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Poisson Weighted Ishita Distribution: Model for Analysis of Over-Dispersed Medical Count Data

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ABSTRACT

A new over-dispersed discrete probability model is introduced, by compounding the Poisson distribution with the weighted Ishita distribution. The statistical properties of the newly introduced distribution have been derived and discussed. Parameter estimation has been done with the application of the maximum likelihood method of estimation, followed by the Monte Carlo simulation procedure to examine the suitability of the ML estimators. In order to verify the applicability of the proposed distribution, a real-life set of data from the medical field has been analysed for modeling a count dataset representing epileptic seizure counts.

Key words: compounding model, coverage probability, simulation, count data, epileptic seizure counts.

1. Introduction

Compounding mechanism for generating new count data probability models has received a great attention from researchers to obtain new probability distributions to fit data sets not adequately fit by common parametric distributions. Compound distributions serve well to describe various phenomena in biology, epidemiology and so on. The work has been done in this particular area since 1920. Using compounding mechanism, Greenwood and Yule (1920) established a relationship between Poisson distribution and a negative binomial distribution by treating the rate parameter in Poisson model as gamma variate. Skellam (1948) proposed a probability distribution from the binomial distribution by regarding the probability of success as a beta variable between sets of trials. Lindely (1958) proposed a one parameter probability distribution to illustrate the difference between fiducial distribution and posterior distribution. Gerstenkorn (1993,1996) introduced several compound distributions and obtained compound of gamma distribution with exponential distribution by treating the

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parameter of gamma distribution as an exponential variate and also obtained compound of polya with beta distribution. Mahmoudi et al. (2010) generalized the Poisson-Lindely distribution of Sankaran (1970) and showed that their generalized distribution has more flexibility in analysing count data. Zamani and Ismail (2010) proposed a new compound distribution by compounding negative binomial with one parameter Lindley distribution that provides good fit for count data where the probability at zero has an inflated value. A new generalized negative binomial distribution was proposed by Gupta and Ong (2004). This distribution arises from Poisson distribution if the rate parameter follows generalized gamma distribution; the resulting distribution so obtained was applied to various data sets and can be used as a better alternative to negative binomial distribution. Rashid, Ahmad and Jan (2016) proposed a new competitive count data model, by compounding negative binomial distribution with Kumaraswamy distribution, which finds its application in biological sciences. Para and Jan (2018) introduced two compounding models with applications to handle count data in medical sciences.

In this paper, we propose a new compounding distribution by compounding Poisson distribution with weighted Ishita distribution. Ishita distribution is a flexible probability model introduced by Shanker and Shukla (2017) and its weighted version was introduced by Shukla and Shanker (2019) as a new life time probability model. The new model is introduced as there is a need to find more flexible models for analyzing over-dispersed count data.

2. Definition of Proposed Model (Poisson Weighted Ishita Distribution)

If $X|\lambda \sim$ Poisson(λ), where λ is itself a random variable following weighted Ishita distribution with parameter c and θ, then determining the distribution that results from marginalizing over λ will be known as a compound of Poisson distribution with that of weighted Ishita distribution, which is denoted by $PWID(X;c,\theta)$. It may be noted that the proposed model will be a discrete since the parent distribution is discrete.

Theorem 2.1: The probability mass function of a Poisson weighted Ishita distribution, i.e. $PWID(X;c,\theta)$ is given by

$$P(X=x)=\frac{(x+c)!\,\theta^{c+3}}{x!\,c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$

$$x=0,1,2,3,\ldots\ ;\ \theta>0, c>0$$

Proof: Using the definition (2), the pmf of a Poisson weighted Ishita distribution, i.e. $PWID(X;c,\theta)$ can be obtained as

$$g(x\mid\lambda)=\frac{e^{-\lambda}\lambda^x}{x!}\quad ;\ x=0,1,2,3,\ldots;\lambda>0$$

When its parameter λ follows weighted Ishita distribution (WID) with pdf

$$h(\lambda;c,\theta)=\frac{\lambda^{c}\theta^{(c+3)}(\theta+\lambda^{2})e^{-\theta\lambda}}{c!(\theta^{3}+(c+1)(c+2))},\quad ;\lambda>0,c>0,\theta>0$$

We have

$$P(X=x)=\int_{0}^{\infty}g(x\,|\,\lambda).h(\lambda;\theta)d\lambda$$

$$P(X=x)=\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^{3}+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^{2}+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right] \tag{2.1}$$

$$x=0,1,2,3,...\quad ;\quad \theta>0,c>0$$

which is the pmf of Poisson weighted Ishita distribution.

