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The construction and analysis of repeated measurement designs (RMD) using the trial and error method

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ABSTRACT

Repeated measurement designs prove broadly applicable in almost all branches of bio-sciences, including agriculture, animal husbandry, botany, zoology. Unbiased estimators for elementary contrasts among direct and residual effects are obtainable in this class of designs, which is considered their important property. In this paper, an attempt was made to provide a new method of overcoming a drawback in the construction method developed by Afsarinejad (1983), where one or more treatments may occur more than once in certain sequences causing the constructed designs to no longer remain uniform in the examined periods. Nine designs were constructed and presented jointly with their corresponding mathematical analyses.

Key words: residual effects, order effects, balanced minimal RMD.

1. Introduction

In many horticultural experiments, for example in high density plantation trials, it is imperative that some treatments like fertilizers, insecticides, etc., applied to the crop are not fully utilized by the crop and the unutilized portion called the residual of these applied treatments results in a carry-over effect, which is an effect that "carries over" from one experimental condition to another. Whenever subjects perform in more than one condition (as they do in within-subject designs), there is a possibility of order effects. In such situations, the biggest drawbacks known as the order effects are caused by exposing the subjects to multiple treatments. Order effects are related to the order in which treatments are given but not due to the treatment itself. Order effects can interfere with the analysis and also challenge the ability to correctly estimate the effect of the treatment itself. To overcome the residual effect of the treatment, the experimenter can introduce a rest period and allow the experimental units to even up.

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Many researchers such as Cochran and Cox (1986), Sheeh and Bross (1961),Westlake (1974), Afsarinejad (1989) and Afsarinejad (1990) have pointed out that if residual effects exist, then the methods applicable to conventional designs are not valid. The experimenter has to design the experiment such that the unbiased estimator for elementary contrasts among direct and residual effects can be obtained. In such situations the repeated measurement designs help us to estimate the order effects, thereby increasing the precision of estimates. Patterson (1952) introduced the concept of using the differences for constructing repeated measurement designs and used the method to construct the designs suggested by Williams (1949). In another attempt Bradley (1958), and Sheehe and Bross (1961) constructed the designs for odd and even number of treatments by using easy algorithms. Gorden (1961) used the group theory to form balanced RMD's. Hedayat and Afsarinejad (1975) defined repeated measurement designs balanced for direct and residual effects. They gave construction methods for balanced minimal RMD. In 1978, they gave the concept of a universal optimal design, completely symmetric design, uniform design and balanced uniform design. Pigeon and Raghavarao (1987) introduced a class of crossover designs known as control balanced residual effect design. They gave their structure in detail and also described the method of construction of these designs. Kenward and Jones (1987) proposed a method for the construction of log-linear models for binary data from cross-over designs. Repeated measurement designs have application in various fields of science such as agricultural science, animal science, medical science and engineering, etc. In intensive agricultural systems it is imperative that some treatments like fertilizer, pesticides, etc., applied to a crop are not fully utilized by the crop and the unutilized portion called the residual of those applied treatments is utilized by the subsequent crop. In such cases, the RMD plays a significant role to study the residual effect. Similarly, in the case of animal experiments, the severe restriction of numbers of animals leaves few degrees of freedom for error, thereby reducing statistical power drastically or preventing multivariate analysis entirely. In such cases, the primary benefit of a repeated measurement design (RMD) is statistical power relative to the sample size, which is important in many real research studies. In clinical trials the data on the patients is recorded more than once. In such situations, using the standard ANOVA procedures is not appropriate as it does not consider dependencies between observations within subjects in the analysis. To deal with such types of study data repeated measurement designs are used.

