Statistical properties and different methods of estimation for extended weighted inverted Rayleigh distribution

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ABSTRACT

The aim of this paper is to introduce a new weighted probability distribution to model the non-monotone failure rate pattern for survival data. The proposed distribution is generalized by considering inverted Rayleigh distribution as a baseline distribution called an extended weighted inverted Rayleigh distribution. Different statistical properties such as moment, quantile function, moment generating function, entropy measurement, Bonferroni and Lorenz curve, stochastic ordering and order statistics have been derived. Different estimation procedures have also been discussed to estimate the unknown parameters of the proposed probability distribution. The Monte Carlo simulation study has been conducted to compare the performances of the proposed estimators obtained through various methods of estimation. Finally, two real data sets have been used to show the applicability of the proposed model in a real-life scenario.

Key words: moments and inverse moments, entropy measurements, order statistics, classical methods of estimation.

1. Introduction

In reliability analysis, numerous methods are available to generalize new probability distribution by adding an extra parameter with specific baseline distributions. For example, the well-known lifetime distributions, namely Weibull and gamma, are generalized by using power and Laplace transform of exponential random variates. Also, Gupta and Kundu (1999) introduced exponentiated exponential distribution by adding a shape parameter as a power of cumulative distribution function (CDF) of an exponential distribution. Nadarajah et al. (2011) proposed an extension of exponential distribution by a simple modification in the survival function of the exponential model. In reliability analysis, Rayleigh distribution (2005) is one of the most popular lifetime distribution and several generalizations based on Rayleigh distribution are advocated from time to time using the similar approach. The inverted versions of these models are also frequently used and well justified for the real life situations. For example, Voda (1972) introduced the inverted version of the Rayleigh model and discussed its different statistical properties. Ahmad et al. (2014) derived the weighted version of Rayleigh distribution and debated about the descriptive measure of statistics and

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estimation procedure for the unknown parameters. Exponentiated inverse Rayleigh distribution (EIRD) is a generalized form of inverse Rayleigh distribution as suggested by Nadarajah and Kotz (2006), etc., although the weighted version of the inverted Rayleigh distribution (IRD) has already been developed by Fatima and Ahmad (2017) using weighting function (length-area biased) approach. In this paper, a new weighted version of IRD has been proposed and studied with IRD as a base line distribution. IRD is the most popular lifetime distribution and frequently used to model the data with non-monotone failure rate. Let us assume that the variable Y is distributed as IRD with parameter θ. The probability density and distribution functions of IRD are given by;

$$f_Y(x) = \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}}, \quad \theta > 0, x > 0 \tag{1}$$

and

$$F_Y(x) = e^{-\frac{\theta}{x^2}} \tag{2}$$

where, θ is scale parameter.

The proposed extended weighted version of IRD has been derived by using the approach discussed by Azzalini (1985). The method mentioned by Azzalini is not new. It was given in 1985 and introduced various skew-symmetric distributions namely skew-normal, skew-chai, skew-Cauchy, skew-t, etc. Recently, Gupta and Kundu (2009) derived a new class of weighted distribution by introducing shape parameters to exponential distributions using the same approach. The lifetime distribution generated by this method possesses several good properties and can be used as a good alternative to other popular distributions such as gamma, Weibull, Rayleigh or generalized exponential distribution, etc.

The organization of the paper is as follows.The introduction of the considered problem is given in Section 1. Section 2 discusses the model genesis and its reliability characteristics. Statistical properties have been discussed in Sections 3, 4 & 5 respectively. Estimation of the unknown parameters is proposed in Section 6. Monte Carlo simulation study has been performed in Section 7. Section 8 has described the applicability of the model and study using two real data sets. Finally, Section 9 concludes the paper.

2. The model

Let X_1 and X_2 be the two i.i.d. random variables, with probability density function (PDF) $f_Y(\cdot)$ and CDF $F_Y(\cdot)$, then for any $\alpha > 0$, consider a new random variable $X = X_1$ given that $\alpha X_1 > X_2$. Then, the PDF of the new random variable X is;

$$f_X(x, \alpha) = \frac{1}{P[\alpha X_1 > X_2]} f_Y(x) F_Y(\alpha x) \quad ; \alpha > 0 \ \& \ x > 0 \tag{3}$$

Now, by using the Equations (1) and (2) in (3), the resulting probability distribution is called as extended weighted inverted Rayleigh distribution (EWIRD). Hence, the PDF and CDF

of the new model are given by;

$$f_w(x, \alpha, \theta) = \left(\frac{1+\alpha^2}{\alpha^2}\right)\left(\frac{2\theta}{x^3}\right) e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \quad ; x > 0 \ \& \ \alpha, \theta > 0 \tag{4}$$

and

$$F_w(x, \alpha, \theta) = e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \tag{5}$$

where, α is the shape parameter of the model.

2.1. Reliability characteristics

- The reliability function $R(t)$ for specified value of t is given by

$$R_w(t) = 1 - F_w(t) = 1 - e^{-\frac{\theta}{t^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \quad ; t > 0 \tag{6}$$

- The hazard rate, i.e. instantaneous failure rate $h(t)$ is the conditional probability of failure in time interval $(t, t + \delta t)$ given that units has survived at least time t. Mathematically, it is given by the following equation:

$$h(t) = \frac{f_w(t)}{R_w(t)} = \frac{\left(\frac{1+\alpha^2}{\alpha^2}\right)\left(\frac{2\theta}{t^3}\right) e^{-\frac{\theta}{t^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}}{1 - e^{-\frac{\theta}{t^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}} \tag{7}$$

- The reverse hazard function $H(t)$ can be interpreted as the ratio of the probability density function to the distribution function and is defined as;

$$H(t) = \frac{f_w(t)}{F_w(t)} = \left(\frac{1+\alpha^2}{\alpha^2}\right)\left(\frac{2\theta}{t^3}\right) \tag{8}$$