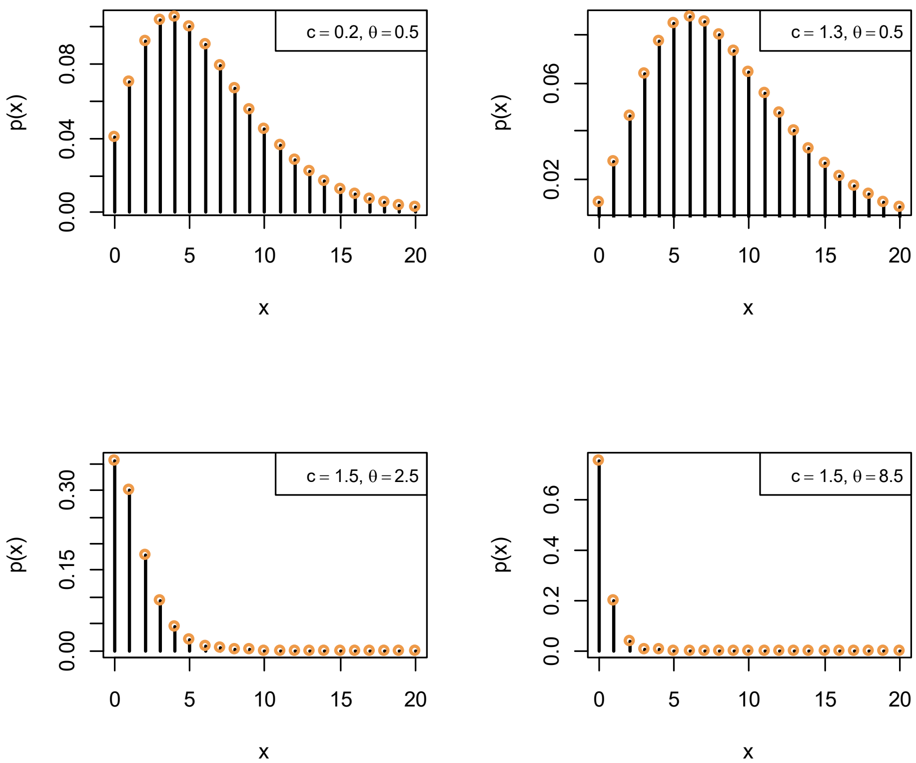

Figure 1. pmf plot of Poisson weighted Ishita distribution for different parameter combinations

The corresponding cdf of Poisson weighted Ishita distribution is obtained as:

$$F_X(x) = \sum_{x=0}^{x} \frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right] \tag{2.2}$$

$$x = 0,1,2,3,... \quad ; \quad \theta > 0, c > 0$$

Cdf is not in the closed form and it can be solved using software like mathematica and MathCAD for getting numerical results.

2.1. Random Data Generation from Poisson weighted Ishita distribution

In order to simulate the data from Poisson weighted Ishita distribution, we employ the discrete version of inverse cdf method. Simulating a sequence of random numbers $y_1, y_2, ..., y_n$ from Poisson weighted Ishita random variable K with pmf $p(K = y_i) = p_i, \sum_{i=0}^{x} p_i = 1$ and a cdf $F(K; c, \theta)$, where x may be finite or infinite can be described as in the following steps:

Step1: Generate a random number *u* from uniform distribution *U(0,1).*

Step2: Generate random number y_i based on

$$\text{if } u \le p_0 = F(y_0; c, \theta) \text{ then } K = y_0$$

$$\text{if } p_0 < u \le p_0 + p_1 = F(y_1; c, \theta) \text{ then } K = y_1$$

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$$.\text{if } \sum_{j=0}^{x-1} p_j < u \le \sum_{j=0}^{x} p_j = F(y_x; c, \theta) \text{ then } K = y_x$$

In order to generate n random numbers $y_1, y_2, ..., y_n$ from Poisson weighted Ishita distribution, repeat step-1 and step-2 *n* times. We have employed R studio software for running the simulation study of the proposed model.