1.1. General Model of Analysis

In RMD certain terms like "direct effect" and "residual effect" are used. Suppose we have't' treatments, which are to be tested and studied via n experimental units. Each experimental unit is used in p periods resulting in $r_i \geq 1$ observations for i^{th} treatment,

i.e. $r_1+r_2+r_3+ \ldots + r_t = np$. Let D denote the set of all such arrangements to which we shall refer as design in D, then let $d_{(i,j)}$ denote the treatment assigned by D in the i^{th} period to the j^{th} experimental unit. Let y_{ij} be response under $d_{(i,j)}$ treatment applied to the j^{th} unit in the i^{th} period and to analyse this set of observations $(y_{ij}'s)$ we take the model:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + e_{ij},\ i=1,2,\ldots,p\ ;\ j=1,2,\ldots,n$$

where

y_{ij} = the observation in the i^{th} period on j^{th} unit
μ = general effect
α_i = i^{th} period effect
β_j = j^{th} unit effect
$\tau_{d(i,j)}$ = direct effect of treatment $d_{(i,j)}$
$\rho_{d(i-1,j)}$ = first order residual effect of treatment $d_{(i-1,j)}$
e_{ij} = uncontrolled random errors which are normally distributed with mean zero and variance σ^2 (i.e. Homocedastic).

1.2. Mathematical Analysis of RMDs

In vector notation np response under the above model can be written as:

$$E(\underline{Y}) = X\left(\underline{\gamma}\right) = [X_1X_2]\begin{bmatrix}\underline{\gamma}\\ \underline{\gamma}\end{bmatrix} = X_1\underline{\gamma}_1 + X_2\underline{\gamma}_2$$

Here, $\underline{\gamma}_1$ is the set of the parameters to be estimated and $\underline{\gamma}_2$ is the set of all other parameters in the model. The normal equations estimating the parameters are:

$$\begin{bmatrix}X_1'\\ X_2'\end{bmatrix}[X_1X_2]\begin{bmatrix}\widehat{\underline{\gamma}_1}\\ \widehat{\underline{\gamma}_2}\end{bmatrix} = \begin{bmatrix}X_1'\underline{Y}\\ X_2'\underline{Y}\end{bmatrix}$$

$$\begin{bmatrix}X_1'X_1 & X_1'X_2\\ X_2'X_1 & X_2'X_2\end{bmatrix}\begin{bmatrix}\widehat{\underline{\gamma}_1}\\ \widehat{\underline{\gamma}_2}\end{bmatrix} = \begin{bmatrix}X_1'\underline{Y}\\ X_2'\underline{Y}\end{bmatrix}$$

$$X_1'X_1\hat{\underline{\gamma}}_1 + X_1'X_2\widehat{\underline{\gamma}_2} = X_1'\underline{Y} \qquad (1)$$

$$X_2'X_1\hat{\underline{\gamma}}_1 + X_2'X_2\widehat{\underline{\gamma}_2} = X_2'\underline{Y} \qquad (2)$$

From normal equation (2) we get

$$\widehat{\underline{\gamma}_2} = (X_2'X_2)^{-1}\left[X_2'\underline{Y} - X_2'X_1\hat{\underline{\gamma}}_1\right] \qquad (3)$$

Putting (3) in equation (1) we get

$$\hat{\underline{\gamma}}_1 = [X_1'\{1 - X_2(X_1'X_2)^{-1}X_2'\}X_1]^{-1}\left[X_1'\{1 - X_2(X_2'X_2)^{-1}X_2'\}\underline{Y}\right] =$$
$$\left[C\left(\hat{\underline{\gamma}}_1\right)\right]^{-1}\left[Q\left(\hat{\underline{\gamma}}_1\right)\right]$$

where

$C\left(\hat{\underline{\gamma}}_1\right)$ is the information matrix associated with $\hat{\underline{\gamma}}$.

1.3. Definition due to Hedayat and Afsarinejad and the design constructed

Hedayat and Afsarinejad (1975), Hedayat and Afsarinejad (1978), Afsarinejad (1983) and Afsarinejad (1990) defined balanced repeated measurement design as: A *RMD (t, n, p)* based on t treatments, *n* experimental units each being used in *p* periods is said to be balanced with respect to the sets of direct treatment effects and the first order residual effects if:

a) Each treatment is tested equally frequently λ_1 times in each period.

b) In the order of application, each treatment is preceded by each other treatment equally frequently λ_2 times.

