n} a_i b_i$ of vectors $a, b \in R^n$ is denoted by $\langle a, b \rangle$.

the input vector that is used in economy in period $t$ and by $y(t) = (y_1(t), y_2, (t), \ldots, y_n(t))$ the output vector that is produced from the input vector $x(t)$ in the admissible process $(x(t), y(t)) \in Z$ is denoted. In the Gale economy inputs $x(t + 1)$ that are used in period $t + 1$ come from the outputs $y(t)$ that are produced in period $t$, i.e. the inequality $x(t + 1) \leqq y(t)$, $t = 0, 1, 2, \ldots, t_1 - 1$ is satisfied. This, under (**G3**), leads to the condition $(y(t), y(t + 1)) \in Z$, $t = 0, 1, \ldots, t_1 - 1$[11]. Moreover, the initial output vector $y^0$ produced in the economy in the period $t = 0$: $y(0) = y^0 > 0$ is determined.

The sequence of output vectors $\{y(t)\}_{t=0}^{t_1}$ satisfying conditions:

$$\begin{gathered}\big(y(t), y(t+1)\big) \in Z, \quad t = 0, 1, \ldots, t_1 - 1, \\ y(0) = y^0 > 0 \\ (\text{vector } y^0 \text{ is determined}),\end{gathered} \tag{4}$$

is called $(y^0, t_1)$-feasible growth process (production trajectory). The $(y^0, t_1)$-feasible processes that are a solution to the problem:

$$\begin{gathered}\max\ u\big(y(t_1)\big) \\ \text{subject to (4)},\end{gathered} \tag{5}$$

are the subject of interest in the turnpike theory. The function $u$: $R_+^n \rightarrow R_+^1$ is a utility function with standard properties defined on the output vectors produced in the economy in the last period $t_1$ of the horizon $T$[12].

In all versions of problem (5) (and similar to it):
- the horizon $T$ (particularly its end period $t_1$) is assumed to be fixed,
- no additional limiting conditions are demanded for the states of the economy, e.g., in the form of the preferable target value (in monetary terms) or the volume of production (in physical units) in the last period of horizon $T$.

Now the problem will be reversed by assuming that not only the production vector $y^0$ (the initial state of the economy) is established, but so is the desirable target production value $V^1$ (measured in von Neumann prices), which the economy should achieve in the shortest possible time. Therefore the interest lies in the solution of the following minimal-time problem:

$$\min t_1 \tag{6}$$

[11] In the literature such an economy is called a closed economy.

[12] Criterion (5) often takes a special (linear) form $\max \langle \bar{p}, y(t_1) \rangle$, i.e. maximization of the output value (measured in the equilibrium prices) produced in the last period of the horizon $T$. The papers with other criterions, e.g. $\max \sum_{t \in T} (1-\gamma)^t u\big(y(t)\big)$ (maximization of the discounted output utility produced in all periods in the horizon $T$; $\gamma \in (0, 1)$ is the consumption discount rate) are rarer, see e. g. Panek (2018).

subject to

$$\begin{gathered}\big(y(t), y(t+1)\big) \in Z, \quad t = 0, 1, \ldots, t_1 - 1,\\ y(0) = y^0 > 0\\ y(t_1) \in Y^1,\end{gathered} \tag{7}$$

where:

$$Y^1 = \left\{ y \in R^n_+ \mid \left\langle \overline{p}, y \right\rangle \geq V^1 > \left\langle \overline{p}, y^0 \right\rangle \right\}. \tag{8}$$

The initial production vector $y^0$ and the set of target states $Y^1$ (production vectors with a total value not less than $V^1$) are determined. Production vectors $y(1), y(2), \ldots, y(t_1)$ and time $t_1$ are the decision variables in the task (6)-(7) – therefore the length of the horizon $T$ becomes the variable as well. About sequence of vectors $\{y(t)\}_{t=0}^{t_1}$, $t_1 < +\infty$ satisfying (7) it is possible to state that it represents ($y^0$, $Y^1$, $t_1$)-feasible growth process. The admissible process $\{y^*(t)\}_{t=0}^{t_1^*}$ that is a solution to problem (6)-(7) is called ($y^0$, $Y^1$, $t_1^*$)-optimal growth process. The economy with the ($y^0$, $Y^1$, $t_1^*$)-optimal growth process achieves the production level $y^*(t_1^*)$ with a value not lower than $V^1$ in the shortest possible time.