3. Statistical properties

In this section, structural properties of the Poisson weighted Ishita model have been evaluated. These include the moment, moment generating function and probability generating function.

3.1. Factorial Moments

Using (2.1), the r[th] factorial moment about origin of the Poisson weighted Ishita distribution (2.1) can be obtained as

$$\mu_{(r)}{}' = E[E(X^{(r)} \mid \lambda)], \; where\, X^{(r)} = X(X-1)(X-2)...(X-r+1)$$

$$\mu_{(r)}{}' = \int_0^\infty \left[\sum_{x=0}^\infty x^{(r)} \frac{e^{-\lambda}\lambda^x}{(x)!} \right] . \frac{\lambda^c \theta^{(c+3)} (\theta+\lambda^2) e^{-\theta\lambda}}{c!(\theta^3+(c+1)(c+2))} d\lambda$$

$$\mu_{(r)}{}' = \frac{\theta^{(c+3)}}{c!(\theta^3+(c+1)(c+2))} \int_0^\infty \left[\lambda^r \left(\sum_{x=r}^\infty \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!} \right) \right] \lambda^c \theta^{(c+3)} (\theta+\lambda^2) e^{-\theta\lambda} d\lambda$$

Taking $u = x - r$, we get

$$\mu_{(r)}{}' = \frac{(r+c)!}{c!(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(r+c+2)(r+c+1)}{\theta^r} \right) \tag{3.1.1}$$

Taking r=1,2,3,4 in (3.1.1), the first four factorial moments about origin of Poisson weighted Ishita distribution can be obtained as

$$\mu_{(1)}{}' = \frac{(c+1)}{(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(c+3)(c+2)}{\theta} \right)$$

$$\mu_{(2)}{}' = \frac{(c+2)(c+1)}{(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(c+4)(c+3)}{\theta^2} \right)$$

$$\mu_{(3)}{}' = \frac{(c+3)(c+2)(c+1)}{(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(c+5)(c+4)}{\theta^3} \right)$$

$$\mu_{(4)}{}' = \frac{(c+4)(c+3)(c+2)(c+1)}{(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(c+6)(c+5)}{\theta^{4r}} \right)$$

3.1.2. Moments about origin (Raw moments)

Using the relationship between factorial moments about origin and the moments about origin of Poisson weighted Ishita distribution (2.1), we have

$$\mu_1{}' = \frac{(c+1)}{(\theta^3+(c+1)(c+2))} \left(\frac{\theta^3+(c+3)(c+2)}{\theta} \right)$$

$$\mu_2' = \mu_{(2)}' + \mu_1' = \frac{(c+2)(c+1)}{(\theta^3+(c+1)(c+2))}\left(\frac{\theta^3+(c+4)(c+3)}{\theta^2}\right) + \frac{(c+1)}{(\theta^3+(c+1)(c+2))}\left(\frac{\theta^3+(c+3)(c+2)}{\theta}\right)$$

$$= \frac{(c+1)}{\theta(\theta^3+(c+1)(c+2))}\left[(c+2)\left(\theta^3+(c+4)(c+3)\right)+\theta\left(\theta^3+(c+3)(c+2)\right)\right]$$

4. Reliability Analysis

In this section, we have obtained the reliability and hazard rate function of the proposed Poisson weighted Ishita distribution.