Clearly, in a balanced RMD (*t, n, p*), the following relation holds:

- $n = \lambda_1 t$
- $n(p-1) = \lambda_2 t(t-1)$

Afsarinejad (1983) gave a construction method for the balanced minimal repeated measurement designs for an odd number of treatments. RMD (15, 105, 3), $\lambda_2 = 1$ and RMD (21, 105, 5), $\lambda_2 = 1$, constructed by this method, have a drawback that treatments can occur more than once in the same sequences. A design constructed by the method given by Afsarinejad is given below in Table 1.

Table 1. RMD (15, 105, 3) given by Afsarinejad

	Initial sequences						
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**
Treatments	1	3	5	7	7	5	3
	15	13	11	9	11	13	15
	3	5	7	7	5	3	1

From the table, it is clear that the treatment "7" is occurring more than once in the 4th initial sequence of RMD (15, 105, 3), $\lambda_2 = 1$. Therefore, on development the final RMD (15, 105, 3), $\lambda_2 = 1$ will have t=15 sequences wherein the treatments will be occurring more than once. Keeping this drawback in view new repeated measurement designs are proposed wherein no sequence has any treatment occurring more than once and which are also suitable for an even number of treatments using the trial and error method of design construction.

1.4. Trial and Error Method

Trial and error is a fundamental method of problem-solving. It is neither a method of finding the best solution nor a method of finding all solutions. It is a technique that

is used simply to find a solution. In this method, a researcher tries the option that has the best possible chances of succeeding. If that didn't work, one can try the next best option until they find a good solution. It is characterized by repeated, varied attempts which are continued until success, or until the practiser stops trying. The trial and error approach is used most successfully with simple problems and in games, and it is often the last resort when no apparent rule applies. Trial and error method is solution-oriented i.e. it makes no attempt to discover why a solution works, merely that it is a solution. It is problem-specific and makes no attempt to generalize a solution to other problems. It is non-optimal and needs little knowledge.

2. New Designs Constructed by Trial and error Method.

The same design by Afsarinejad is constructed using the trial and error method and is a repeated measurement design (15, 105, 3), implying a design having 15 treatments, 105 experimental units and 3 periods. The construction is given in Table 2.

Table 2. RMD (15, 105, 3), $\lambda_2 = 1$

Treatments / Period	Initial sequences						
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**
I	1	3	5	7	2	4	6
II	15	13	11	9	14	12	10
III	3	5	7	10	4	6	8

From the table above it is clear that the repeated measurements design has 15 treatments, 105 experimental units and the experiment lasts for 3 periods. Each experimental unit receives one treatment during each period. As it stands, the design is a 3x105 array containing entries from t = { 1, 2, . . . , 15}. The design is developed cyclically starting from the initial sequences. It can be easily verified that the set of initial sequences given when developed in a cyclic manner, the RMD (15, 105, 3), so obtained is a balanced repeated measurement design according to the definition of Hedayat and Afsarinejad, and has no sequence wherein no treatment occurs more than once in a sequence as shown in Table 3.

Table 3. Design obtained on development sequence of RMD (15, 105, 3)

	Experimental Units														
PERIOD	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
II	15	1	2	3	4	5	6	7	8	9	10	11	12	13	14
III	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2
	16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2
	13	14	15	1	2	3	4	5	6	7	8	9	10	11	12
	5	6	7	8	9	10	11	12	13	14	15	1	2	3	4

Table 3. Design obtained on development sequence of RMD (15, 105, 3) (cont.)