☐ **Lemma 1.** Assuming **(G1)-(G5)**:
**(i)** there exist ($y^0$, $Y^1$, $t_1$) feasible growth processes,
**(2i)** problem (6)-(7) has a solution.
**Proof. (i)** Due to the fact that $y^0 > 0$, $V^1 > \langle \overline{p}, y^0 \rangle$, $\alpha_M > 1$ and the vector $\overline{s}$ of the production structure on the turnpike $N = \{\lambda\overline{s} \mid \lambda > 0\}$ is positive, there exists ($y^0$, $Y^1$, $t_1$)-feasible growth process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} y^0, & t = 0,\\ \overline{y}(t) = \sigma\alpha_M^t\overline{s}, & t = 1, \ldots, t_1,\end{cases} \tag{9}$$

in which $\sigma = \min\limits_i \dfrac{y_i^0}{\overline{s}_i} > 0$ and $t_1$ is the smallest natural number not less than $\dfrac{\ln A}{\ln \alpha_M} > 0$, $A = \dfrac{V^1}{\sigma\left\langle \overline{p}, \overline{s} \right\rangle} > 1$. In this process:

$$\tilde{y}(t_1) \in Y^1 \tag{10}$$

and $\tilde{y}(t_1 - 1) \in Y^1$.

**(2i)** Let $R_{y^0,t}$ denote the set of production vectors that can be produced in period $t$ in the economy starting (in the initial period $t = 0$) with the production level $y(0) = y^0$:

$$R_{y^0,0} = \{y^0\}, \ldots, R_{y^0,t+1} = \left\{ y(t+1) \,|\, \left( y(t), y(t+1) \right) \in Z,\; y(t) \in R_{y^0,t} \right\}, t = 0, 1, \ldots .$$

The sets $R_{y0,\, t}$ are compact (and convex, see e.g. Panek, 2003, Chapter 5, lemma 5.1) and:

$$\exists t_1 > 0(R_{y^0,t_1} \cap Y^1 \neq \varnothing)^{13}.$$

Since $\forall y \in Y^1(\overline{p}, y \geq V^1 > \overline{p}, y^0 > 0)$, thus $R_{y^0,0} \cap Y^1 = \varnothing$. There exists a natural number $t_1^* \leq t_1$, such that

$$R_{y^0,t_1^*} \cap Y^1 \neq \varnothing \text{ and } R_{y^0,t_1^*-1} \cap Y^1 = \varnothing.$$

The number $t_1^*$ is also the earliest period, in which the economy achieves the target set $Y^1$. Since $y^0 \notin Y^1$, so it is a positive number. ■

Note that $(y^0, Y^1, t_1)$—feasible process (9) starting with the period $t = 1$ belongs to the family of processes of a particular form:

$$\overline{y}(t) = \alpha_M^t \overline{y}, \quad t = 0, 1, \ldots \tag{11}$$

$(\overline{y} = \sigma\overline{s}; \sigma > 0)$, which at each period belong to the turnpike $N$:

$$\forall t \geq 0\left( \overline{y}(t) \in N \right).$$

Each process of this form (11) is called an optimal stationary growth (with the rate $\alpha_M$). Since

$$\overline{s}(t) = \frac{\overline{y}(t)}{\left\| \overline{y}(t) \right\|} = \frac{\alpha_M^t \overline{y}}{\left\| \alpha_M^t \overline{y} \right\|} = \frac{\overline{y}}{\left\| \overline{y} \right\|} = \overline{s},$$

this process is characterized by a constant (turnpike) structure. If

$$\forall t \geq 0\left( \overline{y}(t) \in N \right),$$

then also

$$\forall \lambda > 0\, \forall t \geq 0\left( \lambda \overline{y}(t) \in N \right),$$

which means that any positive iteration of the optimal stationary growth process is still the optimal stationary growth process. All of them are on the turnpike $N$.

[13] In the $(y^0, Y^1, t_1)$-feasible process (9) vector $\tilde{y}(t_1)$ belongs simultaneously to $R_{y^0,\, t_1}$ (from the definition) and to $Y^1$ (see 10).

