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Bayesian estimation and prediction based on Rayleigh record data with applications

Raed R. Abu Awwad[1], **Omar M. Bdair**[2], **Ghassan K. Abufoudeh**[3]

ABSTRACT

Based on a record sample from the Rayleigh model, we consider the problem of estimating the scale and location parameters of the model and predicting the future unobserved record data. Maximum likelihood and Bayesian approaches under different loss functions are used to estimate the model's parameters. The Gibbs sampler and Metropolis-Hastings methods are used within the Bayesian procedures to draw the Markov Chain Monte Carlo (MCMC) samples, used in turn to compute the Bayes estimator and the point predictors of the future record data. Monte Carlo simulations are performed to study the behaviour and to compare methods obtained in this way. Two examples of real data have been analyzed to illustrate the procedures developed here.

Key words: Bayesian estimation and prediction, Rayleigh distribution, record values, Markov Chain Monte Carlo samples.

1. Introduction

Assume we have a sequence of independent and identically distributed (iid) random variables from Rayleigh distribution. The two-parameter Rayleigh distribution with parameters λ and μ has the cumulative distribution function (CDF) and the probability density function (PDF), respectively

$$F(x;\lambda,\mu) = 1 - e^{-\lambda(x-\mu)^2}, x > \mu, \tag{1}$$

and

$$f(x;\lambda,\mu) = \begin{cases} 2\lambda\,(x-\mu)\,e^{-\lambda(x-\mu)^2} & if\ x > \mu, \\ 0 & if\ x \leq 0\,, \end{cases} \tag{2}$$

where λ and μ are the scale and location parameters, respectively, and $\lambda > 0$ and $\mu > 0$. From now on the Rayleigh distribution with parameters λ and μ will be denoted by $Ra(\lambda,\mu)$. The Rayleigh distribution has many applications in reliability, life testing and survival analysis. More details on the Rayleigh distribution can be found in Johnson et al. (1994).

[1]Department of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman, Jordan. E-mail: raed_abuawwad@yahoo.com.
[2]Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan. Department of Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4L8, Canada. E-mail: bdairmb@bau.edu.jo.
[3]Department of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman, Jordan. E-mail: ghassan_math@yahoo.com.

The random variable X_j is called a record (upper record) if $X_j > X_i$ for all $i = 1,2,...,j-1$. By convention X_1 is a record. Then, the record times sequence $\{U(n), n \geq 1\}$ is defined as $U(1) = 1$ with probability one, and for $n \geq 2$, $U(n) = \min\{j : j > U(n-1)\}$. The random variables $X_{U(n)}, n \geq 1$ denote the record values from X-sequence. Naturally, record values appear in many real life situations including data related to weather, sports, economics and life-tests. For more details about the applications of record values, we refer the reader to Arnold et al. (1998), Nevzorov (2000), and Gulati and Padgett (2003). Extensive studies for estimating parameters from the Rayleigh distribution based on different types of ordered data are available in the literature, but no attempt has been made for comparing the performances in estimating and predicting under different types of loss functions based on record values. Many authors in the literature worked on record data, among others; Bdair and Raqab (2016) considered the Bayesian prediction of future records from Weibull distribution when one- and two-sequence are used. Raqab et al. (2007) obtained the maximum likelihood estimator and Bayes estimators for the parameters of the Pareto distribution based on the record data. Raqab et al. (2018) studied the estimation and prediction problem of bathtube-shaped distribution based on record values. Madi and Raqab (2004) studied the problem of temperature records as an application to Pareto Bayesian prediction problem. Based on a set of observed records from the exponential distribution, Ahsanullah M. (1980) discussed the problem of predicting the unseen records. Ahmadi and Doostparast (2006) discussed the Bayesian estimation and prediction based on record values for some distributions like Weibull, Pareto and Burr type XII. Bdair and Raqab (2009) studied the mean residual lifetime of record data and many of its mathematical properties.