	Experimental Units														
	31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**
	5	6	7	8	9	10	11	12	13	14	15	1	2	3	4
	11	12	13	14	15	1	2	3	4	5	6	7	8	9	10
	7	8	9	10	11	12	13	14	15	1	2	3	4	5	6
	46	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
	7	8	9	10	11	12	13	14	15	1	2	3	4	5	6
	9	10	11	12	13	14	15	1	2	3	4	5	6	7	8
	10	11	12	13	14	15	1	2	3	4	5	6	7	8	9
	61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1
	14	15	1	2	3	4	5	6	7	8	9	10	11	12	13
	4	5	6	7	8	9	10	11	12	13	14	15	1	2	3
	76	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
	4	5	6	7	8	9	10	11	12	13	14	15	1	2	3
	12	13	14	15	1	2	3	4	5	6	7	8	9	10	11
	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
	91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**	**101**	**102**	**103**	**104**	**105**
	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
	10	11	12	13	14	15	1	2	3	4	5	6	7	8	9
	8	9	10	11	12	13	14	15	1	2	3	4	5	6	7

*Initial Sequences.

2.1. Other designs constructed by the proposed method

A repeated measurement design (10, 30, 3) implies a design having 10 treatments, 30 experimental units and 3 periods. The construction is given in Table 4. The subsequent designs are also given in the following tables.

Table 4. RMD (10, 30, 7), $\lambda_2 = 2$

Treatments / Period	Initial sequences		
	I	**II**	**III**
I	1	4	2
II	10	7	9
III	2	5	3
IV	9	6	4
V	3	1	7
VI	8	10	5
VII	4	2	1

Table 5. RMD (15, 105, 5), $\lambda_2 = 2, \lambda_1 = 7$

Treatments / Period	Initial sequences						
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**
I	1	13	6	1	12	6	1
II	15	4	10	14	4	9	11
III	2	12	7	2	11	7	5
IV	14	5	9	13	5	8	6
V	3	11	8	3	10	3	2

Table 6. RMD (22, 154, 7), $\lambda_1 = 7, \lambda_2 = 2$

Treatments / Period	Initial sequences						
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**
I	1	4	7	10	2	5	8
II	22	19	16	9	21	18	15
III	2	5	8	7	3	6	9
IV	21	18	15	8	20	17	14
V	3	6	9	11	4	7	10
VI	20	17	14	12	19	16	13
VII	4	7	10	14	5	8	11

Table 7. RMD (26, 130, 11), $\lambda_1 = 5, \lambda_2 = 2$

Treatments / Period	Initial sequences				
	I	**II**	**III**	**IV**	**V**
I	1	6	11	3	8
II	26	21	10	24	19
III	2	7	6	4	9
IV	25	20	3	23	18
V	3	8	1	5	10
VI	24	19	5	22	17
VII	4	9	7	6	11
VIII	23	18	12	21	16
IX	5	10	15	7	12
X	22	17	16	20	15
XI	6	11	17	8	13

Table 8. RMD (21, 105, 9), $\lambda_1 = 5, \lambda_2 = 2$

Treatments / Period	Initial sequences				
	I	**II**	**III**	**IV**	**V**
I	1	5	13	19	15
II	21	17	10	3	7
III	2	6	12	18	14
IV	20	7	11	4	8
V	3	15	21	17	13
VI	19	8	1	5	9
VII	4	14	20	16	12
VIII	18	9	2	6	10
IX	5	13	19	15	1

Table 9. RMD (22, 66, 13), $\lambda_2 = 2$

Treatments / Period	Initial sequences		
	I	**II**	**III**
I	1	8	4
II	22	15	19
III	2	9	5
IV	21	14	18
V	3	10	6
VI	20	13	7
VII	4	11	16
VIII	19	12	8
IX	5	1	15
X	18	22	9
XI	6	2	14
XII	17	21	10
XIII	7	3	3
XIV	16	20	11
XV	8	4	1

Table 10. RMD (28, 252, 7), $\lambda_2 = 2$

Treatments / Period	Initial sequences								
	I	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**IX**
I	1	4	7	10	13	27	24	8	11
II	28	25	22	19	16	2	5	20	17
III	2	5	8	11	14	26	23	9	12
IV	27	24	21	18	15	3	6	19	16
V	3	6	9	12	28	25	22	10	13
VI	26	23	20	17	1	4	21	18	15
VII	4	7	10	13	27	24	8	11	1

Similarly the above sequences can be developed cyclically to yield the balanced repeated measurement designs.