4.1. Reliability Function R(x)

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival function of the distribution. The reliability function or the survival function of Poisson weighted Ishita distribution is given by

$$R(x,\theta) = 1 - \sum_{x=0}^{x} \frac{(x+c)!\,\theta^{c+3}}{x!\,c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$

4.2. Hazard Function

The hazard function is also known as the hazard rate, instantaneous failure rate or force of mortality, and is given as:

$$\text{H.R} = h(x,\theta) = \frac{f(x,\theta)}{R(x,\theta)}$$

$$= \frac{(x+c)!\,\theta^{c+3}}{\left(1 - \sum_{x=0}^{x} \frac{(x+c)!\,\theta^{c+3}}{x!\,c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right) x!\,c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$

5. Order statistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2, X_3, \ldots X_n$ drawn from the discrete distribution with cumulative distribution function $F_X(x)$ and probability mass function $P_X(x)$, then the probability mass function of r[th] order statistics $X_{(r)}$ is given by:

$$f_{x(r)}(x,c,\theta)=\frac{n!}{(r-1)!(n-r)!}P(x)[F(x)]^{r-1}[1-F(x)]^{n-r}.\ r=1, 2, 3,\ldots,n$$

Using the equations (2.1) and (2.2), the probability mass function of rth order statistics of Poisson weighted Ishita distribution is given by:

$$f_{(r)}(x,,\theta)=\frac{n!}{(r-1)!(n-r)!}\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$
$$\left[\sum_{x=0}^{x}\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right]^{r-1}$$
$$\left[1-\sum_{x=0}^{x}\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right]^{n-r}$$

Then, the pmf of first order $X_{(1)}$ Poisson weighted Ishita distribution is given by:

$$f_1(x,\theta)=n\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$
$$\left[1-\sum_{x=0}^{x}\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right]^{n-1}.$$

and the pmf of nth order $X_{(n)}$ Poisson Ishita model is given as:

$$f_{w(n)}(x,\theta)=n\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$$
$$\left[\sum_{x=0}^{x}\frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right]^{n-1}$$

6. Estimation of Parameters

In this section, we estimate the parameters of the Poisson weighted Ishita distribution using methods of maximum likelihood estimation.

6.1. Method of Maximum Likelihood Estimation

This is one of the most useful method for estimating the different parameters of the distribution. Let $X_1, X_2, X_3,\ldots, X_n$ be the random size of sample n draw from Poisson weighted Ishita distribution. Then, the likelihood function of Poisson weighted Ishita distribution is given as:

$$L(x\mid\theta)=\prod_{i=1}^{n}\left(\frac{(x+c)!\,\theta^{c+3}}{x!\,c!\,(\theta^{3}+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^{2}+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]\right)$$

$$\log L=\sum_{i=1}^{n}\log(x_i+c)!+n(c+3)\log(\theta)-\sum_{i=1}^{n}\log(x_i!)-n\log(c!)-n\log(\theta^{3}+(c+1)(c+2))+$$
$$\sum_{i=1}^{n}\log(\theta(1+\theta)^{2}+(x_i+c+1)(x_i+c+2))-\sum_{i=1}^{n}(x_i+c+3)\log(1+\theta)$$

By differentiating log-likelihood function with respect to c and θ, and equating them to zero we get normal equations for estimating the parameters of the Poisson weighted Ishita distribution.

$$\frac{\partial}{\partial c}\log L=\frac{\partial}{\partial c}\left(\begin{array}{c}\sum_{i=1}^{n}\log(x_i+c)!+n(c+3)\log(\theta)-\sum_{i=1}^{n}\log(x_i!)-n\log(c!)-n\log(\theta^{3}+(c+1)(c+2))+\\ \sum_{i=1}^{n}\log(\theta(1+\theta)^{2}+(x_i+c+1)(x_i+c+2))-\sum_{i=1}^{n}(x_i+c+3)\log(1+\theta)\end{array}\right)=0$$

$$\frac{\partial}{\partial \theta}\log L=\frac{\partial}{\partial \theta}\left(\begin{array}{c}\sum_{i=1}^{n}\log(x_i+c)!+n(c+3)\log(\theta)-\sum_{i=1}^{n}\log(x_i!)-n\log(c!)-n\log(\theta^{3}+(c+1)(c+2))+\\ \sum_{i=1}^{n}\log(\theta(1+\theta)^{2}+(x_i+c+1)(x_i+c+2))-\sum_{i=1}^{n}(x_i+c+3)\log(1+\theta)\end{array}\right)=0$$