Bayesian estimation of the distribution's parameters as well as prediction of future records are of natural interest in this context. For estimating θ by a decision δ, we consider three types of loss functions. The first one is a symmetric quadratic loss function, which is given by

$$LF_1(\theta, \delta) = (\theta - \delta)^2.$$

The second one is an alternative to the squared loss function, namely the absolute loss function and it is given by

$$LF_2(\theta, \delta) = |\theta - \delta|.$$

Varian (1975) proposed the LINEX loss function which is more commonly used form of asymmetric loss. LINEX loss function can be defined by

$$LF_3(\theta, \delta) = e^{a^*(\delta - \theta)} - a^*(\delta - \theta) - 1, \; a^* \neq 0.$$

To perform a Bayesian estimates of the Rayleigh distribution parameters, their prior distributions should be specified. When both parameters λ and μ are unknown, we assume that λ has the gamma prior distribution. The gamma prior distribution of λ denoted by

$Gamma(a,b)$ and is given by

$$\pi_1(\lambda|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \leq 0. \end{cases} \quad (3)$$

Here, the hyper-parameters $a > 0$, $b > 0$, and $\Gamma(a)$ is the gamma function, *i.e.* $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$. The prior of μ ($\pi_2(\mu)$) is assumed with support $(0, x_{U(1)})$. For more details, one may refer to Kundu (2008) and Abu Awwad et al. (2018) and the reference therein.

The remaining sections of the paper are organized as follows. In Section 2, we propose the maximum likelihood estimator (MLE) of the parameters of Rayleigh distribution. In Section 3, we use the Metropolis-Hastings method with normal proposal (see Metropolis, et al. (1953)) and the Gibbs sampling approach to compute the Bayes estimators (BEs) of λ and μ under different loss functions LF_1, LF_2 and LF_3. The implementation of Gibbs sampling and Metropolis-Hastings methods to compute sample-based estimators for the predictive density functions of the future record values based on some current available records is discussed in Section 4. In Section 5, we show numerical data analyses for illustrative purpose. For this, we employ Monte Carlo simulation to compare the BEs with the corresponding maximum likelihood estimates as well as to predict and compare between the predicted values based on the suggested types of loss functions. We conclude the results obtained in this work in Section 6.

2. Maximum likelihood estimation

Let $x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}$ be a sequence of n Rayleigh upper record values with respective PDF and CDF given in Eq. (1) and Eq. (2). The likelihood function of this sample, see for example Arnold et al. (1998) and Ahsanullah (2004), is given by

$$\begin{aligned} L(\lambda,\mu|data) &= \prod_{i=1}^{n-1} \frac{f(x_{U(i)}|\lambda,\mu)}{1-F(x_{U(i)}|\lambda,\mu)} f(x_{U(n)}|\lambda,\mu) \\ &= 2^n\lambda^n \prod_{i=1}^{n}(x_{U(i)-\mu})\, e^{-\lambda(x_{U(n)}-\mu)^2}. \end{aligned} \quad (4)$$

The natural logarithm of the likelihood function is

$$\ln L(\lambda,\mu|data) = n\ln 2 + n\ln\lambda + \sum_{i=1}^{n}\ln(x_{U(i)}-\mu) - \lambda(x_{U(n)}-\mu)^2. \quad (5)$$

By equating the partial derivatives of Eq. (5), $\frac{\partial \ln L(\lambda,\mu|data)}{\partial\lambda}$ and $\frac{\partial \ln L(\lambda,\mu|data)}{\partial\mu}$, to zero, we readily conclude the following two normal equations

$$\frac{n}{\lambda} - (x_{U(n)}-\mu)^2 = 0, and \quad (6)$$

$$-\sum_{i=1}^{n}(x_{U(i)}-\mu)^{-1}-2\lambda(x_{U(n)}-\mu)=0. \tag{7}$$

Equations (6) and (7) cannot be solved explicitly to obtain exact solutions for λ and μ, hence fixed point iteration method is employed for that. From Eq. (6), we can find the MLE of λ as a function of μ, say $\hat{\lambda}(\mu)$, as follows

$$\hat{\lambda}(\mu)=\frac{n}{(x_{U(n)}-\mu)^2}. \tag{8}$$

Substituting Eq. (8) in Eq. (5), without adding the constant term, we obtain the natural logarithm of the likelihood function of μ as

$$g(\mu)=-n\ln(x_{U(n)}-\mu)^2+\sum_{i=1}^{n}\ln(x_{U(i)}-\mu). \tag{9}$$

By maximizing Eq. (9) with respect to μ, we get the MLE of μ, say $\hat{\mu}_{MLE}$. Applying the fixed point solution method on Eq.'s (10) and (11) below, we can directly obtain the maximum of Eq. (9).