2.2. Analysis of the constructed repeated measurement designs

Let us consider a repeated measurement design in which 't' treatments are applied to 'n' experimental units during 'p' periods. Let the rows correspond to periods and columns to experimental units. For analysis of these *'pn'* observations, we assume the fixed effect model:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + e_{ij} \qquad \text{(A)}$$

where *i=1,2,...,p and j=1,2,...,n*

Y_{ij} is the observation in the i^{th} period on the j^{th} experimental unit receiving $d(i,j)$ treatment.

μ is the general effect.

α_i is the i^{th} period effect.

β_j is the j^{th} experimental unit effect.

$\tau_{d(i,j)}$ is the direct effect of $d(i,j)$ treatment.

$\rho_{d(i-1,j)}$ is the first order residual effect of $d(i,j)$ treatment.

Assuming e_{ij} are homoscedastic with mean (0) and variance(σ^2). The information matrix associated with the entire set of parameters of model (A) can be rewritten as:

$$X'X = \begin{bmatrix} np & nE_{1p} & nE_{1p} & \underline{r}' & \underline{r}^{*\prime} \\ nE_{p1} & nI_p & M & L' & L^{*\prime} \\ pE_{n1} & M' & pI_n & N' & N^{*\prime} \\ \underline{r} & L & N & diag(\underline{r}) & S \\ \underline{r}^* & L^* & N^* & S' & diag(\underline{r}^*) \end{bmatrix}$$

Where E_{ab} is the a x b matrix with all entries as one.

I_p : identity matrix of order p

M : incidence matrix of order p x n for the period unit.

L : incidence matrix of order t x p for the direct effects-period.

N : incidence matrix of order t x n for the direct effectsunits.

L^* : incidence matrix of order t x p for the residual effects-period.

N^* : incidence matrix of order t x n for the residual effects-period.

S : incidence matrix of order t x t for the direct effects-residual effects.

$\underline{r}'$: $[r_1,\ r_2, \dots, r_t]$

$\underline{r}^{*\prime} = [r_1^*,\ r_2^*, \dots, r_3^*]$

$diag(\underline{r}) = diag(r_1, r_2, \dots, r_t)$

$diag(\underline{r}^*) = diag(r_1^*, r_2^*, \dots, r_t^*)$

r_i = No. of observations in which the i^{th} treatment was applied.

r_i^* = No. of treatments in which the i^{th} treatment was applied in the periods other than the last period.

The normal equations of the above model from the information matrix are as follows:

$$np\hat{\mu} + nE_{1p}\underline{\hat{\alpha}} + pE_{1n}\underline{\hat{\beta}} + \underline{r'\hat{\tau}} + \underline{r}^*\underline{\hat{\rho}} = G$$

$$nE_{p1}\hat{\mu} + nI_p\underline{\hat{\alpha}} + M\underline{\hat{\beta}} + L'\underline{\hat{\tau}} + L^{*\prime}\underline{\hat{\rho}} = \underline{A}$$

$$pE_{n1}\hat{\mu} + M'\underline{\hat{\alpha}} + pI_n\underline{\hat{\beta}} + N'\underline{\hat{\tau}} + N^{*\prime}\underline{\hat{\rho}} = \underline{B}$$

$$\underline{r}\hat{\mu} + L\underline{\hat{\alpha}} + N\underline{\hat{\beta}} + \left(diag\underline{r}\right)\underline{\hat{t}} + S\underline{\hat{\rho}} = \underline{T}$$
$$\underline{r^*}\hat{\mu} + L^*\underline{\hat{\alpha}} + N^*\underline{\hat{\beta}} + S'\underline{\hat{t}} + \left(diag\underline{r^*}\right)\underline{\hat{\rho}} = \underline{R}$$

Let

A_i : sum of all the observations in the i^{th} period.

