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STATISTICAL INFERENCE OF EXPONENTIAL RECORD DATA UNDER KULLBACK-LEIBLER DIVERGENCE MEASURE

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ABSTRACT

Based on one parameter exponential record data, we conduct statistical inferences (maximum likelihood estimator and Bayesian estimator) for the suggested model parameter. Our second aim is to predict the future (unobserved) records and to construct their corresponding prediction intervals based on observed set of records. In the estimation and prediction processes, we consider the square error loss and the Kullback-Leibler loss functions. Numerical simulations were conducted to evaluate the Bayesian point predictor for the future records. Finally, data analyses involving the times (in minutes) to breakdown of an insulating fluid between electrodes at voltage 34 kv have been performed to show the performance of the methods thus developed on estimation and prediction.

Key words: Bayes estimation, Bayes prediction, record values, Kullback-Leibler divergence measure, exponential distribution.

1. Introduction

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (iid) random variables from exponential distribution with probability density function (pdf)

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & if\ x > 0, \theta > 0 \\ 0 & if\ x \leq 0\ , \end{cases} \tag{1}$$

and cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\theta x}, x > 0, \theta > 0. \tag{2}$$

Based on the distribution function of exponential distribution, the distribution can be used effectively in analyzing any lifetime data, especially when censoring is used or if the data are grouped. The exponential distribution received considerable attention in the literature during the last three decades and was commonly used in many situations of lifetime data analysis. The exponential distribution was typically used

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to model time intervals between random events, such as the length of time between arrivals at a service station. In queuing theory, the service times of agents in a system are often exponentially distributed. It is worth mentioning here that when times between "random events" follow the exponential distribution with rate θ, then the total number of events in a time period of length t follows the Poisson distribution with parameter θt. Reliability theory and reliability engineering also use the exponential distribution extensively. Many authors have developed inference procedures for exponential distribution. Abufoudeh *et al.* (2017) have obtained the Bayes estimate of the parameter of the exponential distribution under Kullback-Leibler divergence measure. Nasiri *et al.* (2012) have used the upper record range statistic to draw inferences from the parameter of the exponential distribution. Based on record data, Janeen (2004) have discussed the empirical Bayes estimators for the parameter of the exponential distribution. An interested reader may refer to Balakrishnan *et al.* (2005). Balakrishnan *et al.* (1995) have established some recurrence relations for single and product moments from exponential distribution based on record values. Ahsanullah and Kirmani (1991) have obtained some characterizations of the exponential distribution based on lower record values.

In many real life situations, we may be interested in the largest value of data such as stock exchange, weather and sports, because in some cases the decisions may depend on the largest values. Chandler (1952) has introduced the study of record values and has reported many of the basic properties for records. Bdair and Raqab (2009) have studied the mean residual lifetime of records and Bdair and Raqab (2012) have studied the upper bounds of the mean residual lifetime of records. Properties of record values have been extensively studied in the literature by Ahsanullah (1988, 1995), Arnold and Balakrishnan (1989), Arnold *et al.* (1998), Nevzorov (2001), Kamps (1995) and Jaheen (2004).

Let $X_1, X_2, ...$ be a sequence (X-sequence) of iid random variables from the exponential distribution given in Eq. (1). The random variable X_j is called an upper record if $X_j > X_i$ for all $i = 1, 2, ..., j-1$. To formalize this concept, let $X_1, X_2, ..., X_n$ be a sample of size n from X-sequence. By convention X_1 is the first record where $U(1) = 1$ is the first record time. For $n \geq 2$, X_j is an upper record if its value exceeds all of the previous observations. To obtain record data, the n^{th} record time $U(n)$ is defined using the recursive formula $U(n) = \min\left\{j : X_j > X_{U(n-1)}\right\}$, then the n^{th} record is $X_{U(n)}$.