Without any other additional conditions in the Gale economy there may exist more than one turnpike. The uniqueness of the turnpike (von Neumann ray) in the model presented here is assured by the following condition (so called strict equilibrium property):

**(G6)** $$\forall (x, y) \in Z\backslash\{0\}(x \notin N \Rightarrow \beta(x, y, \bar{p}) < \alpha_M).$$

It means that the economic efficiency of any off-turnpike process is less than optimal. Indeed it can be assumed that there exists another turnpike $N'$ alongside the turnpike $N$. Then there exist two such optimal processes $(\bar{x}, \bar{y}) \in Z\backslash\{0\}$, $(\bar{x}', y') \in Z\backslash\{0\}$, such that:

$$\begin{gathered}\alpha_M \bar{x} = \bar{y} > 0, \quad \alpha_M \bar{x}' = \bar{y}' > 0, \\ N = \{\lambda \bar{s} \mid \lambda > 0\}, \quad N' = \{\lambda \bar{s}' \mid \lambda > 0\}, \\ \bar{s} = \frac{\bar{y}}{\|\bar{y}\|}, \quad \bar{s}' = \frac{\bar{y}'}{\|\bar{y}'\|},\end{gathered} \tag{12}$$

$\bar{x}, \bar{y} \in N$ and $\bar{x}', \bar{y}' \in N'$. Since $N \neq N'$, therefore $\bar{s} \neq \bar{s}'$, so $N \cap N' = \emptyset$. Because $\bar{x}' \in N'$, so $\bar{x}' \notin N$. Then, according to **(G6)**, $\beta(\bar{x}', \bar{y}', \bar{p}) < \alpha_M$. However it follows from (12), that $\beta(\bar{x}, \bar{y}, \bar{p}) = \beta(\bar{x}', \bar{y}', \bar{p}) = \alpha_M$. This contradiction confirms that the condition **(G6)** does ensure the uniqueness of the turnpike $N$.

It is not difficult to notice that if condition **(G6)** holds, then the equilibrium price vector is positive and, furthermore, the following lemma is true.

☐ **Lemma 2.** If conditions **(G1)**-**(G6)** hold, then:

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon \in (0, \alpha_M) \forall (x, y) \in Z \backslash \{0\} \left( \left\| \frac{x}{\|x\|} - \bar{s} \right\| \geq \varepsilon \Rightarrow \beta(x, y, \bar{p}) = \frac{\langle \bar{p}, y \rangle}{\langle \bar{p}, x \rangle} \leq \alpha_M - \delta_\varepsilon \right)$$

**Proof**, Cf. Radner (1961), Panek (2003; Chapter 5, lemma 5.2). ■

Let $d(y(t), N)$ denote the (angular) distance of the production vector $y(t)$ from the turnpike $N = \{\lambda \bar{s} | \lambda > 0\}$:

$$d\big(y(t), N\big) = \left\| \frac{y(t)}{\|y(t)\|} - \bar{s} \right\|$$

and now consider $(y^0, Y^1, t_1^*)$-optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$-a solution of the problem (6)-(7). Theorem 2 states about its trajectory in the neighbourhood of the turnpike.

☐ **Theorem 2.** Suppose that the conditions **(G1)-(G6)** hold, then for any positive number $\varepsilon > 0$ there exists a natural number $k_\varepsilon$, such that the number of time periods $\tau_1, \tau_2, \ldots, \tau_k$, in which

$$d(y^*(t), N) \geq \varepsilon^{14} \tag{13}$$

does not exceed $k_\varepsilon$. The number $k_\varepsilon$ does not depend on the postulated target production value $V^1$ or on $t_1^*$.

**Proof.** From (2) and from the definition of $(y^0, Y^1, t_1^*)$—optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$ the following is obtained:

$$\langle \overline{p}, y^*(t+1)\rangle \leq \alpha_M \langle \overline{p}, y^*(t)\rangle, \quad t = 0, 1, \ldots, t_1^* - 1, \tag{14}$$

and hence $\langle \overline{p}, y^*(t_1^*)\rangle \leq \alpha_M^{t_1^*} \langle \overline{p}, y^0\rangle$. If in the periods $\tau_1, \tau_2, \ldots, \tau_k$ the condition (13) holds, then according to Lemma 2, there exists a number $\delta_\varepsilon \in (0, \alpha_M)$, such that:

$$\langle \overline{p}, y^*(t+1)\rangle \leq (\alpha_M - \delta_\varepsilon)\langle \overline{p}, y^*(t)\rangle, \quad t = \tau_1, \tau_2, \ldots, \tau_k. \tag{15}$$

From (14), (15) the upper limit of the output value produced in the period $t_1^*$ is obtained:

$$\langle \overline{p}, y^*(t_1^*)\rangle \leq \alpha_M^{t_1^* - k} (\alpha_M - \delta_\varepsilon)^k \langle \overline{p}, y^0\rangle. \tag{16}$$