B_i : sum of the observations on the j^{th} experimental unit.

T_k : sum of the observations of the k^{th} treatment.

R_p : sum of those observations on which the p^{th} treatment was applied in the immediately preceding period.

G : grand total of all 'np' observations.

Let

$$\underline{A} = \left[A_1, A_2, \dots\dots, A_p\right]'$$
$$\underline{B} = [B_1, B_2, \dots\dots, B_n]'$$
$$\underline{T} = [T_1, T_2, \dots\dots, T_t]'$$
$$\underline{R} = [R_1, R_2, \dots\dots, R_t]'$$

Under the condition that from each unit we have observations in each period (no observation is missing), giving rise to '***pn***' observations, then:

$$M = E_{pn}$$
$$E_{1p}M = \mathrm{p}E_{1n}$$
$$NE_{n1} = LE_{p1} = \underline{r}$$
$$NE_{n1} = LE_{p1} = \underline{r}$$
$$N^*E_{n1} = L^*E_{p1} = \underline{r}^*$$
$$E_{1t}N = pE_{1n}$$
$$E_{1p}\underline{A} = G$$
$$E_{1p}\underline{B} = G$$
$$E_{1t}\underline{T} = G$$
$$E_{tt}N\underline{B} = pE_{t1}G$$
$$E_{tt}N^*\underline{B} = (p-1)E_{t1}G$$
$$E_{1t}N^*\underline{B} = (p-1)G$$

From Equation: we have

$$\underline{\hat{\alpha}} = \frac{\left[\underline{A} - nE_{p1}\hat{\mu} - M\underline{\hat{\beta}} - L'\underline{\hat{t}} - L^{*\prime}\underline{\hat{\rho}}\right]}{n}$$

Put $\underline{\hat{\alpha}}$ in equation:

$$PE_{n1}\hat{\mu} + \frac{1}{n}M'\underline{A} - M'E_{p1}\hat{\mu} - \frac{1}{n}M'M\underline{\hat{\beta}} - \frac{1}{n}M'L'\underline{\hat{\tau}} - \frac{1}{n}M'L^{*\prime}\underline{\hat{\rho}} + pI_n\underline{\hat{\beta}} + N'\underline{\hat{\tau}} + N^{*\prime}\underline{\hat{\rho}} = \underline{B}$$

$$\left(pI_n - \frac{p}{n}E_{nn}\right)\underline{\hat{\beta}} + \left(N' - \frac{1}{n}M'L'\right)\underline{\hat{\tau}} + \left(N^{*\prime} - \frac{1}{n}M'L^{*\prime}\right)\underline{\hat{\rho}} = \underline{B} - \frac{1}{n}M'\underline{A}$$

$$\underline{\hat{\beta}} = \frac{1}{p}I_n\left(\underline{B} - \frac{1}{n}E_{np}\underline{A}\right) - \left(N' - \frac{1}{n}E_{np}L'\right)\underline{\hat{\tau}} - (N^* - \frac{1}{n}E_{np}L^{*\prime})\,\underline{\hat{\rho}}$$

From Equation substituting $\underline{\hat{\beta}}$ in equation:

$$\underline{\hat{\alpha}} = \frac{\underline{A}}{n} - E_{p1}\hat{\beta} - \frac{L'\underline{\hat{\tau}}}{n} - \frac{L^{*\prime}\underline{\hat{\rho}}}{n}$$

From the equations and substituting $\underline{\hat{\alpha}}$ and $\underline{\hat{\beta}}$ *in equation* gives:

$$\left[(diag.\underline{r} - \frac{1}{n}LL' - \frac{1}{p}NN' + \frac{1}{np}\,\underline{r}\,\underline{r}')\right]\underline{\hat{\tau}} + (S - \frac{1}{n}LL^{*\prime} - \frac{1}{p}NN^{*\prime} - \frac{1}{np}\underline{r}\,\underline{r}^{*\prime})\hat{\rho} = \underline{T} - \frac{1}{n}L\underline{A} - \left(\frac{1}{p}N - \frac{1}{np}\underline{r}E_{1n}\right)\underline{B}$$