In this research work, based on record values we estimate the parameter θ of the exponential distribution using both classical and Bayesian methods of estimation, as well as we predict the future record values depending on a sequence of past records. In the Bayesian estimation method and in prediction of future values, we use two types of loss functions; the first is the square error loss (SEL) function, which is defined as

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2. \tag{3}$$

The second is the Kullback-Leibler divergence measure (KL) as an alternative loss function. The Kullback-Leibler divergence measure (also called relative entropy measure) has been introduced by Kullback and Leibler (1951). Unlike the square error loss function, KL does not measure the discrepancy between an unknown parameter and its estimate, but between the actual distribution $f(x|\theta)$ of the record sample $\underset{\sim}{x}$ of size n from X-sequence and the approximate distribution $\hat{f}(x|\hat{\theta})$. As a consequence, it is invariant with one-to-one reparametrization of the parameters and, hence, becomes a serious competitor to square error loss function. An interesting property of the KL divergence is $KL(f,\hat{f}) \geqslant 0$ with equality if and only if $f(x|\theta) = \hat{f}(x|\hat{\theta})\ \forall x \in \underset{\sim}{x}$. For more details, one may refer to Abufoudeh *et al.* (2018) and Singh *et al.* (2014). The Kullback-Leibler divergence measure of the true distribution $f(x|\theta)$ from the approximate distribution $\hat{f}(x|\hat{\theta})$ is defined as

$$\begin{aligned} KL(f,\hat{f}) &= E_f\left[\log \frac{f(x|\theta)}{\hat{f}(x|\hat{\theta})}\right] \\ &= E_f\left[\log \frac{\theta e^{-\theta x}}{\hat{\theta} e^{-\hat{\theta} x}}\right] \\ &= \log \frac{\theta}{\hat{\theta}} - (\theta - \hat{\theta})\, E_f(X) \\ &= \frac{\hat{\theta}}{\theta} - \log \frac{\hat{\theta}}{\theta} - 1. \end{aligned} \tag{4}$$

This measure is called Kullback-Leibler error loss (KEL) function and it is denoted by $KL(\theta,\hat{\theta})$.

The rest of the article is organized as follows. In Section 2, based on record data from the exponential distribution of parameter θ, we find the maximum likelihood estimate and the Bayes estimate of θ, under both SEL and KEL functions. In Section 3, we find the point and credible interval of the future records based on previously known records generated from the exponential distribution. A simulation study based on different sizes of record samples from the exponential distribution and real life example is presented in Section 4. Simulation studies that compare all classical and Bayes estimates along with a real life example are presented and discussed in Section 5. Finally, we conclude the results thus obtained in Section 6.

2. Classical method

The most common classical technique in estimating the unknown parameters of a distribution is the maximum likelihood (ML) estimation method. The ML estimation method chooses, as an estimate of θ, the value $\hat{\theta}$, which maximizes the likelihood function. Suppose we observe n upper record values $\underset{\sim}{x} = (x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)})$ from X-sequence of iid random variables following the exponential distribution with pdf and cdf given in (1) and (2), respectively. According to Arnold *et al.* (1998), the likelihood function of records based on exponential distribution is given as

$$\begin{aligned} L(\theta|\underset{\sim}{x}) &= \prod_{i=1}^{n-1} \frac{f(x_{U(i)}|\theta)}{1-F(x_{U(i)}|\theta)} f(x_{U(n)}|\theta) \\ &= \theta^n \, e^{-\theta x_{U(n)}}. \end{aligned} \tag{5}$$

Applying $\ln(.)$ for both sides, we obtain the log-likelihood function

$$\ln L(\theta|\underset{\sim}{x}) = n \ln \theta - \theta x_{U(n)}.$$

Differentiating the above equation with respect to θ and equating the resulting term to zero, we obtain the ML estimator of θ as follows

$$\hat{\theta} = \frac{n}{x_{U(n)}}.$$

3. Bayesian method

In this section, we introduce the Bayesian point estimation and credible interval for the unknown parameter of the exponential distribution based on upper record values. The KEL and SEL functions are used to approximate the point estimation of the unknown parameter θ.