$$h(\mu)=\mu, \tag{10}$$

where

$$h(\mu)=x_{U(n)}+2n\left(\sum_{i=1}^{n}(x_{U(i)}-\mu)^{-1}\right)^{-1}. \tag{11}$$

Very simple iterative procedure $h(\mu^{(j)})=\mu^{(j+1)}$, where $\mu^{(j)}$ is the j-th iterative, can be used to solve Eq. (10). Once $\hat{\mu}_{MLE}$ is obtained, the MLE of λ, say $\hat{\lambda}_{MLE}$, can be calculated from Eq. (8) as $\hat{\lambda}_{MLE}=\hat{\lambda}(\hat{\mu}_{MLE})$.

3. Bayesian estimation and corresponding CIs

Let us first consider the case when the location parameter μ is known. Based on the n observed upper record data $x_{U(1)},x_{U(2)},\ldots,x_{U(n)}$, and by combining the likelihood function Eq. (4) and the prior density Eq. (3), the marginal density of λ given μ and data can be obtained to be $Gamma(a+n,\ b+(x_{U(n)}-\mu)^2)$ of the form

$$\pi_1(\lambda|\mu,data)=\frac{(b+(x_{U(n)}-\mu)^2)^{a+n}}{\Gamma(a+n)}\lambda^{a+n-1}\,e^{-\lambda(b+(x_{U(n)}-\mu)^2)}. \tag{12}$$

Under the squared error loss function LF_1, the BE $\hat{\lambda}_{B_1}$ of λ is the posterior mean which is given by

$$\hat{\lambda}_{B_1}=E_{posterior}(\lambda|\mu,data)=\frac{a+n}{b+(x_{U(n)}-\mu)^2}.$$

Clearly, the BE under LF_1 loss function, $\widehat{\lambda}_{B_1}$ is the same as the the corresponding MLE of λ when Jeffrey's prior ($a = b = 0$) is employed. The median of the posterior density, $\hat{\lambda}_{B_2}$, is the BE of λ in case of absolute error loss function LF_2. Since the median of the posterior density cannot have an explicit expression, a numerical solution is required by solving the following equation in w:

$$\Gamma\left(a+n,\ (b+(x_{U(n)}-\mu)^2)w\right)-\frac{\Gamma(a+n)}{2}=0,$$

where

$$\Gamma(a,c)=\int_c^{\infty} x^{a-1}\, e^{-x}\, dx,\ a>0, c>0,$$

is the incomplete gamma function. Under the LINEX loss function LF_3 and for any given $a^* \neq 0$, the BE $\hat{\lambda}_{B_3}$ of λ can be computed using the PDF of the gamma distribution as follows:

$$\begin{aligned}
\hat{\lambda}_{B_3} &= -\frac{1}{a^*}\ln\left[E_{posterior}[e^{-a^*\lambda}|data]\right] \\
&= -\frac{1}{a^*}\ln\left[\int_0^{\infty} e^{-a^*\lambda}\ \pi_1(\lambda|\mu,data)\, d\lambda\right] \\
&= -\frac{a+n}{a^*}\ln\left[\frac{b+(x_{U(n)}-\mu)^2}{a^*+b+(x_{U(n)}-\mu)^2}\right].
\end{aligned}$$

Since the posterior distribution of λ given μ and data follows a gamma distribution, a credible interval of λ can be easily obtained using the percentiles from the gamma distribution. In particular, if a is positive integer, then the chi-square table values can be easily used for constructing credible interval for λ.