On the other hand, according to Lemma 1, there exists $(y^0, Y^1, t_1)$—feasible growth process $\{\tilde{y}(t)\}_{t=0}^{t_1}$ of the form (9). In this process the production vector $\tilde{y}(t_1)$ in the period $t_1 \geq t_1^*$ satisfies the condition (1), so $\langle \overline{p}, \tilde{y}(t_1)\rangle \geq V^1$. At the same time $t_1$ is the earliest period in which this condition is satisfied, which means that $\tilde{y}(t_1 - 1)$, in other words $\langle \overline{p}, \tilde{y}(t_1 - 1)\rangle < V^1$. Then, under condition **(G3)**, there exists another $(y^0, Y^1, t_1)$—feasible growth process $\{\tilde{\tilde{y}}(t)\}_{t=0}^{t_1}$:

$$\tilde{\tilde{y}}(t) = \begin{cases} y^0, & t = 0, \\ \sigma \alpha_M^t \overline{s}, & t = 1, \ldots, t_1 - 1, \\ \sigma \alpha_M^{t_1 - 1} \overline{s} + y', & t = t_1, \end{cases} \tag{17}$$

where $\tilde{\tilde{y}}(t_1) = \tilde{\tilde{y}}(t_1 - 1) + y'$, $y'$ is a vector that satisfies the condition $0 \leq y' \leqq \sigma(\alpha_M - 1)\alpha_M^{t_1 - 1}\overline{s} (> 0)$ and:

[14] The production structure in the $(y^0, Y^1, t_1^*)$—optimal process differs from the production structure on the turnpike by at least $\varepsilon$.

$$\left\langle \overline{p}, y' \right\rangle = V^1 - \left\langle \overline{p}, \tilde{\tilde{y}}(t_1 - 1) \right\rangle > 0^{15}, \tag{18}$$

which give the following lower limit of the production value in the period $t_1^*$:

$$\left\langle \overline{p}, y^*(t_1^*) \right\rangle \geq V^1 = \left\langle \overline{p}, \tilde{\tilde{y}}(t_1) \right\rangle = \left\langle \overline{p}, \tilde{y}(t_1 - 1) + y' \right\rangle = \sigma \alpha_M^{t_1 - 1} \left\langle \overline{p}, \overline{s} \right\rangle + \left\langle \overline{p}, y' \right\rangle > 0. \tag{19}$$

From (16), (19) the condition results:

$$\alpha_M^{t_1^* - k} (\alpha_M - \delta_\varepsilon)^k \left\langle \overline{p}, y^0 \right\rangle \geq \sigma \alpha_M^{t_1 - 1} \left\langle \overline{p}, \overline{s} \right\rangle + \left\langle \overline{p}, y' \right\rangle > 0,$$

from which (due to the fact that $t_1 \geq t_1^*$, $\alpha_M > 1$ and $\left\langle \overline{p}, y' \right\rangle > 0$) the inequality follows:

$$\left( \frac{\alpha_M}{\alpha_M - \delta_\varepsilon} \right)^k < \frac{\alpha_M \left\langle \overline{p}, y^0 \right\rangle}{\sigma \left\langle \overline{p}, \overline{s} \right\rangle}.$$

This allows an estimation of the number

$$k < B = \frac{\ln A}{\ln \alpha_M - \ln(\alpha_M - \delta_\varepsilon)},$$

where $A = \dfrac{\alpha_M \left\langle \overline{p}, y^0 \right\rangle}{\sigma \left\langle \overline{p}, \overline{s} \right\rangle} > 1$, in other words $B > 0$. If $B$ is a natural number, then it is enough to take $k_\varepsilon = B - 1$ to complete the proof. When $B$ is not a natural number, then it is sufficient to take the smallest natural number greater than $B - 1$ as the number $k_\varepsilon$ that is referred in the theorem. The number $B$, and therefore also the number $k_\varepsilon$, do not depend on $V^1$ or on $t_1^*$. ■

In the literature theorems similar to the proven one (called "weak" turnpike theorems) indicate the fundamental significance of the turnpike in the economic growth process. They prove that over long periods of time the production structure in each optimal growth process "almost always" becomes similar to the production structure on the turnpike, where the economy achieves both the maximum technological and economic production efficiency and the highest growth rate[16]. An even more distinctive, special situation occurs when optimal growth process reaches the turnpike in a certain period. This problem is addressed in another "very strong" turnpike theorem.