3.1. Bayesian estimation

The inference problem concerning the unknown parameter θ can easily be dealt with using the Bayesian method, since the posterior distribution supposedly contains all available information about θ (both sample and prior information). The posterior distribution of θ given $\underset{\sim}{x}$ is defined as

$$\pi(\theta|\underset{\sim}{x}) = \frac{L(\theta|\underset{\sim}{x})\, \pi_1(\theta)}{\int_0^\infty L(\theta|\underset{\sim}{x})\, \pi_1(\theta)\, d\theta}. \tag{6}$$

Assume that θ has the conjugate gamma prior with pdf

$$\pi_1(\theta|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} & \text{if } \theta > 0, \\ 0 & \text{if } \theta \le 0, \end{cases} \tag{7}$$

where $a > 0$, $b > 0$ are the hyper-parameters. By substituting Eq. (5) and Eq. (7), we immediately obtain

$$\pi(\theta|\underset{\sim}{x}) = \frac{(b + x_{U(n)})^{a+n}}{\Gamma(a+n)}\, \theta^{a+n-1}\, e^{-\theta(b+x_{U(n)})}. \tag{8}$$

That is, the posterior distribution of θ given $\underset{\sim}{x}$, is $Gamma(a+n, b+x_{U(n)})$.
The Bayesian estimator of θ under the SEL function is then given by

$$\hat{\theta}_{B_1} = E_{posterior}(\theta|\underset{\sim}{x}) = \frac{a+n}{b+x_{U(n)}}.$$

The Bayesian estimator of θ under the KEL function $\hat{\theta}_{B_2}$ is obtained by minimizing the risk function

$$E_{posterior}(KL(\theta,\hat{\theta})) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1\right)\pi(\theta|\underset{\sim}{x})\, d\theta.$$

By differentiating $E_{posterior}(KL(\theta,\hat{\theta}))$ with respect to $\hat{\theta}$ and setting its derivative to zero, we get the equation

$$\int_0^\infty \left(\frac{1}{\theta} - \frac{1}{\hat{\theta}}\right)\pi(\theta|\underset{\sim}{x})\, d\theta = 0.$$

Solving for $\hat{\theta}$ we conclude

$$\hat{\theta}_{B_2} = \frac{1}{E_{posterior}(\frac{1}{\theta}|\underset{\sim}{x})} \tag{9}$$

Using Eq. (8) and Eq. (9), the Bayes estimator under the KEL function is then

$$\hat{\theta}_{B_2} = \frac{\Gamma(a+n)}{(b+x_{U(n)})\Gamma(a+n-1)}.$$

3.2. Credible interval

Since the posterior distribution of θ follows gamma distribution, a credible interval of θ can be obtained as follows:
The $(1-\beta)100\%$ credible interval of θ, (C_L, C_U), satisfies the following two conditions

$$P(C_L < \theta < \infty) = 1 - \frac{\beta}{2}, \tag{10}$$

$$P(C_U < \theta < \infty) = \frac{\beta}{2}. \tag{11}$$

Now, from Eq. (10), we have

$$\int_{C_L}^{\infty} \frac{\left(b+x_{U(n)}\right)^{a+n}}{\Gamma(a+n)}\, \theta^{a+n-1}\, e^{-\theta\left(b+x_{U(n)}\right)}\, d\theta = 1 - \frac{\beta}{2}.$$

Using the transformation $u=\theta(b+x_{U(n)})$, we immediately obtain

$$\int_{(b+x_{U(n)})C_L}^{\infty} u^{a+n-1}\, e^{-u}\, du = \left(1-\frac{\beta}{2}\right)\Gamma(a+n).$$

Based on the incomplete gamma function, which is defined as

$$\Gamma(c,d)=\int_{d}^{\infty} x^{c-1}e^{-x}\, dx,\ c>0,\ d>0, \tag{12}$$

we immediately obtain

$$\Gamma\left(a+n,\ (b+x_{U(n)})C_L\right)=\left(1-\frac{\beta}{2}\right)\Gamma(a+n). \tag{13}$$

Similarly from Eq. (11), we obtain

$$\Gamma\left(a+n,\ (b+x_{U(n)})C_U\right)=\frac{\beta}{2}\Gamma(a+n). \tag{14}$$

Consequently, we conclude the lower and upper credible interval C_L and C_U by solving Eqs. (13) and (14) using a suitable numerical method, with respect to C_L and C_U, respectively.
In particular, if a is a positive integer, then the chi-square table values can be used to construct the credible interval for θ as follows:
Since θ has $Gamma(a+n,\ b+x_{U(n)})$, then a pivotal statistic $Q=2\theta(b+x_{U(n)})$ has $\chi^2_{2(a+n)}$. Hence, the $(1-\beta)100\%$ credible interval for θ is given by

$$\frac{\chi^2_{(1-\frac{\beta}{2},\,2(a+n))}}{2(b+x_{U(n)})}<\theta<\frac{\chi^2_{(\frac{\beta}{2},\,2(a+n))}}{2(b+x_{U(n)})},$$

where $\chi^2_{(\beta,r)}$ is the $100\beta^{th}$ upper percentile of chi-square with r degrees of freedom.