From the equations and substituting $\underline{\hat{\alpha}}$ and $\underline{\hat{\beta}}$ *in equation* gives:

$$\underline{r}^*\hat{\mu} + \left(\frac{1}{n}L^* + \frac{1}{np}L^*E_{pp}\right)\underline{A} - L^*E_{p1}\hat{\beta} - \frac{1}{np}L^*E_{pn}\underline{\hat{\beta}} - \frac{1}{n}L^*L'\underline{\hat{\tau}} - \frac{1}{n}L^*L^{*\prime}\underline{\rho} + \frac{1}{p}N^*\underline{B} - \frac{1}{np}N^*E_{np}\underline{A} - \frac{1}{p}\left(N^*N' - \frac{1}{np}N^*E_{np}L'\right)\underline{\hat{\tau}} - \frac{1}{p}\left(N^*N^{*\prime} - \frac{1}{np}N^*E_{np}L^{*\prime}\right)\underline{\hat{\rho}} + S'\underline{\hat{\tau}} + \left(diag.\underline{r}^*\right)\underline{\hat{\rho}} = \underline{R}$$

or

$$(S' - \frac{1}{n}L^*L' - \frac{1}{p}N^*N' + \frac{1}{np}\underline{r}^*\underline{r}')\underline{\hat{\tau}} + \left(diag\underline{r}^* - \frac{1}{n}L^*L^{*\prime} - \frac{1}{p}N^*N^{*\prime} + \frac{1}{np}\underline{r}^*\underline{r}^{*\prime}\right)\underline{\hat{\rho}} = \underline{R} - \frac{1}{n}L^*\frac{A}{p} - (\frac{1}{np}N^* - r^*E_{1n})\underline{B}$$

Equations are the reduced normal equations for estimating the direct and residual effects of a general *RMD* (*t, n, p*). As a special case, let each treatment occurs *v* times in each row (period), then:

$$\underline{r} = pvE_{11}$$
$$\underline{r}^* = (pv - v)E_{t1}$$
$$\underline{r}\,\underline{r}' = p^2v^2E_{tt}$$
$$\underline{r}\,\underline{r}^{*\prime} = pv^2(p-1)E$$
$$E_{np}L^{*\prime} = (p-1)vE_{nt}$$
$$L^*\underline{A} = v(G - A_1)E_{t1}$$
$$L = vE_{tp}$$
$$L^* = v\left[O_{t,1}E_{t,p-1}\right]$$
$$LL' = pv^2E_{tt}$$

$$L^*L^{*\prime} = (p-1)v^2E_{tt}$$
$$E_{tp}L^{*\prime} = (p-1)vE_{tt}$$

Substituting these relations in equations respectively gives:

$$\underline{\hat{\alpha}} = \left(\frac{1}{n} + \frac{1}{np}E_{pp}\right)\underline{A} - E_{p1}\hat{\mu} - \frac{1}{np}E_{pn}\underline{B} - \frac{v}{n}E_{pt}\hat{\tau} - \frac{1}{n}L^{*\prime}\underline{\hat{\rho}}$$

$$\underline{\hat{\beta}} = \frac{1}{p}\underline{B} - \frac{1}{np}E_{np}\underline{A} - \left(\frac{1}{p}N' - \frac{v}{n}E_{nt}\right)\underline{\hat{\tau}} - \left(\frac{1}{p}N^{*\prime} - \frac{(p-1)v}{np}E_{nt}\right)\underline{\hat{\rho}}$$

$$\left(pvI_t - \frac{1}{p}NN'\right)\underline{\hat{\tau}} + \left(S - \frac{1}{p}NN^*\right)\underline{\hat{\rho}} = \underline{T} - \frac{1}{p}N\underline{B}$$

$$\left(S' - \frac{1}{p}N^*N'\right)\underline{\hat{\tau}} + \left[v(p-1)I_t - \frac{(p-1)v}{pt}E_{tt} - \frac{1}{p}N^*N'\right]\underline{\hat{\rho}} = \underline{R} - \frac{1}{p}N^*\underline{B} + \frac{1}{t}E_{t1}A_1 - \frac{1}{pt}E_{t1}G$$