The different shape of the distribution, i.e. curve of density function and reliability function are presented in Figure 1. The shape of the hazard is presented in Figure 2. From Figure 2, it is clear that the proposed model is unimodal and exhibits the pattern of non-monotone failure rate. Usually, the problem of non-monotone failure rate is arising in medical and engineering sciences. In survival analysis, several times it has been realized that the failure rate of survival data is reached to a pick in the beginning stage and then declined abruptly until it stabilized. Such behaviour of the hazard rate is called a non-monotone failure rate, and the same behaviour of the failure rate is accommodated by the proposed model which, would be more flexible and used as a alternative survival model.
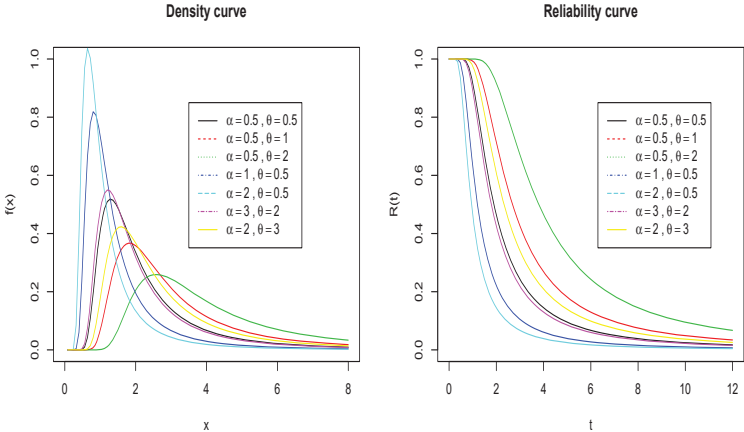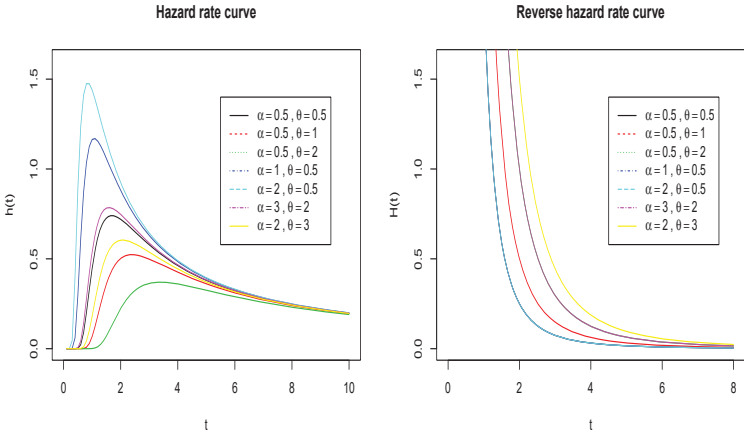
Figure 1: Density and reliability curve.

Figure 2: Hazard rate and reverse hazard rate.

3. Statistical properties

The different statistical properties of EWIRD are discussed in the following subsections.

3.1. Moments

The rth moment about the origin is defined as;

$$\mu'_r = E(X^r) = \int_{x=0}^{\infty} x^r f_w(x, \alpha, \theta) dx$$

$$= \int_{x=0}^{\infty} \left(\frac{1+\alpha^2}{\alpha^2}\right) 2\theta x^{r-3} e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} dx \qquad (9)$$

$$= \Gamma\left(1 - \frac{r}{2}\right) \left[\frac{\theta(1+\alpha^2)}{\alpha^2}\right]^{\frac{r}{2}}$$

The above expression is valid only for $r \leq 1$. Therefore, only mean of the distribution will exist in closed form and obtained by setting $r = 1$ in the Equation (9).

$$Mean = E(X) = \sqrt{\frac{\pi\theta(1+\alpha^2)}{\alpha^2}} \qquad (10)$$

3.2. Inverse moments

The rth inverse moment about origin (M'_{r-1}) is evaluated by the following expression:

$$M'_{r-1} = E\left(\frac{1}{X}\right)^r = \int_{x=0}^{\infty} x^{-r} f_w(x, \alpha, \theta) dx$$

$$= \int_{x=0}^{\infty} \left(\frac{1+\alpha^2}{\alpha^2}\right) 2\theta x^{-r-3} e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} dx \qquad (11)$$

$$= \Gamma\left(1 + \frac{r}{2}\right) \left[\frac{\theta(1+\alpha^2)}{\alpha^2}\right]^{\frac{-r}{2}}$$

- The values of different inverse moments are obtained by putting $r = 1, \cdots, 4$ in the above equation and we get,

$$E\left(\frac{1}{X}\right) = \frac{1}{2}\sqrt{\frac{\pi\alpha^2}{\theta(1+\alpha^2)}}$$

$$E\left(\frac{1}{X^2}\right) = \frac{\alpha^2}{\theta(1+\alpha^2)}$$

$$E\left(\frac{1}{X^3}\right) = \frac{3}{4}\sqrt{\frac{\pi\alpha^6}{\theta^3(1+\alpha^2)^3}}$$

and

$$E\left(\frac{1}{X^4}\right) = \frac{2\alpha^4}{\theta^2(1+\alpha^2)^2}$$

respectively.

- Variance of the inverse variable is

$$Var\left(\frac{1}{X}\right) = \frac{\alpha^2(4-\pi)}{4\theta(1+\alpha^2)}$$

- Coefficient of variation (CV) is evaluated as

$$CV = \frac{\sqrt{Var\left(\frac{1}{X}\right)}}{E\left(\frac{1}{X}\right)} = \sqrt{\frac{4-\pi}{\pi}} \approx 0.5223$$

3.3. Quantile function

The quantile function for the proposed model is computed by the following expression:

$$F(Q_i) = \frac{i}{\zeta} \tag{12}$$

where, i and ζ indicates the position and total partition respectively. After simplification, the quartile function is

$$Q_i = \sqrt{\frac{\theta}{[\ln\zeta - \ln i]}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \tag{13}$$

- **Quartile:** The first, second and third quartiles are evaluated by taking $\zeta = 4$ & $i = 1, 2, 3$ respectively.

- **Decile:** The deciles are calculated by assuming $\zeta = 10$, then the consecutive deciles are obtained by taking $i = 1, 2, \cdots, 9$.

- **Percentile:** The percentiles are evaluated by assuming $\zeta = 100$, then the consecutive deciles are obtained by taking $i = 1, 2, \cdots, 99$.

3.4. Median and Mode

The median for the PDF (4) is evaluated by using the following expression:

$$P[X \leq M] = P[X \geq M] = \frac{1}{2} \tag{14}$$

where, M is the median value. Also, it can be directly extracted from quantile expressions, e.g. 2nd (in quartile), 5th (in deciles) and 50th (in percentile) quantiles are median. Thus, the

median is

$$M = \sqrt{\frac{\theta(1+\alpha^2)}{(\ln 2)\alpha^2}} \tag{15}$$

If the proposed probability distribution is moderately skewed then it has been verified that the difference between mean and mode is almost equal to three times the difference between the mean and median. Hence, we have

Mode=3 Median-2 Mean

which yield

$$Mode = 0.06\sqrt{\frac{\theta(1+\alpha^2)}{\alpha^2}} \tag{16}$$

3.5. Sample generation

The random sample for EWIRD can be generated using an inverse CDF transformation method as follows.