These two derivative equations cannot be solved analytically, therefore $\hat{c}$ *and* $\hat{\theta}$ will be obtained by maximizing the log likelihood function numerically using the Newton-Raphson method, which is a powerful technique for solving equations iteratively and numerically.

6.2. Monte Carlo Simulation

In order to investigate the performance of the maximum likelihood estimators for a finite sample size n using Monte Carlo simulation procedure. Using the inverse cdf method discussed in sub-section 2.1, random data is generated from Poisson weighted Ishita distribution. We took four random parameter combinations as $c=0.4,\ \theta=0.4$, $c=0.8,\ \theta=0.9$, $c=1.5,\ \theta=1.7$ and $c=2.5,\ \theta=3.2$, to carry out the simulation study and the process was repeated 1000 times by going from small to large sample sizes n = (10,25,75,200,300,600). From Table 1, it is clear that the estimated variances and MSEs decrease when the sample size n increases. The coverage probabilities (CP) are near to 0.95 when the sample size increases. Thus, the agreement between theory and practice improves as the sample size *n* increases. Hence, the maximum likelihood method performs quite well in estimating the model parameters of the Poisson weighted Ishita distribution.

Table 1. Simulation study of ML estimators of Poisson weighted Ishita distribution

Sample size (n)	Parameters	$c=0.4,\ \theta=0.4$				$c=0.8,\ \theta=0.9$			
		Bias	Variance	MSE	Coverage Probability	Bias	Variance	MSE	Coverage Probability
10	c	-0.0948	0.008298	0.017285	0.899	0.054353	0.016773	0.0197272	0.919
	θ	0.214834	0.091871	0.1380246	0.922	0.033567	0.039843	0.0409697	0.911
25	c	-0.07802	0.005691	0.0117781	0.939	-0.00876	0.005471	0.0055477	0.929
	θ	0.078574	0.043124	0.0492979	0.943	0.037584	0.010208	0.0116206	0.922
75	c	-0.06463	0.003127	0.007304	0.949	0.016155	0.000861	0.001122	0.942
	θ	-0.05445	0.031839	0.0348038	0.953	0.011167	0.000993	0.0011177	0.949
200	c	-0.04363	0.003372	0.0052756	0.959	-0.00825	0.001847	0.0019151	0.956
	θ	-0.01411	0.007967	0.0081661	0.957	0.007271	0.001054	0.0011069	0.955
300	c	-0.02236	0.001644	0.002144	0.958	0.001713	0.000411	0.0004139	0.959
	θ	-0.00354	0.004377	0.0043895	0.961	0.006739	0.000308	0.0003534	0.966
600	c	-0.01746	0.000182	0.0004869	0.959	0.006854	0.000149	0.000196	0.963
	θ	-0.00046	0.002975	0.0029752	0.971	0.002234	0.000175	0.000180	0.969
Sample size (n)	Parameters	$c=1.5,\ \theta=1.7$				$c=2.5,\ \theta=3.2$			
		Bias	Variance	MSE	Coverage Probability	Bias	Variance	MSE	Coverage Probability
10	c	-0.050778	0.000497	0.003075	0.939	0.054253	0.016673	0.054253	0.829
	θ	0.040750	0.003891	0.005552	0.889	0.033467	0.039743	0.033467	0.915
25	c	-0.044688	0.000719	0.002716	0.938	-0.00886	0.005371	-0.00886	0.927
	θ	-0.016717	0.002150	0.002429	0.952	0.037484	0.010108	0.037484	0.943
75	c	-0.032848	0.000382	0.001461	0.956	0.016055	0.000761	0.016055	0.949
	θ	0.000015	0.000310	0.000310	0.953	0.011067	0.000893	0.011067	0.943
200	c	0.003141	0.000628	0.000638	0.962	-0.00835	0.001747	-0.00835	0.959
	θ	0.003232	0.000001	0.000011	0.958	0.007171	0.000954	0.007171	0.959
300	c	-0.005717	0.000003	0.000036	0.961	0.001613	0.000311	0.001613	0.962
	θ	-0.001419	0.000012	0.000014	0.959	0.006639	0.000208	0.006639	0.961
600	c	0.002955	0.000040	0.000049	0.965	0.006754	0.000049	0.006754	0.972
	θ	0.000943	0.000036	0.000037	0.962	0.002134	0.000075	0.002134	0.968