Now, we consider the case when both parameters λ and μ are unknown. By using the prior distributions $\pi_1(\lambda|a,b)$ and $\pi_2(\mu)$, the joint posterior function of λ and μ is given by

$$\pi(\lambda,\mu|data)=\frac{L(\lambda,\mu|data)\,.\,\pi_1(\lambda|\mu,a,b)\pi_2(\mu)}{\int_0^{\infty}\int_0^{\infty} L(\lambda,\mu|data)\,.\,\pi_1(\lambda|\mu,a,b)\pi_2(\mu)\,d\lambda\,d\mu}. \tag{13}$$

The marginal density of μ is obtained to be

$$\pi(\mu|\lambda,data)\propto\prod_{i=1}^{n}(x_{u(i)}-\mu)\ e^{\lambda(x_{U(n)}-\mu)^2}\ \pi_2(\mu), \tag{14}$$

where $\pi_2(\mu)$ is a prior distribution with support $(0, x_{U(1)})$. Here, we follow the approach suggested by Berger and Sun (1993) that no specific form of prior $\pi_2(\mu)$ on μ is assumed. For more details about this type of prior, the reader is referred to Abu Awwad et al. (2018). Under the squared error loss function LF_1, the BE of $\theta = g(\lambda,\mu)$, a function λ and μ, can be presented as

$$\hat{\theta}_{B_1} = E_{posterior}(\theta|data) = \int_0^\infty \int_0^\infty \theta\, \pi(\lambda,\mu|data)\, d\lambda\, d\mu.$$

The BE of θ ($\hat{\theta}_{B_2}$), when the absolute error loss function LF_2 is used, is just the median of the posterior distribution, *i.e.*

$$\hat{\theta}_{B_2} = Med_{posterior}(\theta|data).$$

The BE $\hat{\theta}_{B_3}$ of θ, under the LINEX loss function LF_3, can be obtained as

$$\hat{\theta}_{B3} = -\frac{1}{a^*} \ln\left[E_{posterior}(e^{-a^*\theta}|data)\right] = -\frac{1}{a^*} \ln\left[\int_0^\infty \int_0^\infty e^{-a^*\theta}\, \pi(\lambda,\mu|data)\, d\lambda\, d\mu\right].$$

Here, the Bayes point estimators $\hat{\theta}_{B_1}$, $\hat{\theta}_{B_2}$ and $\hat{\theta}_{B_3}$ cannot be obtained in closed forms. It can be easily checked that λ can be generated directly using Eq. (12), while μ cannot be generated directly from Eq. (14). For this, we implement the Metropolis-Hastings (M-H) method (see Metropolis, et al. (1953)) with normal proposal distribution to generate random values of μ from Eq. (14). The MLEs of λ and μ can be considered as initial values. We can apply this method of generation, M times, to obtain $\{(\lambda_i,\mu_i); i = 1,...,M\}$. We use these MCMC samples to obtain the Bayes estimates of $\theta = g(\lambda,\mu)$ and the corresponding credible intervals. The M-H algorithm proceeds as follows.

M-H algorithm for prediction problem:

1. Start with an initial values $(\lambda^{(0)},\mu^{(0)})$ and set $k = 1$;
2. Given $\mu^{(k-1)}$, generate μ from $\pi(\mu|\text{data})$ appeared in Eq. (14) with the $N(\mu^{(k-1)},S_\mu^2)$ proposal distribution, where S_μ^2 is the variance of μ. The values of μ can be generated as follows:
   a. Generate ζ_k from $\Omega(.|\mu^{(k-1)},S_\mu^2) = N(\mu^{(k-1)},S_\mu^2)$ and u from the uniform distribution $U(0,1)$
   b. If $u < \min(1,\nu)$ then let $\mu^{(k)} = \zeta_k$, else go to (a), where
   $$\nu = \frac{\pi(\zeta_k|\text{data})}{\pi(\mu^{(k-1)}|\text{data})} \frac{\Omega(\mu^{(k-1)}|\zeta_k,S_\mu^2)}{\Omega(\zeta_k|\mu^{(k-1)},S_\mu^2)}.$$
3. Given μ, generate λ from $Gamma(a+n,\ b+(x_{U(n)}-\mu)^2)$;
4. Set $k = k+1$.
5. Repeat steps 2-4, M times.