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[15] This condition ensures that in the $(y^0, Y^1, t_1)$—feasible growth process of the form (17) the economy in the last period $t_1$ reaches the production value $\left\langle \overline{p}, \tilde{\tilde{y}}(t_1) \right\rangle = V^1$.

[16] This is inherent quality of each $(y^0, Y^1, t_1^*)$—optimal process, independent of target, desired (arbitrarily high) production value $V^1$ or of (minimal) time $t_1^*$ needed by the economy to achieve this production level.

☐ **Theorem 3.** If:
- conditions **(G1)-(G6)** are satisfied**,**
- solution $\{y^*(t)\}_{t=0}^{t_1^*}$ to the problem (6)-(7) is unique and
- in a certain period $\check{t} < t_1^*$

$$y^*(\check{t}) \in N,$$

then: $$\forall t \in \{\check{t}+1, \ldots, t_1^* - 1\} \; (y^*(t) \in N).$$

**Proof.** If $y^*(\check{t}) \in N$, then (similar to (17)) there exists $(y^0, Y^1, t_1)$-feasible process:

$$\tilde{\tilde{y}}(t) = \begin{cases} y^*(t), & t = 0, 1, \ldots, \check{t}, \\ \alpha_M^{t-\check{t}} y^*(\check{t}), & t = \check{t}+1, \ldots, t_1 - 1, \\ \alpha_M^{t_1-\check{t}-1} y^*(\check{t}) + y', & t = t_1, \end{cases}$$

in which:
- $t_1$ is the smallest natural number not less than $\check{t} + A$ where
  $A = \dfrac{\ln V^1 - \ln\langle \overline{p}, y^*(\check{t})\rangle}{\ln \alpha_M} > 0,$
- $0 \leq y' \leqq (\alpha_M - 1)\, \alpha_M^{t_1-\check{t}-1} y^*(\check{t}) \;\; (> 0),$
- condition (18) is satisfied.

Therefore:

$$\langle \overline{p}, y^*(t_1^*)\rangle \geq V^1 = \langle \overline{p}, \tilde{\tilde{y}}(t_1)\rangle = \langle \overline{p}, \tilde{\tilde{y}}(t_1 - 1) + y'\rangle = \alpha_M^{t_1-\check{t}-1} \langle \overline{p}, y^*(\check{t})\rangle + \langle \overline{p}, y'\rangle > 0. \quad (20)$$

According to (2) the optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$ satisfies the condition:

$$\langle \overline{p}, y^*(t_1^*)\rangle \leq \alpha_M^{t_1^*-\check{t}} \langle \overline{p}, y^*(\check{t})\rangle.$$

If $y^*(\tau) \notin N$ in a certain period $\tau \in \{\check{t}+1, \ldots, t_1^* - 1\}$, then according to Lemma 2,

$$\exists \delta_\varepsilon \in (0, \alpha_M) \langle \overline{p}, y^*(\tau+1)\rangle \leq (\alpha_M - \delta_\varepsilon)\langle \overline{p}, y^*(\tau)\rangle,$$

which leads to the inequality:

$$\langle \overline{p}, y^*(t_1^*)\rangle \leq \alpha_M^{t_1^*-\check{t}-1} (\alpha_M - \delta_\varepsilon) \langle \overline{p}, y^*(\check{t})\rangle. \quad (21)$$

From (20), (21) the result is:

$$\alpha_M^{t_1^*-\check{t}-1} (\alpha_M - \delta_\varepsilon) \langle \overline{p}, y^*(\check{t})\rangle > \alpha_M^{t_1-\check{t}-1} \langle \overline{p}, y^*(\check{t})\rangle > 0, \quad (22)$$

where $t_1 \geq t_1^*$. If $t_1 = t_1^*$, then the process $\left\{\tilde{\tilde{y}}(t)\right\}_{t=0}^{t_1}$ is $(y^0, Y^1, t_1^*)$-optimal[17]. If $t_1 > t_1^*$, then it follows from (22) that:

$$\alpha_M - \delta_\varepsilon > \alpha_M^{t_1 - t_1^*}. \tag{23}$$

For $t_1 = t_1^* + 1$ we have $\alpha_M - \delta_\varepsilon > \alpha_M$, therefore $\delta_\varepsilon < 0$, in contradiction to our assumption. When $t_1 - t_1^* = k > 1$, then from (23) we get $\alpha_M - \delta_\varepsilon > \alpha_M^k$, where $k \geq 2$, and this contradicts the condition $\alpha_M > 1$. The obtained contradictions conclude the proof. ■

The optimal process $\left\{y^*(t)\right\}_{t=0}^{t_1^*}$ a solution to the problem (6)-(7) with a set of target states (8) which in a certain period $\check{t} < t_1^*$ reaches the turnpike $N$ in a two-dimensional commodity space is illustrated in the following figure.