4. Bayesian prediction

In this section, we consider the problem of one sample prediction. The idea of this problem is to find the Bayes predictors and bounds of future record values based on observed records which have been taken from X-sequence. We consider the two loss functions SEL and KEL to find the predictors and bounds. One sample prediction problem has been studied by many authors, see Ahsanullah (1980), Dunsmore (1983), Berred (1998) and Bdair and Raqab (2016).

4.1. Predictors of future records

Let $\underset{\sim}{x} = (x_{U(1)}, x_{U(2)}, ..., x_{U(m)})$ be the m observed upper records. To find the Bayes predictor of the n^{th} future upper record $X_{U(n)},\ 1 \leq m < n$, we need to derive the posterior predictive density at any point $y > x_{U(m)}$, as follows:
The conditional probability density function of $y = x_{U(n)}$ given that the observed upper record data $\underset{\sim}{x}$ is indicated by $f_{X_{U(n)}|\underset{\sim}{x}}(y|\theta)$. Since the upper record values satisfy the Markovian property, then $f_{X_{U(n)}|\underset{\sim}{x}}(y|\theta) = f_{X_{U(n)}|x_{U(m)}}(y|\theta)$. The conditional probability density function of $y = x_{U(n)}$ given that $x_{U(m)}$ is given (see Ahsanullah (1995)) by

$$
\begin{aligned}
f_{X_{U(n)}|x_{U(m)}}(y|\theta) &= \frac{[H(y) - H(x_{U(m)})]^{n-m-1}}{(n-m-1)!} \frac{f(y|\theta)}{1 - F(x_{U(m)|\theta})} \\
&= \frac{\theta^{n-m}\, e^{\theta x_{U(m)}}}{(n-m-1)!} (y - x_{U(m)})^{n-m-1}\, e^{-\theta y},
\end{aligned}
$$

where $H(.) = -\ln(1 - F(.))$. Using the well-known Binomial expansion, we immediately obtain

$$
f_{X_{U(n)}|x_{U(m)}}(y|\theta) = \frac{\theta^{n-m}\, e^{\theta x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i\, y^{n-m-1-i}\, e^{-\theta y}.
$$

The posterior predictive density at any point $y > x_{U(m)}$, is then

$$
\begin{aligned}
f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\theta) &= E_{posterior}\left[f_{X_{U(n)}|x_{U(m)}(y|\theta)}\right] \\
&= \int_0^\infty \frac{\theta^{n-m}\, e^{\theta x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i\, y^{n-m-1-i}\, e^{-\theta y}\, \pi(\theta|\underset{\sim}{x})\, d\theta.
\end{aligned}
$$

The Bayes estimator of future records under the SEL function, is given by

$$
\begin{aligned}
X^{BP1}_{U(n)} &= E_{f^P}(Y|\underset{\sim}{x}) \\
&= \int_{x_{U(m)}}^\infty \left[\int_0^\infty \frac{\theta^{n-m}\, e^{\theta x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i\, y^{n-m-1-i}\, e^{-\theta y}\, \pi(\theta|\underset{\sim}{x})\, d\theta\right] dy \\
&= \int_0^\infty \frac{\theta^{n-m}\, e^{\theta x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i \left(\int_{x_{U(m)}}^\infty y^{n-m-1-i}\, e^{-\theta y}\, dy\right) \pi(\theta|\underset{\sim}{x})\, d\theta.
\end{aligned}
$$

Using the transformation $u=\theta y$ and Eq. (12), we obtain

$$X_{U(n)}^{BP1}=\int_0^\infty \frac{e^{\theta x_{U(m)}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i(x_{U(m)})^i\,\frac{\Gamma(n-m-i+1,\theta x_{U(m)})}{\theta^{1-i}}\,\pi(\theta|\underset{\sim}{x})\,d\theta.$$