The BE of $\theta = g(\lambda, \mu)$ under the squared error loss function LF_1 is obtained as

$$\widehat{\theta}_{B_1} = \frac{1}{M}\sum_{i=1}^{M} g(\lambda_i, \mu_i).$$

To obtain the BE of $\theta = g(\lambda, \mu)$, under the absolute error loss function LF_2, we compute $\theta_i = g(\lambda_i, \mu_i)$, $i = 1, 2, ..., M$ and order $\theta_1, \theta_2, ..., \theta_M$ as $\theta_{(1)}, \theta_{(2)}, ..., \theta_{(M)}$, then the BE of $\theta = g(\lambda, \mu)$ is $\widehat{\theta}_{B_2} = Median\{\theta_{(1)}, \theta_{(2)}, ..., \theta_{(M)}\}$.

We evaluate the Bayes estimator of $\theta = g(\alpha, \lambda)$ with respect to the LINEX loss function LF_3 with $a^* \neq 0$ as

$$\widehat{\theta}_{B_3} = -\frac{1}{a^*}\ln\left[\frac{1}{M}\sum_{i=1}^{M} e^{-a^* g(\lambda_i, \mu_i)}\right].$$

Obtain the posterior variance of $\theta = g(\lambda, \mu)$ as

$$\hat{V}ar(\theta|data) = \frac{1}{M}\sum_{i=1}^{M}(\theta_i - \hat{\theta}_{B_{1,2,or3}})^2$$

To compute the CI of $\theta = g(\lambda, \mu)$, we order $\theta_1, \theta_2, ..., \theta_M$ as $\theta_{(1)}, \theta_{(2)}, ..., \theta_{(M)}$. Then, $(1-\gamma)100\%$ symmetric CI of θ is given by $\left(\theta_{([\frac{M\gamma}{2}])}, \theta_{([\frac{M(1-\gamma)}{2}])}\right)$.

4. Bayesian prediction for future records and corresponding PIs

Here, we predict the future unseen records based on a sequence of observed records, under different loss functions LF_1, LF_2 and LF_3 with $a^* \neq 0$, when the one-sample prediction problem is used. Naturally, we can notice the prediction problems in many real life situations such as the prediction of extremes of rainfall, water levels and sea surface. In the past two decades, many improvements have been done to this field. The readers may refer to Ahsanullah (1980) and Nagaraja (1984). Al-Hussaini and Ahmad (2003) studied the Bayesian prediction interval for the future generalized order statistics.
Suppose that we can only notice the first m upper records $\underset{\sim}{x} = (x_{U(1)}, x_{U(2)}, ..., x_{U(m)})$. Our goal is to obtain the Bayes point prediction of unobserved records under different loss functions LF_1, LF_2 and LF_3, as well as to construct the Bayes predictive interval for the nth future upper record $X_{U(n)}$, where $1 \leq m < n$. The posterior predictive density of $X_{U(n)}$ at any point $y > x_{U(m)}$ is given by

$$f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha, \lambda) = E_{posterior}\left[f_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda, \mu)\right],$$

where $f_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda, \mu)$ is the conditional PDF of $X_{U(n)}$ given the records data $\underset{\sim}{x}$. Applying the Markovian property on the record values, then $f_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda, \mu) = f_{X_{U(n)}|x_{U(m)}}(y|\lambda, \mu)$ and the posterior predictive density of $X_{U(n)}$ at any point $y > x_{U(m)}$ is obtained as

$$\begin{aligned} f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda) &= E_{posterior}\left[f_{X_{U(n)}|x_{U(m)}}(y|\lambda,\mu)\right] \\ &= \int_0^\infty\int_0^\infty f_{X_{U(n)}|x_{U(m)}}(y|\lambda,\mu)\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda\,d\mu \\ &= \int_0^\infty\int_0^\infty \frac{[H(x_{U(n)})-H(x_{U(m)})]^{n-m-1}}{(n-m-1)!}\,\frac{f(x_{U(n)})}{1-F(x_{U(m)})}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda\,d\mu \end{aligned}$$

where $H(x) = -\ln(1-F(x))$. Using Eq's (1) and (2) and the binomial expansion, we have

$$\begin{aligned} f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu) &= \int_0^\infty\int_0^\infty \frac{2\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2} \\ &\quad\times (y-\mu)^{2(n-m-i-\frac{1}{2})}e^{-\lambda(y-\mu)^2}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu,\; y > x_{U(m)}. \end{aligned}$$