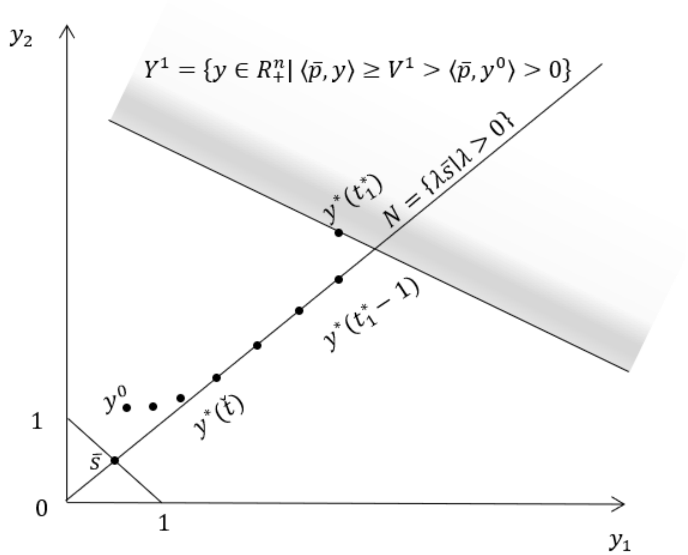


**Figure 1. The illustration of Theorem 3. The trajectory $\left\{y^*(t)\right\}_{t=0}^{t_1^*}$ (solution to the problem (6)-(7)) in the neighbourhood of the turnpike** $N = \{\lambda\bar{s} \mid \lambda > 0\} \subset \mathbb{R}_+^2$

Source: Own elaboration.

## 3. The second minimal-time growth problem

In the problem (6)-(7), the set of target states $Y^1$ includes all production vectors lying on or above the hyperplane

[17] When $t_1 = t_1^*$, then it follows from (22) that $\delta_\varepsilon < \alpha_M - 1$. This situation cannot be ruled out because $\alpha_M > 1$. Thus, when $t_1 = t_1^*$, there may exists another $(y^0, Y^1, t_1^*)$—feasible process with a production trajectory in at least one period different from $\left\{\tilde{\tilde{y}}(t)\right\}_{t=0}^{t_1}$). Therefore the hypothesis of the theorem includes the solution's condition of uniqueness.

$$H_{\overline{p}} = \{y \in R^n \mid \langle \overline{p}, y \rangle = V^1\},$$

with the normal vector $\overline{p} > 0$ and parameter (absolute term) $V^1 > \langle \overline{p}, y^0 \rangle > 0$ (see Figure 1). From the economic point of view this is a relatively weak limitation. A set defined this way does not take into consideration social expectations / demands / needs formulated for the target production volume of (at least some) commodities in the economy[18]. Now this condition will be included. A solution to the problem (6)-(7), in which the current target states set (8) will be replaced with the set

$$Y^1 = \{y \in R^n_+ \mid y \geqq y^1\}, \tag{24}$$

will be found. Vector $y^1$ a is a minimal, socially desirable level of production ($y^1 > y^0$). Lemma 1 remains valid, but now the set $Y^1$ has the form (24) and period $t_1$, in which the condition (10) is satisfied, is the smallest natural number satisfying the condition (25)[19]. It is assumed that vector $y^1 > y^0$ that bounds from below the set of target states $Y^1$ meets the following condition:

**(G7)**
$$\exists M \left( \frac{\max_i y_i^1}{\min_i y_i^1} \leq M \right)^{20}.$$

□ **Lemma 3.** If the conditions **(G1)-(G7)** are satisfied, then:

**(i)** $\forall y^1 > y^0$ there exists the following $(y^0, Y^1, t_1)$—feasible process $\left\{\tilde{y}(t)\right\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} y^0, & t = 0, \\ \sigma \alpha_M^t \overline{s}, & t = 1, 2, \ldots, t_1, \end{cases}$$

(see (9)), in which $\sigma = \min_i \frac{y_i^0}{\overline{s}_i} > 0$, $\tilde{y}(t_1) \geqq y^1$ (in other words $\tilde{y}(t_1) \in Y^1$) and $t_1$ is the minimal natural number that satisfies the condition:

$$t_1 \geq \frac{\ln A_1}{\ln \alpha_M} > 0, \quad \text{where} \quad A_1 = \sigma^{-1} \max_i \frac{y_i^1}{\overline{s}_i} > 1, \tag{25}$$

**(ii)** there exists a natural number $l$ independent of $y^1$ (nor of $t_1$), such that if $t_1 > l$, then $\tilde{y}(t_1 - l) \leqq y^1$.