- Generate random deviates (U) from uniform distribution.

- Equate $F_w(X) = U \Rightarrow X = F^{-1}(U)$

- After simplification, we get

$$X = \sqrt{\frac{\theta(1+\alpha^2)}{\ln(U^{-1})\alpha^2}} \tag{17}$$

3.6. Moment generating function

The moment generating function (mgf) of a continuous r.v. x is defined by the following equation:

$$M_X(t) = E(e^{tX}) = \int_{x \in R^+} e^{tx} f_w(x, \alpha, \theta) dx \tag{18}$$

Thus, using PDF (4) we get;

$$M_X(t) = \int_{x=0}^{\infty} e^{tx} \left[\left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{x^3}\right) e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right] dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \Gamma\left(1 - \frac{r}{2}\right) \sqrt[r]{\left(\frac{\theta + \theta\alpha^2}{\alpha^2}\right)} \tag{19}$$

Also, the moment can be obtained by using the above expression of m.g.f. Also, the characteristics function is simply obtained by replacing t by it in the above equation.

4. Entropy measurements

In information theory, entropy measurement plays a vital role in studying the uncertainty associated with the probability distribution. In this section, we discuss a different measure of change. For more detail about entropy, measurement see, Reniyi (1961).

4.1. Renyi entropy

Renyi entropy (RE) of a r.v. X is defined as

$$
\begin{aligned}
RE &= \frac{1}{(1-\kappa)} \ln \left[\int_{x=0}^{\infty} f_w^{\kappa}(x, \alpha, \theta) dx \right] \\
&= \frac{1}{(1-\kappa)} \ln \left[\int_{x=0}^{\infty} \left\{ \left(\frac{1+\alpha^2}{\alpha^2} \right) \left(\frac{2\theta}{x^3} \right) e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2} \right)} \right\}^{\kappa} dx \right]
\end{aligned}
\tag{20}
$$

Hence, after solving the internal, we get the following

$$
RE = \frac{1}{2}\ln(\theta + \theta\alpha^2) - \ln\alpha - \ln 2 - \frac{(3\kappa - 1)}{2(\kappa - 1)}\ln\kappa + \frac{1}{(1-\kappa)}\ln\Gamma\left(\frac{(3\kappa - 1)}{2} \right)
\tag{21}
$$

4.2. β-Entropy

The β-entropy (BE) is obtained as follows:

$$
BE = \frac{1}{\beta - 1}\left[1 - \int_{x=0}^{\infty} f_w^{\beta}(x, \alpha, \theta) dx \right]
\tag{22}
$$

Using PDF (4) and after simplification the expression for β-entropy is given by

$$
BE = \frac{1}{\beta - 1}\left[1 - \phi(\alpha, \theta, \beta) \right]
\tag{23}
$$

where, $\phi(\alpha, \theta, \beta) = \left(\dfrac{\theta + \theta\alpha^2}{\alpha^2} \right)^{\frac{1-\beta}{2}} \dfrac{2^{\beta - 1}}{\beta^{(3\beta - 1)/2}} \Gamma\left(\dfrac{(3\beta - 1)}{2} \right)$

4.3. Generalized entropy

The generalized entropy (GE) is obtained by

$$
GE = \frac{v_{\lambda}\mu^{-\lambda} - 1}{\lambda(\lambda - 1)} \quad ; \lambda \neq 0, 1
\tag{24}
$$

where, $\nu_\lambda = \int_{x=0}^{\infty} x^\lambda f_w(x, \alpha, \theta)dx$. The value of ν_λ is calculated as;

$$
\begin{aligned}
\nu_\lambda &= \int_{x=0}^{\infty} x^\lambda \left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{x^3}\right) e^{-\frac{\theta}{x^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} dx \\
&= \left(\frac{\theta(1+\alpha^2)}{\alpha^2}\right)^{\lambda/2} \Gamma\left(1 - \frac{\lambda}{2}\right)
\end{aligned}
\tag{25}
$$

Using the above expression

$$
GE = \frac{\left(\frac{\theta(1+\alpha^2)}{\alpha^2}\right)^{\lambda/2} \Gamma\left(1 - \frac{\lambda}{2}\right)\mu^{-\lambda} - 1}{\lambda(\lambda - 1)} \qquad ; \lambda \neq 0, 1
\tag{26}
$$

4.4. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves have good link-up to each other, and their extensive applications can be found in economics to study the income and poverty level. In the present time, these are frequently used in reliability theory also. These were initially proposed and analysed by Bonferroni (1930) and are defined by

$$
B(c) = \frac{1}{cm} \int_0^a x f_w(x, \alpha, \theta)dx
\tag{27}
$$

$$
L(c) = \frac{1}{m} \int_0^a x f_w(x, \alpha, \theta)dx
\tag{28}
$$

respectively. where $a = F_w^{-1}(c)$ and $m = E(x)$. Hence, using the Equation (4), the above two equations are reduced as

$$
B(c) = \frac{1}{c\sqrt{\pi}} IG\left(\frac{1}{2}, \frac{\theta + \theta\alpha^2}{a^2\alpha^2}\right)
\tag{29}
$$

$$
L(c) = \frac{1}{\sqrt{\pi}} IG\left(\frac{1}{2}, \frac{\theta + \theta\alpha^2}{a^2\alpha^2}\right)
\tag{30}
$$

where, $IG(a_1, b_1)$ stands for incomplete gamma function.

4.5. Stochastic ordering

The concept of stochastic ordering is used to show the ordering mechanism in lifetime distributions. For more detail about stochastic ordering see, Shaked and Shanthikumar (1988). The random variable X and Y is said to possess the following ordering behaviour:

- stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.

- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.

- mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x.

- likelihood ratio order $(X \leq_{lr} Y)$ if $\left(\dfrac{f_w^X(x)}{f_w^Y(x)} \right)$ decreases in x.

From the above relations, we analyzed that;

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Downarrow (X \leq_{st} Y) \Rightarrow (X \leq_{mrl} Y)$$

The proposed distribution is also ordered with respect to the strongest likelihood ratio ordering as shown in the following lemma.

Lemma: Let $X \sim f_w(\alpha_1, \theta_1)$ and $Y \sim f_w(\alpha_2, \theta_2)$. If $\alpha_1 > \alpha_2$, then $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y)$, $(X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$.