Under LF_1, the BP of $Y = X_{U(n)}$ can be evaluated as

$$\begin{aligned} X^{BP1}_{U(n)} &= E_{f^P}(Y|\underset{\sim}{x}) \\ &= \int_{x_{U(m)}}^\infty y\left[\int_0^\infty\int_0^\infty \frac{2\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2}\right. \\ &\qquad\left.\times\,(y-\mu)^{2(n-m-i-\frac{1}{2})}e^{-\lambda(y-\mu)^2}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu\right] dy. \\ &= \int_0^\infty\int_0^\infty \frac{e^{\lambda(x_{U(m)}-\mu)^2}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i} \\ &\times \left[\lambda^{i-\frac{1}{2}}\Gamma\left(n-m-i+\frac{1}{2},\lambda(x_{U(m)}-\mu)^2\right)+\mu\lambda^i\Gamma\left(n-m-i,\lambda(x_{U(m)}-\mu)^2\right)\right] \\ &\quad \pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu. \end{aligned}$$

Based on the MCMC samples $\{(\lambda_j,\mu_j);\ j=1,2,...,M\}$ obtained in Section 3, a simulation predictor $\hat{X}_{U(n)}^{BP_1}$ of $Y = X_{U(n)}$ can be computed as

$$\hat{X}_{U(n)}^{BP_1} = \frac{1}{M}\sum_{j=1}^{M}\frac{e^{\lambda_j(x_{U(m)}-\mu_j)^2}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu_j)^{2i}\,\lambda_j^{i-\frac{1}{2}}$$
$$\times\left[\Gamma\left(n-m-i+\frac{1}{2},\lambda_j(x_{U(m)}-\mu_j)^2\right)+\mu_j\lambda_j^i\Gamma\left(n-m-i,\lambda_j(x_{U(m)}-\mu_j)^2\right)\right]. \tag{15}$$

Usually, it is important to predict the first unseen record value $X_{U(m+1)}$, the simulation-based consistent predictor of the first unseen record value can be evaluated by submitting $n = m+1$ in Eq. (15) as

$$\hat{X}_{U(n)}^{BP_1} = \frac{1}{M}\sum_{j=1}^{M}e^{\lambda_j(x_{U(m)}-\mu_j)^2}\left[\lambda_j^{-\frac{1}{2}}\Gamma\left(\frac{3}{2},\lambda_j(x_{U(m)}-\mu_j)^2\right)+\mu_j\Gamma\left(1,\lambda_j(x_{U(m)}-\mu_j)^2\right)\right].$$

Under LF_2, the BP of $Y = X_{U(n)}$ is given by

$$X_{U(n)}^{BP2} = Med_{f^P}(Y|\underset{\sim}{x}),$$

which is obtained by solving the equation

$$\int_{x_{U(m)}}^{X_{U(n)}^{BP2}} f_{X_{U(n)}|\underset{\sim}{x}}^{P}(y|\lambda,\mu)\,dy = \frac{1}{2},$$

or the equation

$$\int_{X_{U(n)}^{BP2}}^{\infty} f_{X_{U(n)}|\underset{\sim}{x}}^{P}(y|\lambda,\mu)\,dy = \frac{1}{2}.$$

Which are equivalent to the simultaneous equations

$$\int_{X_{U(n)}^{BP2}}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\frac{2\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2}\right.$$
$$\left.\times\,(y-\mu)^{2(n-m-i-\frac{1}{2})}e^{-\lambda(y-\mu)^2}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda\,d\mu\right]dy = \frac{1}{2},$$

and

$$\left[\int_0^\infty\int_0^\infty \frac{1}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2}\right.$$
$$\left.\times\,\lambda^i\Gamma\left(n-m-i,\lambda(X_{U(n)}^{BP2}-\mu)^2\right)\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu\right]=\frac{1}{2}.$$