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[18] For example, resulting from the consumer needs of the society.

[19] See further, Lemma 3(i).

[20] As the vector $y^1$ (determining the form of the target states set $Y^1$) any production vector greater than the initial vector $y^0$, in which the range (distance) between the coordinate values does not increase „too rapidly" (grows not faster than linearly) may be accepted. Under other assumptions here $M \geq 1$.

**Proof** of part **(i)** is obtained by repeating directly the proof of Lemma 1(i), but $t_1$ now is the smallest natural number satisfying the condition (25).

**(ii)** Let $l_{y1}$ the smallest natural number such that:

$$\tilde{y}(t_1 - l_{y^1}) = \sigma \alpha_M^{t_1 - l_{y^1}} \overline{s} \leqq y^1. \tag{26}$$

Such a number exists because $\tilde{y}(0) = y^0 < y^1 \leqq \tilde{y}(t_1)$. From (26) results:

$$l_{y^1} \geq t_1 - \frac{\ln A_2}{\ln \alpha_M}, \quad A_2 = \sigma^{-1} \min_i \frac{y_i^1}{\overline{s}_i} > 1, \tag{27}$$

$(A_2 \leq A_1)$. Since $t_1$ is the smallest natural number satisfying the condition (25), then:

$$t_1 \leq \frac{\ln A_1}{\ln \alpha_M} + 1. \tag{28}$$

Similarly, since is the smallest natural number satisfying the condition (27), therefore:

$$l_{y^1} \leq t_1 - \frac{\ln A_2}{\ln \alpha_M} + 1. \tag{29}$$

From (28), (29) the inequality follows:

$$l_{y^1} \leq \frac{\ln \dfrac{A_1}{A_2}}{\ln \alpha_M} + 2. \tag{30}$$

From the construction of numbers $A_1$, $A_2$ and from **(G7)** we get:

$$1 \leq \frac{A_1}{A_2} = \frac{\max\limits_i \dfrac{y_i^1}{\overline{s}_i}}{\min\limits_i \dfrac{y_i^1}{\overline{s}_i}} \leq \frac{\dfrac{\max\limits_i y_i^1}{\min\limits_j \overline{s}_j}}{\dfrac{\min\limits_i y_i^1}{\max\limits_j \overline{s}_j}} = \frac{\max\limits_j \overline{s}_j}{\min\limits_j \overline{s}_j} \cdot \frac{\max\limits_i y_i^1}{\min\limits_i y_i^1} = C \frac{\max\limits_i y_i^1}{\min\limits_i y_i^1} \leq C \cdot M,$$

where $C = \dfrac{\max\limits_j \overline{s}_j}{\min\limits_j \overline{s}_j} \geq 1$ and the number $M \geq 1$ satisfies the condition **(G7)**. Condition (30) implies the following conclusion:

$$l_{y^1} \leq Q = \frac{\ln(C \cdot M)}{\ln \alpha_M} + 2.$$

As the number $l$ mentioned in the Lemma 3, it is enough to take the smallest natural number not less than $Q$. ■

The equivalent of the „weak" turnpike theorem is the following Theorem 4.

☐ **Theorem 4.** If conditions **(G1)-(G7)** are satisfied and the sequence of production vectors $\left\{y^*(t)\right\}_{t=0}^{t_1^*}$ is ($y^0$, $Y^1$, $t_1^*$)—optimal growth process[21] then for any positive number $\varepsilon > 0$ there exists a natural number $k_\varepsilon$, such that the number of time periods, in which the condition (13) is satisfied does not exceed $k_\varepsilon$. Number $k_\varepsilon$ does not depend on the minimum target production volume $y^1$ (i.e. the form of the target state set $Y^1$) nor on $t_1^*$.