Proof: According to the definition of likelihood ratio order, first we obtain the ratio $\left[\dfrac{f_w^X(x)}{f_w^Y(x)} \right]$ i.e.

$$\psi = \frac{f_w^X(x)}{f_w^Y(x)} = \frac{\left(\dfrac{1+\alpha_1^2}{\alpha_1^2} \right) \left(\dfrac{2\theta_1}{x^3} \right) e^{-\dfrac{\theta_1}{x^2}\left(\dfrac{1+\alpha_1^2}{\alpha_1^2} \right)}}{\left(\dfrac{1+\alpha_2^2}{\alpha_2^2} \right) \left(\dfrac{2\theta_2}{x^3} \right) e^{-\dfrac{\theta_2}{x^2}\left(\dfrac{1+\alpha_2^2}{\alpha_2^2} \right)}}$$

$$= \frac{\theta_1 \alpha_2^2 (1+\alpha_1^2)}{\theta_2 \alpha_1^2 (1+\alpha_2^2)} e^{-\dfrac{1}{x^2} \left[\dfrac{\theta_1(1+\alpha_1^2)}{\alpha_1^2} + \dfrac{\theta_2(1+\alpha_2^2)}{\alpha_2^2} \right]}$$

Therefore,

$$\psi' = \frac{2\theta_1 \alpha_2^2 (1+\alpha_1^2)}{x^3 \theta_2 \alpha_1^2 (1+\alpha_2^2)} \left[\frac{\theta_1(1+\alpha_1^2)}{\alpha_1^2} + \frac{\theta_2(1+\alpha_2^2)}{\alpha_2^2} \right] e^{-\dfrac{1}{x^2} \left[\dfrac{\theta_1(1+\alpha_1^2)}{\alpha_1^2} + \dfrac{\theta_2(1+\alpha_2^2)}{\alpha_2^2} \right]} \quad (31)$$

from above equation, we observed that if $\psi' > 0 \ \forall \ \alpha_1, \alpha_2$, hence $(X \leq_{lr} Y)$. The remaining statements can be established in the same way.

5. Order statistics

Let us consider $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are the n ordered random sample from (4) then $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics. $X_{(1)}$, $X_{(n)}$ and $X_{(r)}$ denote the minimum, maximum and r^{th} order statistics respectively. Then the PDF of rth order statistics, as suggested by (1970), is given by

$$f_r(X_{(r)}, \alpha, \theta) = \frac{n!}{(r)!(n-r)!} \left[F_w(x_{(r)}) \right]^{r-1} \left[1 - F_w(x_{(r)}) \right]^{n-r} f_w(x_{(r)}, \alpha, \theta) \quad (32)$$

Using the expressions (4) and (5) for $X_{(r)}$ in the above equation we get the expression for r^{th} order statistics, i.e.

$$f_r(X_{(r)}, \alpha, \theta) = \frac{n!}{(r)!(n-r)!} \left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{x_{(r)}^3}\right) \left[e^{-\frac{\theta}{x_{(r)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\right]^r$$

$$\times \left[1 - e^{-\frac{\theta}{x_{(r)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\right]^{n-r} \tag{33}$$

5.1. Distribution of minimum, maximum and median

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the n independent ordered random sample observed, then the distribution of minimum $X_{(1)}$, maximum $X_{(n)}$ order statistics are obtained by putting $r = 1$ & $r = n$ in the Equation (35). Hence, after simplification we get

$$f_{mini}(X_{(1)}, \alpha, \theta) = n\left[1 - F_w(x_{(1)})\right]^{n-1} f_w(x_{(1)}, \alpha, \theta)$$

$$= n\left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{x_{(1)}^3}\right) \left[e^{-\frac{\theta}{x_{(1)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\right] \left[1 - e^{-\frac{\theta}{x_{(1)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\right]^{n-1}$$

$$\tag{34}$$

$$f_{max}(X_{(n)}, \alpha, \theta) = n\left[F_w(x_{(n)})\right]^{n-1} f_w(x_{(n)}, \alpha, \theta)$$

$$= n\left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{x_{(n)}^3}\right) \left[e^{-\frac{\theta}{x_{(n)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}\right]^n \tag{35}$$

Now, the density function for sample median for order statistics is given by $X_{(m+1:n)}$. It is computed by

$$f_{m+1}(\tilde{X}_{(m+1:n)}, \alpha, \theta) = \frac{(2m+1)!}{(m)!(m)!} [F_w(\tilde{x}_{m+1})]^m [1 - F_w(\tilde{x}_{m+1})]^m f_w(\tilde{x}_{m+1}, \alpha, \theta)$$

$$= \frac{(2m+1)!}{(m)!(m)!} \left(\frac{1+\alpha^2}{\alpha^2}\right) \left(\frac{2\theta}{\tilde{x}_{m+1}^3}\right) \left[e^{-\frac{\theta}{\tilde{x}_{m+1}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right]^{m+1}$$

$$\times \left[1 - e^{-\frac{\theta}{\tilde{x}_{m+1}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right]^m$$

(36)

5.2. Joint distribution of rth and s^{th} order statistics

The joint density function of rth and s^{th} $(r < s)$ order statistics is obtained by considering the following expression:

$$f_{r:s:n}(X_r, X_s, \alpha, \theta) =$$
$$= xi \left[F_w(x_{(r)})\right]^{r-1} \left[1 - F_w(x_{(s)})\right]^{n-s} \left[F_w(x_{(s)}) - F_w(x_{(r)})\right]^{s-r-1} f_w(x_{(r)}) f_w(x_{(s)})$$

(37)

Using PDF and CDF of EWIRD, the density function for r^{th}, s^{th} is given by

$$f_{r:s:n}(X_r, X_s, \alpha, \theta) = \xi \left[e^{-\frac{\theta}{x_{(r)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right]^r \left[1 - e^{-\frac{\theta}{x_{(s)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right]^{n-s}$$

$$\times \left[e^{-\frac{\theta}{x_{(s)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} - e^{-\frac{\theta}{x_{(r)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)} \right]^{s-r-1}$$

$$\times \left(\frac{1+\alpha^2}{\alpha^2}\right)^2 \left(\frac{4\theta^2}{x_{(r)}^3 x_{(s)}^3}\right) e^{-\frac{\theta}{x_{(s)}^2}\left(\frac{1+\alpha^2}{\alpha^2}\right)}$$