7. Applications of Poisson Weighted Ishita Distribution

In this section, we fit our proposed model and other related models to a vaccine adverse event count data studied by Rose et al. (2006). The data are the frequencies which correspond to 4020 observed systemic adverse events for four injections for each of the 1005 study participants. The data set is given in Table 2.

Table 2. Data set representing vaccine adverse event count data studied by Rose et al. (2006)

Vaccine adverse event	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	1437	1010	660	428	236	122	62	34	14	8	4	4	1

Maximum likelihood estimation method is used in estimating the parameters for all the suggested models using R software. Parameter estimates with standard errors in parenthesis for each fitted model are given in Table 3.

Table 3. Estimated Parameters by ML method for fitted distributions for data set representing epileptic seizure counts

Distribution	Parameter Estimates (Standard Error)	Model function
Poisson Weighted Ishita	$\hat{c} = 0.33\ (0.08)$ $\theta = 1.51(0.06)$	$P(X=x) = \frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3+(c+1)(c+2))}\left[\frac{\theta(1+\theta)^2+(x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$ $x = 0,1,2,3,\ldots\ ;\ \theta > 0, c > 0$
Poisson Ishita	$\theta = 1.29(0.01)$	$P(X=x) = \frac{\theta^3}{\theta^3+2}\left[\frac{\theta(1+\theta)^2+(x+1)(x+2)}{(1+\theta)^{x+3}}\right]$ $x = 0,1,2,3,\ldots, \theta > 0$
Poisson	$\lambda = 1.50(0.01)$	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \lambda > 0,\ x = 0,1,2,\ldots$
Poisson Lindley	$\theta = 0.99\ (0.016)$	$p(x) = \frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}\ x = 0,1,2,\ldots, \theta > 0$
Geometric	$p = 0.398\ (0.004)$	$p(x) = q^x p \quad 0 < q < 1, q = 1\text{-}p, x = 0,1,2,\ldots$
Negative Binomial	$p = 0.50\ (0.013)$ $r = 1.53\ (0.08)$	$p(x) = \binom{x+r-1}{x} p^r q^x,\ x = 0,1,2,\ldots$ $r > 0, 0 < p < 1$

Zero Inflated Poisson	$\alpha = 0.26\ (0.009)$ $\lambda = 2.04\ (0.031)$	$p(x) = \begin{cases} \alpha + (1-\alpha)\dfrac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0, x = 0 \\ (1-\alpha)\dfrac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0, x = 0,1,2,... \end{cases}$ $0 < \alpha < 1,\ \lambda > 0$
Discrete Weibull	$q = 0.65\ (0.007)$ $\gamma = 1.15\ (0.017)$	$p(x) = q^{x^\gamma} - q^{(x+1)^\gamma},\quad x = 0,1,2,...$ $0 < q < 1,\ \gamma > 0$

We compute the expected frequencies for fitting Poisson weighted Ishita, Poisson Ishita, Poisson, Geometric, Negative Binomial, Zero Inflated Poisson, Poisson Lindley and discrete Weibull distributions with the help of R studio statistical software, and Pearson's chi-square test is applied to check the goodness of fit of the models discussed. The calculated expected frequencies for each fitted model are given in Table 4. For Poisson weighted Ishita, negative binomial and discrete Weibull distributions, p-value is >0.05, hence it fits the data statistically good. Poisson Ishita, Poisson, Geometric, zero inflated Poisson and Poisson Lindley does not fit the data at all as p-value in the case of these models is <0.05. Based on the chi-square, we observe that Poisson weighted Ishita distribution has the highest p-value (0.8162), which signifies that Poisson weighted Ishita provides a better fit for the data set representing vaccine adverse event count data studied by Rose et al. (2006) as compared to other fitted models.