Based on the MCMC samples $\{(\lambda_j,\mu_j);\ j=1,2,...,M\}$ obtained in Section 3, a simulation predictor $\hat{X}_{U(n)}^{BP_2}$ of $Y=X_{U(n)}$ can be obtained by solving the equation, for $\hat{X}_{U(n)}^{BP_2}$

$$\frac{1}{M}\sum_{j=1}^{M}\left[\frac{1}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu_j)^{2i}\,e^{\lambda_j(x_{U(m)}-\mu_j)^2}\right.$$
$$\left.\times\,\lambda_j^i\Gamma\left(n-m-i,\lambda_j(\hat{X}_{U(n)}^{BP_2}-\mu_j)^2\right)\right]=\frac{1}{2}.$$

In the similar way of the BP under the square error loss function, the BP of $Y=X_{U(n)}$ under the the LINEX loss function LF_3 can be obtained as

$$\begin{aligned}
X_{U(n)}^{BP3} &= -\frac{1}{a^*}\ln\left[E_{f^P}(e^{-a^*Y}|\underset{\sim}{x})\right]\\
&= -\frac{1}{a^*}\ln\left[\int_{x_{U(m)}}^\infty e^{-a^*y}\int_0^\infty\int_0^\infty\frac{2\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\right.\\
&\quad\left.\times\, e^{\lambda(x_{U(m)}-\mu)^2}(y-\mu)^{2(n-m-i-\frac{1}{2})}e^{-\lambda(y-\mu)^2}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu\,dy\right]\\
&= -\frac{1}{a^*}\ln\left[\int_0^\infty\int_0^\infty\frac{\lambda^{n-m-1}e^{-a^*\mu+\frac{a^{*2}}{4\lambda}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\right.\\
&\quad\times e^{\lambda}(x_{U(m)}-\mu)^2\sum_{k=0}^{2(n-m-i-\frac{1}{2})}\binom{2(n-m-i-\frac{1}{2})}{k}(-1)^k(\frac{a^*}{2\lambda})^k\\
&\quad\left.\times\frac{\Gamma\left(n-m-i-k,\lambda((x_{U(m)}-\mu)+\frac{a*}{2\lambda})\right)}{\lambda^{n-m-i-k-1}}\,\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda d\mu\right].
\end{aligned}\tag{16}$$

Based on the MCMC samples $\{(\lambda_j,\mu_j);\ j=1,2,...,M\}$, a simulation predictor $\hat{X}^{BP_3}_{U(n)}$ of $Y=X_{U(n)}$ can be obtained as

$$\begin{aligned}
\hat{X}^{BP_3}_{U(n)} \quad = \quad & -\frac{1}{a^*}\ln\left[\frac{1}{M}\sum_{j=1}^{M}\frac{\lambda_j^{n-m-1}e^{-a^*\mu_j+\frac{a^{*2}}{4\lambda_j}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu_j)^{2i}\right.\\
& \times e^{\lambda_j}(x_{U(m)}-\mu_j)^2\sum_{k=0}^{2(n-m-i-\frac{1}{2})}\binom{2(n-m-i-\frac{1}{2})}{k}(-1)^k(\frac{a^*}{2\lambda_j})^k\\
& \left.\times\frac{\Gamma\Big(n-m-i-k,\lambda_j((x_{U(m)}-\mu_j)+\frac{a*}{2\lambda_j})\Big)}{\lambda_j^{n-m-i-k-1}}\right]. \qquad (17)
\end{aligned}$$

The method for obtaining prediction intervals for the nth record value $Y=X_{U(n)}$, $1\leq m<n$ under different loss functions depends on the predictive survival function of $Y=X_{U(n)}$ at any point $y>x_{U(m)}$, which is defined as follows:

$$S^P_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu) \quad = \quad E_{posterior}\Big(S_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu)\Big),$$

where $S_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu)$ is the survival function of $Y=X_{U(n)}$. Based on the Markovian property of record values, we have

$$\begin{aligned}
S_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu) \quad &= \quad S_{X_{U(n)}|x_{U(m)}}(y|\lambda,\mu)\\
&= \int_y^\infty f_{X_{U(n)}|x_{U(m)}}(z|\lambda,\mu)\\
&= \frac{2\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2}\\
&\quad\times\int_y^\infty(z-\mu)^{2(n-m-i-\frac{1}{2})}e^{-\lambda(z-\mu)^2}\,dz\\
&= \frac{1}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i}\,e^{\lambda(x_{U(m)}-\mu)^2}\\
&\quad\times\frac{\Gamma\big(n-m-i,\lambda(y-\mu)^2\big)}{\lambda^{-i}}
\end{aligned} \qquad (18)$$