In particular, when $r = 1$ and $s = n$, we have the joint distribution of minimum and maximum order statistics and it is written as

$$
f_{1:n:n}(X_1, X_n, \alpha, \theta) = \frac{n!}{(n-2)!} \left(\frac{1+\alpha^2}{\alpha^2} \right)^2 \left(\frac{4\theta^2}{x_{(1)}^3 x_{(n)}^3} \right) e^{-\frac{\theta}{x_{(n)}^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)} \left[e^{-\frac{\theta}{x_{(1)}^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)} \right]
$$

$$
\times \left[e^{-\frac{\theta}{x_{(n)}^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)} - e^{-\frac{\theta}{x_{(1)}^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)} \right]^{n-2}
$$

$$(38)$$

where, $\xi = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$

6. Estimation of the parameters

In this section, we discuss different estimation procedures for estimating the unknown model parameters of the proposed model. These methods are presented below;

6.1. Maximum Likelihood Estimation

Let $X_1, X_2, \cdots, X_n$ be the random sample of size n from density function (4). The likelihood function is written as;

$$
\begin{aligned}
L(\alpha, \theta) &= \prod_{i=1}^{n} f_w(x_i, \alpha, \theta) \\
&= \left(\frac{1+\alpha^2}{\alpha^2} \right)^n \frac{(2\theta)^n}{\prod_{i=1}^{n} x_i^3} e^{-\Sigma_{i=1}^{n} \frac{\theta}{x_i^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)}
\end{aligned}
$$

$$(39)$$

Hence, log-likelihood by ignoring the constant is written as;

$$
L_1 = \ln L(\alpha, \theta) = n \ln(1+\alpha^2) - 2n \ln \alpha + n \ln \theta - 3 \sum_{i=1}^{n} \ln x_i - \left(\frac{\theta + \theta \alpha^2}{\alpha^2} \right) \sum_{i=1}^{n} \frac{1}{x_i^2} \quad (40)
$$

Thus, the MLEs are obtained by maximizing the above equation w.r.t. to the parameters and as a result we have two likelihood equations which yield the MLEs of the unknown parameters.

$$
\frac{n}{\theta} - \frac{1+\alpha^2}{\alpha^2} \sum_{i=1}^{n} \frac{1}{x_i^2} = 0 \tag{41}
$$

and

$$\frac{n\alpha}{1+\alpha^2} - \frac{n}{\alpha} + \frac{\theta}{\alpha^3} \sum_{i=1}^{n} \frac{1}{x_i^2} = 0 \tag{42}$$

6.1.1   Interval estimate based on MLEs

From above, it is clear that the exact distribution for MLEs is not easy to find. Thus, we considered the asymptotic distribution of MLE to construct $100(1-\alpha)\%$ approximate confidence interval. Thus, for this purpose we evaluate the Fisher information matrix and it is obtained as

$$I(\hat{\alpha}, \hat{\theta}) = \begin{pmatrix} -l_{\alpha\alpha} & -l_{\alpha\theta} \\ -l_{\theta\alpha} & -l_{\theta\theta} \end{pmatrix}_{\hat{\alpha}, \hat{\theta}} \tag{43}$$

where, $l_{\alpha\alpha} = \dfrac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha^2}, l_{\alpha\theta} = \dfrac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha \partial \theta}, l_{\theta\alpha} = \dfrac{\partial^2 \ln L(\alpha, \theta)}{\partial \theta \partial \alpha}, l_{\theta\theta} = \dfrac{\partial^2 \ln L(\alpha, \theta)}{\partial \theta^2}$

The above matrix is inverted and its diagonal elements provide the asymptotic variance of the estimates for the parameter. Hence, approximate CI is given by

$$[\hat{\alpha}_L, \hat{\alpha}_U] \in [\hat{\alpha} \mp Z_{\gamma/2} \sqrt{\hat{\sigma^2}_{\alpha\alpha}}]$$

and

$$[\hat{\theta}_L, \hat{\theta}_U] \in [\hat{\theta} \mp Z_{\gamma/2} \sqrt{\hat{\sigma^2}_{\theta\theta}}]$$

6.2.   Maximum Product Spacing method of estimation

In this subsection, we described a very effective and alternative method to MLEs named maximum product spacing method. It was initially introduced and extensively studied by Chen and Amin (1979). Coolen and Newby (1990) studied its invariance properties and concluded that it possesses the similar features as MLEs. Recently, the utility of this method has been nicely explained by Singh et al. (2014). Under this method of estimation techniques the likelihood function is defined on the basis of spacing of two consecutive CDFs and is given by

$$L'(\alpha, \theta) = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \tag{44}$$

such that $\sum_{i=1}^{n} D_i = 1$. Taking log both side, we get;

$$\begin{aligned} \ln L'(\alpha, \theta) &= \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i \\ &= \frac{1}{n+1} \left[\ln D_1 + \sum_{i=2}^{n} \ln D_i + \ln D_{n+1} \right] \end{aligned} \tag{45}$$

where, $D_1 = F(x_1)$, $D_i = F(x_i) - F(x_{i-1})$ and $D_{n+1} = 1 - F(x_n)$

The MPS estimates of the parameter α, θ are obtained by maximizing the above equation w. r. t. the parameters.

6.2.1 Interval estimate based on MPS

Here, we consider the asymptotic confidence intervals based on maximum product spacing estimates. It was mentioned by Cheng and Amin (1979), Ghosh and Jammalamadaka (2001) that the MPS method also shows asymptotic properties like the Maximum likelihood estimator and is asymptotically equivalent to MLE. Keeping this in mind, we have to consider the Fisher information matrix and it is obtained as;

$$I^{'}(\hat{\alpha}, \hat{\theta}) = \begin{pmatrix} -W_{\alpha\alpha} & -W_{\alpha\theta} \\ -W_{\theta\alpha} & -W_{\theta\theta} \end{pmatrix}_{\hat{\alpha}_{MP}, \hat{\theta}_{MP}} \tag{46}$$

where, $W_{\alpha\alpha} = \dfrac{\partial^2 \ln L^{'}(\alpha, \theta)}{\partial \alpha^2}, W_{\alpha\theta} = \dfrac{\partial^2 \ln L^{'}(\alpha, \theta)}{\partial \alpha \partial \theta}, W_{\theta\alpha} = \dfrac{\partial^2 \ln L^{'}(\alpha, \theta)}{\partial \theta \partial \alpha}, W_{\theta\theta} = \dfrac{\partial^2 \ln L^{'}(\alpha, \theta)}{\partial \theta^2}$