Table 4. Fitted proposed distribution and other competing models to a data set representing epileptic seizure counts

Epileptic seizure (X)	Observed	Poisson Weighted Ishita	Poisson Ishita	Poisson	Geometric	Negative Binomial	Zero Inflated Poisson	Poisson Lindley	Discrete Weibull
0	1437	1427.4	1518.0	890.8	1603.5	1409.1	1437.0	1500.1	1410.7
1	1010	1035.2	965.9	1342.3	963.9	1068.7	787.3	1003.5	1065.4
2	660	665.7	620.3	1011.4	579.4	670.7	803.3	629.2	667.7
3	428	401.0	386.5	508.1	348.3	391.6	546.4	378.7	393.1
4	236	229.5	231.9	191.4	209.4	220.2	278.7	221.6	222.6
5	122	126.0	134.4	57.7	125.9	120.9	113.8	127.0	122.6
6	62	66.9	75.5	14.5	75.7	65.3	38.7	71.7	66.0
7	34	34.5	41.4	3.1	45.5	34.9	11.3	39.9	34.9
8	14	17.4	22.2	0.6	27.3	18.5	2.9	22.0	18.2
9	8	8.6	11.7	0.1	16.4	9.7	0.7	12.1	9.3
10	4	4.2	6.1	0.0	9.9	5.1	0.1	6.6	4.7
11	4	2.0	3.1	0.0	5.9	2.6	0.0	3.5	2.4
12	1	1.8	3.1	0.0	9.0	2.8	0.0	4.1	2.3
P-value		0.8162	0.0036	0.0003	0.0001	0.2619	<0.0001	0.0322	0.3564

Furthermore, in order to compare our proposed distribution and other competing models, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, AICC and BIC values. From Table 5, it is observed that the Poisson weighted Ishita distribution has lesser AIC, AICC and BIC values as compared to other competing models. Hence, we can conclude that the Poisson weighted Ishita distribution leads to a better fit than the other competing models for analysing the data set given in Table 2.

Table 5. Model comparison criterion for fitted models to a data set

Criterion	Poisson Weighted Ishita	Poisson Ishita	Poisson	Geometric	Negative Binomial	Zero Inflated Poisson	Poisson Lindley	Discrete Weibull
-logL	6737.2	6747.5	7231.1	6778.0	6740.6	6868.8	6746.0	6739.7
AIC	13478.4	13496.9	14464.3	13558.1	13485.2	13741.6	13494.0	13483.4
BIC	13491.0	13503.2	14470.6	13564.4	13497.8	13754.2	13500.3	13496.0

We also use Likelihood Ratio (LR) test to check whether the fitted Poisson weighted Ishita distribution for a given data set is statistically "superior" to the fitted Poisson Ishita distribution. In any case, hypothesis tests of the type $H_0 : \Theta = \Theta_0$ versus $H_1 : \Theta \neq \Theta_0$ can be performed using LR statistics. In this case, the LR statistic for testing H_0 versus H_1 is $\omega = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$ where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically ($as\, n \to \infty$) distributed as χ_k^2, with k degrees of freedom, which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.05 at 95% confidence level.