**Proof.** Following the proof of Theorem 2 and assuming that in the periods $\tau_1, \tau_2, \ldots, \tau_k$ condition (13) is satisfied the condition (16) is deduced. Under the assumptions **(G1)-(G7)**, according to Lemma 3, there exists a natural number $l$ and an ($y^0$, $Y^1$, $t_1$)-feasible process of the form $\left\{\tilde{y}(t)\right\}_{t=0}^{t_1}$ (9), such that $\tilde{y}(t_1)=\sigma\alpha_M^{t_1}\overline{s}\in Y^1$, $t_1 > l$, and $\tilde{y}(t_1-l)=\sigma\alpha_M^{t_1-l}\overline{s}\leqq y^1$(number $l$ does not depend on $y^1$ nor on $t_1$). Since $Y^1 \ni y^*(t_1^*)\geqq y^1 \geqq \tilde{y}(t_1-l)$, thus

$$\left\langle\overline{p},y^*(t_1^*)\right\rangle\geq\left\langle\overline{p},\tilde{y}(t_1-l)\right\rangle=\sigma\alpha_M^{t_1-l}\left\langle\overline{p},\overline{s}\right\rangle>0. \tag{31}$$

Combining (16), (31) we get the inequality:

$$\alpha_M^{t_1^*-k}(\alpha_M-\delta_\varepsilon)^k\left\langle\overline{p},y^0\right\rangle\geq\sigma\alpha_M^{t_1-l}\left\langle\overline{p},\overline{s}\right\rangle>0.$$

From which it follows (given that $t_1 \geq t_1^*$), that:

$$0<\left(\frac{\alpha_M}{\alpha_M-\delta_\varepsilon}\right)^k\leq\frac{\alpha_M^l\left\langle\overline{p},y^0\right\rangle}{\sigma\left\langle\overline{p},\overline{s}\right\rangle},$$

or

$$k\leq B=\frac{\ln A}{\ln\alpha_M-\ln(\alpha_M-\delta_\varepsilon)},$$

where $A=\dfrac{\alpha_M^l\left\langle\overline{p},y^0\right\rangle}{\sigma\left\langle\overline{p},\overline{s}\right\rangle}>1$ ($B > 0$). As a number $k_\varepsilon$ it is enough to take the smallest natural number not less than $B$. The number $B$, similarly to $l$, does not depend on $y^1$ nor on $t_1$. ■

Theorems 2-4 remain true if in the problem (6)-(7) we replace the initial condition $y(0) = y^0 > 0$ with a weaker condition:

$$y(0) = y^0 \geq 0$$

[21] There is a solution of the problem (6)-(7) with the set of target states (24).

and one of the following demands:

- there exists an $(y^0, \check{t})$—feasible process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, such that $\check{y}(\check{t}) > 0$ (leading to a positive production vector),
- there exists an $(y^0, \check{t})$—feasible process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, such that $\check{y}(\check{t}) \in N$ (leading to the turnpike).

The solution of the problem (6)-(7) with the set of target states (23), when $y^1 \ngtr y^0$ remains to be found. A special case when $y^1 \leqq y^0$ leads to the trivial solution $y^*(0)$. The optimal process—solution to the problem (6)-(7) with the set of target states (24) creates in this case the first and only element of a sequence $\{y^*(t)\}_{t=0}^{t_1^*}$, in which $t_1^* = 0$.

The solution to the minimal-time growth of the type (6)-(7) is probably also characterized by the so-called strong turnpike effect. This proof is beyond the scope of this work.

## Conlusions

The novelty of the approach proposed in the paper consists in replacing the standard-used criterion for assessing the quality of economic growth processes in the form of an abstract function of utility by natural, simple and at the same time having a high social acceptance the criterion for which is the time needed by an economy starting from a given state to reach the postulated target state. Although the theory of utility (preference) occupies a prominent place in mathematical economics—utility is the adopted measure of assessing the quality of functioning of economic entities, both on the macro and micro scale—but due to its abstract form this measure is mainly used in theoretical considerations. In empirical research both its exemplification and form are not so obvious.

The main contribution of the work is the proof that a change in the growth criterion does not deprive the economy of its fundamental turnpike properties. The results may be particularly inspiring for those interested in mathematical economics due to the emerging prospect of further possible extensive theoretical research because the article contains evidence of the main properties of optimal processes with a minimal-time growth criterion only in the simplest, stationary version of the Gale economy with a single bus. It is highly probable that they are valid also in the case of both non-stationary economy with variable technology and multi-lane production bus, as well as a Gale-type economy with investments (an example of the latest is presented in Panek (2021). However this requires further research. There is also a vast research field for econometricians who are naturally more interested in empirical research, e.g. on the basis of the dynamic Leontief model with the minimal time increment criterion.

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