Using similar approach as MLE the $100(1-\gamma)$ asymptotic confidence interval is given by

$$\hat{\alpha}_{MP} \mp Z_{\gamma/2} \sqrt{(\sigma_{\hat{\alpha}\alpha}^2)_{MP}}$$

and

$$\hat{\theta}_{MP} \mp Z_{\gamma/2} \sqrt{(\sigma_{\theta\theta}^2)_{MP}}$$

6.3. Estimators based on percentile

Estimation of the parameters based on percentile is not the new one and frequently used when the distribution function is in closed form. It was proposed and extensively studied by Kao (1958, 1959). Recently, this method has gained some popularity in the statistical literature and used was by Gupta and Kundu (2001), Kundu, and Raqab (2005) based on different lifetime models. Most of the time estimators obtained by this method have a nice closed form. The percentile-based estimators are mainly obtained by minimizing the Euclidean distance between the sample percentile and population percentile points. Hence, using the expression of CDF, we get

$$x = \sqrt{\frac{\theta(1+\alpha^2)}{\ln[F^{-1}(x,\alpha,\theta)]\alpha^2}} \tag{47}$$

The percentile estimators of α, θ are obtained by minimizing

$$PE = \sum_{i=1}^{n} \left[x_i - \sqrt{\frac{\theta(1+\alpha^2)}{\ln[F^{-1}(x,\hat{\alpha},\hat{\theta})]\alpha^2}} \right]^2 \tag{48}$$

where, $F(x, \hat{\alpha}, \hat{\theta})$ denotes the estimated value of CDF. It can be assumed as

$$E[F(x_i, \alpha, \theta)] = \frac{i}{n+1} = p_i$$

6.4. Ordinary and Weighted Least Squares Estimation

The theory of least squares estimation was proposed by Swain et al. (1988) to estimate the parameters of Beta distribution using the principal of least squares. The LSEs of the unknown parameters of EWIRD are evaluated by minimizing

$$LSE = \sum_{i=1}^{n} \left[F(x_i, \alpha, \theta) - \frac{i}{n+1} \right]^2 \tag{49}$$

using the Equation (5) in the above equation, we get

$$LSE = \sum_{i=1}^{n} \left[e^{-\frac{\theta}{x^2} \left(\frac{1+\alpha^2}{\alpha^2} \right)} - \left(\frac{i}{n+1} \right) \right]^2 \tag{50}$$

Hence, the least square estimators of the parameter α and θ are obtained by minimizing the above equation w.r.t α and θ respectively.

7. Simulation study

In this section, Monte Carlo simulation study has been performed to compare the performance of the obtained estimators in previous subsections. The comparisons of these estimators are made in terms of average mean square error (mse) based on 5000 replications. Since, all estimators are not in closed form, thus non-linear optimization iterative procedure has been used to obtain the estimates of the parameters. In result Tables, $(\hat{\theta}_{ml}, \hat{\alpha}_{ml})$, $(\hat{\theta}_{mp}, \hat{\alpha}_{mp})$, $(\hat{\theta}_{lse}, \hat{\alpha}_{lse})$, $(\hat{\theta}_{pse}, \hat{\alpha}_{pse})$ denotes the estimators obtained by the method of MLE, MPSE, LSE and PSE for scale and shape parameters respectively. The simulation study has been carried out for $n = 10, 20, 30, 50, 80, \& 120$ when $\theta = 2, \alpha = 3$. Average estimates of the parameters and corresponding mse are reported in Table 1. From Table 1, it has been noticed that the mse of all estimator is decreasing when the sample size is increasing, which guarantees the consistency of the estimators. Also, among the estimators obtained by different method of estimation the following patterns have been noticed in terms of their average mse.

$$mse(\hat{\theta}_{lse}) < mse(\hat{\theta}_{mp}) < mse(\hat{\theta}_{mle}) < mse(\hat{\theta}_{pse})$$

and

$$mse(\hat{\alpha}_{ml}) < mse(\hat{\alpha}_{mp}) < mse(\hat{\alpha}_{pse}) < mse(\hat{\alpha}_{lse})$$

respectively. Thus, for the scale parameter θ, the LSE method performs better as compared to the other methods of estimation, however in the case of the shape parameter α, MLE

Table 1: Average estimate and corresponding mse (in each second row) of the estimators when $\theta = 2$, $\alpha = 3$.

n	$\hat{\theta}_{ml}$	$\hat{\theta}_{mp}$	$\hat{\theta}_{lse}$	$\hat{\theta}_{pse}$	$\hat{\alpha}_{ml}$	$\hat{\alpha}_{mp}$	$\hat{\alpha}_{lse}$	$\hat{\alpha}_{pse}$
10	2.2359	1.8034	1.8182	2.0608	3.2627	2.2831	1.9998	2.4456
	0.6672	0.3576	0.3439	0.8425	0.3391	0.5349	1.0549	0.6935
20	2.1047	1.8103	1.8181	2.2620	3.0717	2.3244	2.0647	2.4320
	0.2843	0.1999	0.1963	0.8403	0.2866	0.4674	0.9078	0.5880
30	2.0877	1.8238	1.8185	2.0987	3.1902	2.3333	2.0858	2.2867
	0.1622	0.1327	0.1274	0.7143	0.2479	0.4513	0.8958	0.5624
50	2.0733	1.8299	1.8348	2.0993	3.3729	2.3333	2.0624	2.3249
	0.0923	0.0922	0.0907	0.6439	0.1439	0.4485	0.8860	0.5584
80	2.0856	1.8499	1.7936	2.1270	3.1526	2.3263	2.0719	2.5151
	0.0561	0.0596	0.0546	0.5961	0.0389	0.4456	0.8646	0.4825
120	2.0720	1.8495	1.8430	2.2823	3.1546	2.3245	2.0567	2.5827
	0.0442	0.0483	0.0421	0.5926	0.0379	0.4358	0.8593	0.4576

method provides a better result.

8. Application to lifetime data

In this section, survival/reliability data applications of the proposed model are provided. For this purpose, we have considered two data sets and checked the suitability of the proposed model.