Based on the MCMC samples $\{\theta_j;\ j=1,2,...,M\}$ generated from Eq. (8), the Bayes predictor of future records becomes

$$\hat{X}_{U(n)}^{BP1}=\frac{1}{M}\sum_{j=1}^{M}\frac{e^{\theta_j x_{U(m)}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i(x_{U(m)})^i\,\frac{\Gamma(n-m-i+1,\theta_j x_{U(m)})}{\theta_j^{1-i}}.$$

The Bayes predictor of future records under the KEL function, is given by

$$X_{U(n)}^{BP2}=\frac{1}{E_{f^P}(\frac{1}{Y}|\underset{\sim}{x})}$$

Using a similar argument, the Bayes predictor of future records will be

$$\hat{X}_{U(n)}^{BP2}=\frac{M}{\sum_{j=1}^{M}\frac{e^{\theta_j x_{U(m)}}}{(n-m-1)!}\sum\limits_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i(x_{U(m)})^i\,\frac{\Gamma(n-m-i-1,\theta_j x_{U(m)})}{\theta_j^{-1-i}}}.$$

4.2. Bounds of future records

Under SEL function, we present the Bayesian predicted bounds of the $(1-\beta)100\%$ interval of the future record value, $Y=X_{U(n)}$, (Y_L,Y_U).
The lower bound Y_L can be obtained by solving the following equation for Y_L

$$\int_{Y_L}^{\infty} f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\theta)\,dy=1-\frac{\beta}{2},$$

or equivalently

$$\int_{Y_L}^{\infty} f^P_{X_{U(n)}|X_{U(m)}}(y|\theta)\,dy=1-\frac{\beta}{2}.$$

This is equivalent to solve the equation

$$\int_{Y_L}^{\infty}\left[\int_0^\infty \frac{\theta^{n-m}\,e^{\theta x_{U(m)}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i(x_{U(m)})^i\,y^{n-m-1-i}\,e^{-\theta y}\,\pi(\theta|\underset{\sim}{x})\,d\theta\right]dy=1-\frac{\beta}{2},$$

which yields to

$$\int_0^\infty \left[\frac{e^{\theta x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i \frac{\Gamma(n-m-i, \theta Y_L)}{\theta^{-i}} \right] \pi(\theta|\underset{\sim}{x})\, d\theta = 1 - \frac{\beta}{2}.$$

Based on the MCMC samples $\{\theta_j\, ;\ j = 1,2,...,M\}$, the lower bound Y_L can be found by solving the equation

$$\frac{1}{M} \sum_{j=1}^{M} \left[\frac{e^{\theta_j x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i \frac{\Gamma(n-m-i, \theta_j Y_L)}{\theta_j^{-i}} \right] = 1 - \frac{\beta}{2}.$$

Following the same approach, the upper bound Y_U can be found by solving the equation

$$\int_{Y_U}^\infty f^P_{X_{U(n)}|X_{U(m)}}(y|\theta)\, dy = \frac{\beta}{2}.$$

Under the KEL function, the lower and upper bounds can be obtained by solving the following two equations for Y_L and Y_U, respectively.

$$\left[\frac{1}{M} \sum_{j=1}^{M} \left[\frac{e^{\theta_j x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i \frac{\Gamma(n-m-i, \theta_j Y_L)}{\theta_j^{-i}} \right]^{-1} \right]^{-1} = 1 - \frac{\beta}{2},$$

and

$$\left[\frac{1}{M} \sum_{j=1}^{M} \left[\frac{e^{\theta_j x_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^i (x_{U(m)})^i \frac{\Gamma(n-m-i, \theta_j Y_U)}{\theta_j^{-i}} \right]^{-1} \right]^{-1} = \frac{\beta}{2}.$$

5. Simulation study and illustrative example

Here, we perform a simulation study based on the exponential distribution with $\theta = 2$ $(E(2))$. From $(E(2))$ we generate different sample size cases of records $n = 5, 7, 10$. A sample of size n upper record can be generated using the transformation

$$X_{U(k)} = \frac{\sum_{i=1}^{k} e(i)}{\theta}, k = 1, 2, ..., n,$$

where $\{e(i), i \geq 0\}$ is a sequence of iid $E(1)$ [see Arnold *et al.* (1998), p.20].
Using the mean square error (MSE), we investigate the performance of the maximum likelihood estimator (MLE) and the Bayesian estimator of the parameter θ, based on 1000 replications. In Bayesian method, we use an informative prior $\pi_1(\theta)$ of gamma distribution with hyper parameters $a = 2$ and $b = 1$, based on the two