Using the notation $C_{a/\beta,\gamma..}$ to denote the matrix for estimating the parameter vector of a linear model after eliminating the general effect and the parameter vectors, γ... and ignoring any other parameter in the model which are not listed in the subscript. From equations the coefficient matrices $C_{\tau/\alpha,\beta,..}$, $C_{\rho/\alpha,\beta,..}$ in the equation estimating the direct and residual effects are respectively:

$$C_{\tau/\alpha,\beta,\rho..} = C_{\tau/\alpha,\beta,..} - F_1\overline{C}_{\rho/\alpha,\beta}F_1'$$

where

$$C_{\rho/\alpha,\beta,\tau..} = C_{\rho/\alpha,\beta,..} - F_1'\overline{C}_{\tau/\alpha,\beta}F_1$$

$$C_{\tau/\alpha,\beta,..} = pvI_t - \frac{1}{p}NN'$$

$$C_{\rho/\alpha,\beta.} = v(p-1)I_t - \frac{(p-1)}{np}v^2E_{tt} - \frac{1}{p}N^*N^{*\prime}$$

And $$F_1 = S - \frac{1}{p}NN^{*\prime}$$

2.3. Inverse of circulant matrix

The estimate of $\underline{\hat{\gamma}}_1$is

$$[X_1'\{1 - X_2(X_1'X_2)^{-1}X_2'\}X_1]^{-1}\left[X_1'\{1 - X_2(X_2'X_2)^{-1}X_2'\}\underline{Y}\right] = \left[C\left(\underline{\hat{\gamma}}_1\right)\right]^{-1}\left[Q\left(\underline{\hat{\gamma}}_1\right)\right]$$

Where

$C\left(\underline{\hat{\gamma}}_1\right)$ is the information matrix associated with $\underline{\hat{\gamma}}$.

For estimating the parameter vector of a linear model we have:

$$C_{\tau/\alpha,\beta,..} = pvI_t - \frac{1}{p}NN'$$

$$C_{\rho/\alpha,\beta.} = v(p-1)I_t - \frac{(p-1)}{np}v^2E_{tt} - \frac{1}{p}N^*N^{*\prime}$$

And to subsequently estimate the parameters we need to find the inverse of $C\left(\underline{\hat{\gamma}}_1\right)$, which is circulant in nature. Hence, to find the inverse a direct method proposed by Lin Fuyong (2011) is used to get the inverse matrix of the circulant matrix. The elements of the inverse matrix are functions of zero points of the characteristic polynomial $g(z)$ and $g'(z)$of the circulant matrix.

3. Conclusion

The initial sequences for the constructed designs have been given in the form of tables. A desirable property of the designs constructed by trial and error is that in all the given sequences of treatments, no treatment occurs more than once in a sequence, whereas in the method of construction given by Afsarinejad (1983) for same parametric combinations in some sequences treatments occur more than once. A matrix approach to the mathematical analysis of the designs is given along with the coefficient matrices $C_{\tau/\alpha,\beta,..}$, $C_{\rho/\alpha,\beta,..}$ in the equation estimating the direct and residual effects. And to subsequently estimate the parameters we need to find the inverse of $C\left(\underline{\hat{\gamma}}_1\right)$, which is circulant in nature, a direct method proposed by Lin Fuyong (2011) is used. Although the elementary method to find the inverse can also be used, in the case of matrices of higher order, the method given by Lin Fuyong is a feasible one.

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