Data Set 1: Cancer data

These data represent the survival times (in days) of 45 head and neck cancer patients treated with combined radiotherapy and chemotherapy. Firstly, the data set is reported by Efron (1988). To check the suitability of the considered data set for the proposed model different model selection tools are used such as AIC, BIC, and log-likelihood criterion. These statistical tools are defined as follows:

$$AIC = -2 * \ln L(x, \hat{\alpha}, \hat{\theta}) + 2 * k$$

$$BIC = -2 * \ln L(x, \hat{\alpha}, \hat{\theta}) + k * \ln(n)$$

where, k is the number of parameters involved in the probability distribution, and n is the sample size. Smaller values of AIC, BIC and the LogL test statistic are indicators of better fit of distributions. The proposed model is compared with the most commonly used non-monotone failure rate models namely inverted exponential distribution (IED), generalized inverted exponential distribution (GIED) and Inverse Weibull distribution (IWD). Among these models, it has been observed that the proposed model has the least AIC, BIC and negative LogL, see Table 4. Hence, the proposed model can be taken as an alternative to these models when data have the non-monotone failure rate.

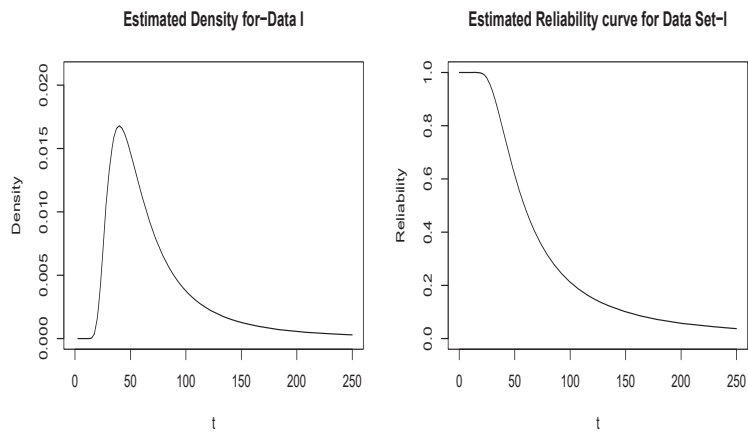 **Data Set 2: Ball bearing failure data**

Estimated Density for−Data I

Estimated Reliability curve for Data Set−I

Figure 3: Estimated plot for the real data-I

Estimated Hazard rate curve for Data−I

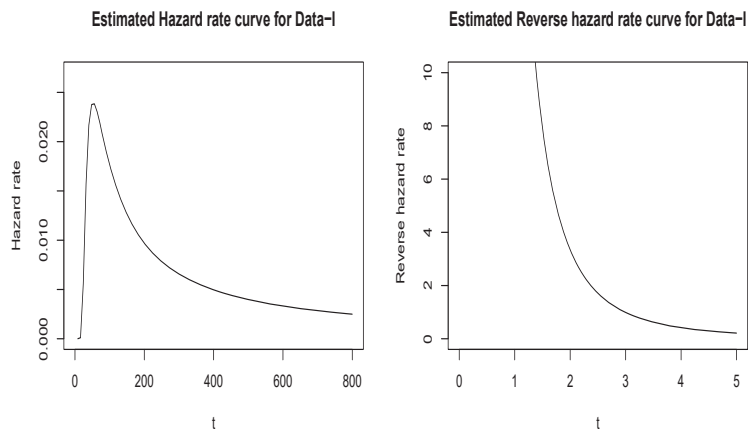**Estimated Reverse hazard rate curve for Data−I**

Figure 4: Estimated plot for the real data-I

Table 2: Estimates and values of various measures for data set-I

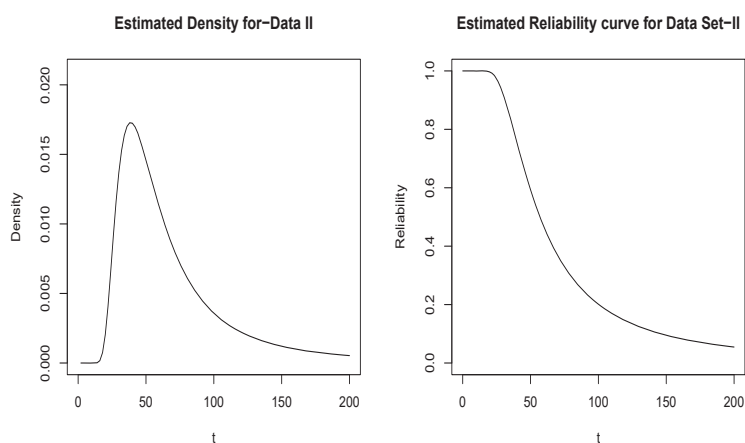| Models	Estimate	AIC	BIC	-LogL
IED	59.125	773.37	775.44	385.69
GIED	(0.777,49.241)	773.18	777.30	384.59
IWD	(28.505,0.786)	767.16	771.28	381.58
EWIRD	(6.679,0.053)	691.04	694.65	343.52

Figure 5: Estimated plot for the real data-II

The considered data set represents the 23 ball bearing failure times (millions of cycles) for units tested at one level of stress and it was firstly reported and analysed by Lawless (1982). The summary of the ball bearing data set is

Minimum	First Quartile	Median	Mean	Third Quartile	Maximum
17.88	47.20	67.80	72.22	95.88	173.40

To check the validity of the proposed model, we used Kolmogrov-Smirnov test. Thus, the hypothesis is

$$H_0 : \text{Samples are observed from proposed model}$$

$$H_1 : \text{Samples are not observed from proposed model}$$

Hence, test statistic for testing the null hypothesis is

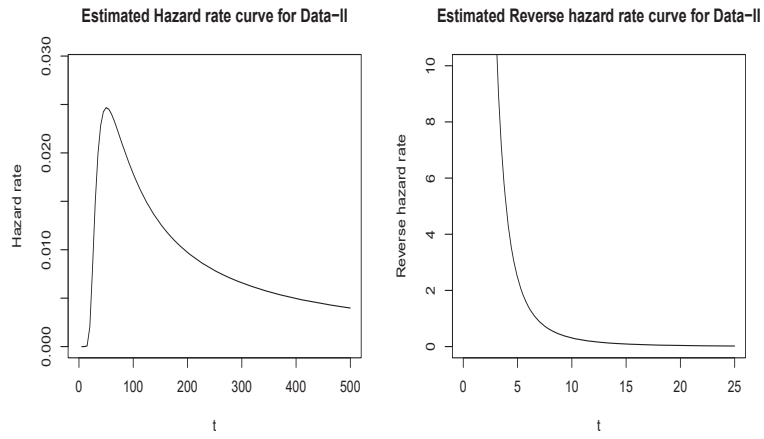$$KS_{Cal} = Sup_x |\hat{F}_n(x) - \hat{F}(x)| = 0.098 \text{ and } KS_{tab} = 0.276$$