suggested types of error loss functions SEL and KEL, to estimate the unknown parameter. The MCMC samples, which are used in computing the Bayes estimates, are generated from gamma distribution with hyper parameters $a=2$ and $b=1$. We compare the performance of the Bayes estimates of θ under two different priors for θ; the noninformative prior $(a=b=0)$ (Prior 0), where the prior density becomes improper and not specifically related to the gamma density, and the informative prior $(a=2, b=1)$ (Prior 1). In the prediction process, we consider only the one sample prediction problem to find the predictors of the future records as well as the predicted intervals of the predictors based on the informative prior under the suggested loss functions SEL and KEL for all cases of records $n=5,7,10$.

In Tables 1 and 2, we present the MLEs and the Bayes estimators of θ under SEL and KEL functions when Prior 0 and 1 are used.

Table 1. MLEs and Bayes estimators when $\theta=1$, MSEs are reported in parentheses.

		Prior 0		Prior 1	
Cases	MLE	SEL	KEL	SEL	KEL
$n=5$	1.2268	1.2268	0.9814	1.3337	1.1432
	(0.6233)	(0.6233)	(0.3664)	(0.3801)	(0.2180)
$n=7$	1.1530	1.1530	0.9883	1.2439	1.1057
	(0.2605)	(0.2605)	(0.1743)	(0.2397)	(0.1535)
$n=10$	1.1067	1.1067	0.9960	1.1731	1.0754
	(0.1795)	(0.1795)	(0.1362)	(0.1692)	(0.1227)

Table 2. MLEs and Bayes estimators when $\theta=2$, MSEs are reported in parentheses.

		Prior 0		Prior 1	
Cases	MLE	SEL	KEL	SEL	KEL
$n=5$	2.3990	2.3990	1.9192	2.1909	1.8779
	(1.7493)	(1.7493)	(1.0242)	(0.5327)	(0.3795)
$n=7$	2.3899	2.3899	2.0484	2.2062	1.9611
	(1.2336)	(1.2336)	(0.7970)	(0.4922)	(0.3568)
$n=10$	2.2073	2.2073	1.9866	2.1596	1.9796
	(0.5432)	(0.5432)	(0.4054)	(0.4094)	(0.3230)

It can be observed from Tables 1 and 2 that the Bayes estimators show superior behaviour over the MLEs of θ as these estimates provide smaller MSEs. Furthermore, it is evident that the Bayes estimators obtained under Prior 1 compete quite well with those obtained under Prior 0 in terms of the MSE criterion. It can be also noted, as expected, that the MSEs tend to be smaller as the number of observed records increases.

In Table 3, we present the average credible interval length (AL) and coverage probability (CP) for the 95% confidence interval when $\theta=2$ under SEL and KEL for $n=5,7,10$. Table 4 contains different percentiles of the generated value of θ, which are basically generated when $\theta=2$ as an initial value.

Table 3. AL and CP for the 95% confidence intervals when $\theta = 2$.

Cases		Prior 0		Prior 1	
		AL	CP	AL	CP
$n=5$	θ	6.1430	0.93	4.1050	0.96
$n=7$	θ	4.7036	0.95	3.6353	0.96
$n=10$	θ	3.8140	0.94	3.1574	0.95

Table 4. Percentiles for the generated values of θ.

	0.005	0.025	0.05	0.5	0.95	0.975	0.995
Percentiles	0.3685	0.6226	0.7772	1.9565	3.7570	4.1501	5.046

It can be noticed from Table 3 that the ALs of the 95% confidence intervals are better when using Prior 1 than that when using Prior 0, and that the ALs decrease as the number of observed records increases.

Table 5 contains the predicted values and the corresponding 95% predicted intervals for the future record $X_{U(n)}$, $1 \leq m < n$ based on observed records under SEL and KEL functions when Prior 1 is used.

Table 5. Predicted values and the corresponding 95% predicted intervals for $X_{U(n)}$, $1 \leq m < n$ under SEL and KEL functions when Prior 1 is used.