Table 6. Likelihood Ratio test of Poisson weighted Ishita distribution versus Poisson Ishita distribution

Model	Model Function	-logl	Likelihood Ratio Statistic
Poisson weighted Ishita	$P(X = x) = \frac{(x+c)!\theta^{c+3}}{x!c!\,(\theta^3 + (c+1)(c+2))} \left[\frac{\theta(1+\theta)^2 + (x+c+1)(x+c+2)}{(1+\theta)^{x+c+3}}\right]$ $x = 0,1,2,3,...,\ \theta > 0, c > 0$	6737.2	20.60
Poisson Ishita	$P(X = x) = \frac{\theta^3}{\theta^3 + 2}\left[\frac{\theta(1+\theta)^2 + (x+1)(x+2)}{(1+\theta)^{x+3}}\right]$ $x = 0,1,2,3,...,\ \theta > 0$	6747.5	

We have $\chi_1^2 = 6.35$ < Likelihood Ratio Statistic (20.60), thus the null hypothesis is rejected and it is concluded that parameter c is playing a significant role in Poisson weighted Ishita distribution for analysing the data set given in Table 2.

8. Conclusion

A new over-dispersed probability distribution is introduced using the compounding technique. Statistical properties of the proposed model are studied and application in handling count data set representing epileptic seizure counts is analyzed.

REFERENCES

ADIL, R., ZAHOOR, A., JAN, T. R., (2016). A new count data model with application in genetics and ecology. *Electronic Journal of Applied Statistical Analysis,* 9(1), pp. 213–226.

GERSTENKORN, T., (1993). A compound of the generalized gamma distribution with the exponential one, *Recherches surles deformations,* 16(1), pp. 5−10.

GERSTENKORN, T., (1996). A compound of the Polya distribution with the beta one, *Random Operators and Stochastic Equations*, 4(2), pp. 103−110.

GREENWOOD, M., YULE, G. U., (1920). An inquiry into the nature of frequency distribution representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents, *Journal of Royal Statistical Society*, 83, pp. 255−279.

GUPTA, R. C., ONG, S. H., (2004). A new generalization of the negative binomial distribution, *Journal of Computational Statistics and Data Analysis,* 45, pp. 287−300.

LINDELY, D. V., (1958). Fiducial Distributions and Bayes theorem. *Journal of the Royal Statistical Society*, 20(1), pp. 120-107.

MAHMOUDI E., ZAKERZADEH H., (2010). Generalized Poisson-Lindely Distribution. *Communications in statistics – Theory and Methods,* 39(10*),* pp. 1785−1798.

PARA, B. A., JAN, T. R., (2018). An Advanced Discrete Model with Applications in Medical Science, *Journal of Multiscale Modelling*, 9(1), DOI: 10.1142/S1756973718500014, ISSN: 1756-9737 (print).

PARA, B. A., JAN, T. R., (2018). Discrete Inverse Weibull Beta Model: Properties And Applications In Health Science. *Pakistan Journal of Statistics,* 34(3), pp. 229−349.

PLACKETT, R. L., (1953). The truncated Poisson distribution. *Biometrics*, 9(4), pp. 485−488.

R CORE TEAM, (2019). R version 3.5.3: A language and environment for statistical computing. *R Foundation for Statistical Computing*, Vienna, Austria. URL https://www.R-project.org/.

ROSE, C. E., MARTIN, S. W., WANNEMUEHLER, K. A., PLIKAYTIS, B. D., (2006), On the use of zero-inflated and hurdle models for modeling vaccine adverse event count data, *Journal of Biopharmaceutical Statistics,* 16, pp. 463−481.

SHANKER, R., SHUKLA, K. K., (2017). Ishita distribution and its Applications, *Biometrics & Biostatistics International Journal*, 5(2), pp. 1–9.

SHUKLA, K. K., SHANKER, R., (2019). Weighted Ishita Distribution and Its Application to Survival Data, *International Journal of Mathematics and Statistics*, 20(1), pp. 67–82.

SKELLER, J. G., (1948). A probability distribution derived from the binomial distribution by regarding the probability of success as variable between the sets of trials, *Journal of the Royal Statistical Society, Series B,* 10, pp. 257−261.

ZAMANI, H., ISMAIL, N., (2010). Negative Binomial–Lindley Distribution And Its Application. *Journal of Mathematics and Statistics*, 1, pp. 4–9.