Figure 6: Estimated plot for the real data-II

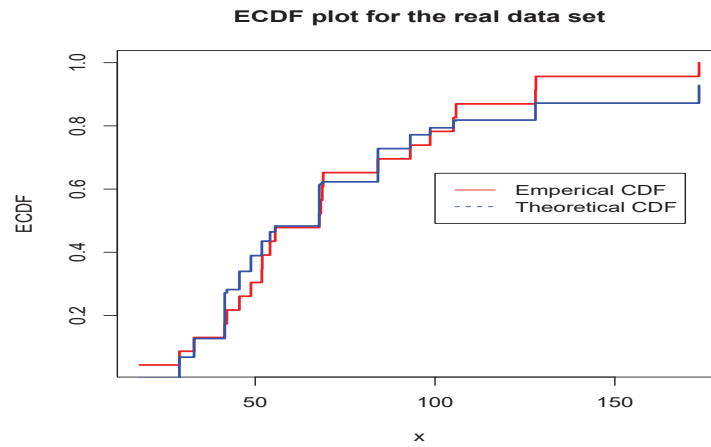
Figure 7: Empirical CDF plot for the real data

We see that the calculated value of is less than the tabulated value. Hence the null hypothesis may be accepted at $\alpha = 5\%$ level of significance. Also, from the empirical cumulative distribution function plot (see Figure 7) it is clear that the data set-II provides excellent fit to the proposed model and hence, one may use EWIRD as an alternative lifetime model. The estimated plots for the density function, reliability function, hazard function and reverse hazard rate function are given in Figure 3, 4, 5 and 6 respectively. These plots indicate that cancer data and ball bearing data both adequately accommodate the new model.

9. Conclusion

In this article, a new version of weighted probability distribution, named EWIRD has been introduced. Different statistical properties such as moments, inverse moments, moment generating function, entropy, stochastic ordering and order statistics have been discussed. Different estimation procedures are also described to estimate the unknown parameters, and their performances are compared through Monte Carlo simulations. The applications of the proposed model are provided based on two real data sets and it has been found that it can efficiently be used to model the data with a non-monotone failure rate pattern.

Acknowledgments

REFERENCES

AZZALINI, A., (1985). A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics*, 12, pp. 171–178.

AHMAD, A., AHMAD, S. P., AHMED, A., (2014). Characterization and Estimation of Double Weighted Rayleigh Distribution, *Journal of Agriculture and Life Sciences*, 1(2), pp. 2375–4222.

BONFERRONI, C. E., (1930). Elementi di Statistica General, *Seeber, Firenze.*

CHENG, R. C. H., AMIN, N. A. K., (1979). Maximum product of spacings estimation with applications to the lognormal distribution, *University of Wales IST, Math Report*, pp. 79–1.

COOLEN, F. P. A., NEWBY, M. J., (1990). A note on the use of the product of spacings in Bayesian inference, *Memorandum COSOR*; Vol. 9035), Eindhoven: Technische Universiteit Eindhoven.

DAVID, H. A., (1970). Order Statistics, *New York, Wiley*.

EFRON, B., (1988). Logistic regression, survival analysis, and the Kaplan-Meier curve, *Journal of American Statistical Association*, 83, pp. 414–425.

FATIMA, K., AHMAD, S. P., (2017). Weighted Inverse Rayleigh Distribution, *International Journal of Statistics and Systems*, 12(1), pp. 119–137.

GHOSH, K., JAMMALAMADAKA, S., R., (2001). A general estimation method using spacings, *Journal of Statistical Planning and Inference*, 93.

GUPTA, R. D., KUNDU, D., (1999). Generalized exponential distributions, *Australia and New Zealand Journal of Statistics*, 41, pp. 173–188.

GUPTA, R. D., KUNDU, D., (2001). Generalized exponential distributions: different methods of estimation, *Journal of Statistical Computation and Simulation*, 69, pp. 315–338.

GUPTA, R. D., KUNDU, D. (2009,). A new class of weighted exponential distributions, *Statistics*, 43, pp. 621–634.

KAO, J. H. K., (1958). Computer methods for estimating Weibull parameters in reliability studies, *IRE Transactions on Reliability and Quality Control*, 13, pp. 15–22.

KAO, J. H. K., (1959). A graphical estimation of mixed Weibull parameters in life testing electron tube, *Technometrics*, 1, pp. 389–407.

KUNDU, D., RAQAB, M. Z., (2005). Generalized Rayleigh distribution: different methods of estimation, *Computational Statistics and Data Analysis*, 49, pp. 187–200.

LAWLESS, J. F., (1982). Statistical Models and Methods for Lifetime Data, *New York: John Wiley & Sons*.

MAKKAR, P., SRIVASTAVA, P. K. , SINGH R. S., UPADHYAY, S. K., (2014). Bayesian survival analysis of head and neck cancer data using log normal model, *Communications in Statistics - Theory and Methods*, 43, pp. 392–407.

NADARAJAH, S., HAGHIGHI, F., (2011). An extension of the exponential distribution, *Statistics: Journal of Theoretical and Applied Statistics*, 45(6), pp. 543–558.

NADARAJAH, S., KOTZ, S., (2006). The exponentiated type distributions, *Acta Applicandae Mathematicae*, 92(2), pp. 97–111.

RAYLEIGH, J. W. S., (1880). On the resultant of a large number of vibrations of the some pitch and of arbitrary phase. *Philosophical Magazine*, 5-th Series, 10, pp. 73–78.

RANNEBY, B., (1984). The Maximum Spacings Method. An Estimation Method Related to the Maximum Likelihood Method, *Scandinavian Journal of Statistics*, 11, pp. 93–112.

RENYI, A., (1961). On measures of entropy and information, in: Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, *University of California Press, Berkeley*.

SINGH, U., SINGH, S. K., SINGH, R. K., (2014). Comparative study of traditional estimation method and maximum product spacing method in Generalized inverted exponential Distribution, *Journal of Statistics Application and Probability*, 3(2), pp. 1–17.

SWAIN, J. J., VENKATARAMAN, S., WILSON, J. R., (1988). Least squares estimation of distribution functions in Johnson's translation system, *Journal of Statistical Computation and Simulation*, 29, pp.271–297.

SHAKED, M., SHANTHIKUMAR, J. G., (1994). Stochastic Orders and Their Applications, *New York, Academic Press.*

VODA, V. G., (1972). On the inverse Rayleigh random variable, *Rep. Statistical Application Res. Jues*, 19 (4), pp. 13–21.