Cases	$X_{U(n)}$	SEL		KEL	
		Predicted value	95% predicted interval	Predicted value	95% predicted interval
$n=5$	$X_{U(6)}$	0.5595	$(0.3363, 1.2840)$	0.4883	$(0.3362, 0.8862)$
	$X_{U(7)}$	0.7878	$(0.3761, 1.8599)$	0.6587	$(0.3757, 1.1092)$
	$X_{U(8)}$	1.0160	$(0.4413, 2.3957)$	0.8371	$(0.4402, 1.2898)$
$n=7$	$X_{U(8)}$	2.7196	$(2.3172, 3.9666)$	2.6615	$(2.3171, 3.4318)$
	$X_{U(9)}$	3.1313	$(2.3937, 4.9242)$	3.0183	$(2.3933, 3.9143)$
	$X_{U(10)}$	3.5431	$(2.5208, 5.8033)$	3.3774	$(2.5191, 4.3131)$
$n=10$	$X_{U(11)}$	2.6875	$(2.3913, 3.5917)$	2.6546	$(2.3912, 3.2473)$
	$X_{U(12)}$	2.9908	$(2.4490, 4.2793)$	2.9258	$(2.4487, 3.6192)$
	$X_{U(13)}$	3.2940	$(2.5454, 4.9082)$	3.1978	$(2.5444, 3.9231)$

We can observe from Table 5 that all predicted values are located within the predicted intervals and it is worth to note that the predicted intervals get to be wider as the values n increases, i.e. when we try to predict future values that are much wider than the observed data.

Example (real data):

To illustrate the results of this work thus obtained, we analyse the real data of times (in minutes) to breakdown of an insulating fluid between electrodes at voltage 34 kv. These data are originally reported in Lawless (1982, Table 1.1, p.3). The complete data set consists of 19 times to breakdown: 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89 and this involves a substantial extrapolation from the exponential data. From these data, we extract $n = 7$ upper record values which are: 0.96, 4.15, 8.01, 31.75, 33.91, 36.71, 72.89. Table 6 contains the ML estimator, the Bayes estimator of θ under the SEL and KEL functions when Prior 0 and 1 are used, and the corresponding 95% credible interval of the unknown parameter θ.

Table 6. MLE and Bayes estimates of θ using SEL and KEL functions under Prior 0 and 1.

Cases		Prior 0		Prior 1		
	MLE	SEL	KEL	SEL	KEL	95% credible interval
$n=7$	0.9604	0.9604	0.8232	1.0858	0.9651	$(0.4274, 2.0254)$

Table 7 contains the 8th, 9th and 10th future records, and also their 95% predicted intervals based on the $n=7$ observed upper records.

Table 7. 8th, 9th, 10th future records and their 95% predicted intervals using SEL and KEL functions.

Cases	$X_{U(n)}$	SEL		KEL	
		Predicted value	95% predicted interval	Predicted value	95% predicted interval
$n=7$	$X_{U(8)}$	85.136	$(73.185, 119.869)$	83.616	$(73.185, 111.765)$
	$X_{U(9)}$	97.382	$(75.653, 145.063)$	94.522	$(75.645, 130.039)$
	$X_{U(10)}$	109.628	$(79.830, 167.541)$	105.563	$(79.803, 145.677)$

Based on the previously known records, we find that the MLE of θ to be 0.9604 while the Bayes estimators under Prior 1 are 1.0858, 0.9651 using SEL and KEL, respectively. The 8th, 9th and 10th future records are computed to be 85.136, 97.382, 109.628 under SEL function and 83.616, 94.522, 105.563 under KEL function. The predicted intervals are also computed for all cases and it is found that they include the predicted values.

6. Conclusion

In this work, we have considered the problem of estimating the parameter of exponential distribution and predicting the future (unobserved) records based on an observed set of exponential record data. We have computed the maximum likelihood estimator of the parameter of the exponential distribution and also the Bayes estimator under both SEL and KEL functions. We have computed the MSEs to make a comparison between the MLEs and Bayes estimators. The MCMC samples are used to compute the predictors and the predicted intervals of the future records. Simulation and data analyses are performed to study the behaviour of the proposed methods on estimation and prediction, as well as real data example is presented for illustrative purposes. Based on our study, we recommend the use of KEL function over the well-known SEL function in both estimation and prediction problems depending on the values of the MSEs, which are reported in the previous tables, used in making our comparisons.

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