The predictive survival function of $Y = X_{U(n)}$ at any point $y > x_{U(m)}$, is then

$$S^P_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda,\mu) = \int_0^\infty\int_0^\infty \left[\frac{1}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu)^{2i} \times e^{\lambda(x_{U(m)}-\mu)^2}\frac{\Gamma\left(n-m-i,\lambda(y-\mu)^2\right)}{\lambda^{-i}}\right]\pi(\lambda,\mu|\underset{\sim}{x})\,d\lambda\,d\mu. \tag{19}$$

Eq. (19) cannot be evaluated analytically. Based on the MCMC samples $\{(\lambda_j,\mu_j); j=1,...,M\}$ and under the square error loss function LF_1, the estimate of the predictive survival function for $X_{U(n)}$ can be written as

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = \frac{1}{M}\sum_{j=1}^{M}\left[\frac{1}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^i\,(x_{U(m)}-\mu_j)^{2i}\,e^{\lambda_j(x_{U(m)}-\mu_j)^2} \times \frac{\Gamma\left(n-m-i,\lambda_j(y-\mu_j)^2\right)}{\lambda_j^{-i}}\right].$$

Under the absolute error loss function LF_2, and based on the MCMC samples $\{(\lambda_i,\mu_i); i=1,...,M\}$, we find the estimate predictive survival function of $X_{U(n)}$ as follows: Evaluate Eq. (18) for each $\{(\lambda_i,\mu_i), i=1,...,M\}$ to get S_1 , S_2 , ... , S_M, where $S_i = S_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda_i,\mu_i)$. Order S_1 , S_2 , ... , S_M to get $S_{(1)} < S_{(2)} < ... < S_{(M)}$, then the estimate predictive survival function of $X_{U(n)}$ is

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = Median\left[S_{(1)}\,,\,S_{(2)}\,,\,...\,,\,S_{(M)}\right]$$

Under the LINEX loss function LF_3, the estimate predictive survival function of $X_{U(n)}$ is obtained as

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = -\frac{1}{a^*}\ln\left[\frac{1}{M}\sum_{j=1}^{M} e^{-a^* S_{X_{U(n)}|\underset{\sim}{x}}(y|\lambda_j,\mu_j)}\right].$$

Consider

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(L) = 1-\frac{\gamma}{2}, \tag{20}$$

and

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(U) = \frac{\gamma}{2}. \tag{21}$$

Solving the non-linear equations (20) and (21) for L and U under different loss functions

LF_1, LF_2 and LF_3, give the $(1-\gamma)100\%$ prediction interval for $X_{U(n)}$, $n > m$. We need to apply a suitable numerical technique to solve these non-linear equations as they cannot be solved analytically.

5. Simulation and data analysis

Here, we conduct a simulation study to examine the behaviour of the MLEs and BEs as well as BPs that developed in the previous sections under different loss functions based on record data and study a real life examples with the Rayleigh fitting distribution. All computations are performed using Mathematica 11 software.

5.1. Simulation study

In this simulation, the values of the Rayleigh parameters are considered as $\lambda = 2$, $\mu = 1$ to generate record data from $Ra(\lambda,\mu)$. The first n observed records are generated by using the transformation:

$$X_{U(k)} = \left(\frac{\sum_{i=1}^{k} e(i)}{\mu}\right)^{\frac{1}{\lambda}}, k = 1,2,...,n,$$

where $\{e(i), i \geq 0\}$ is a sequence of *iid* $Exp(1)$, [see Arnold et al. (1998), p.20]. For the Rayleigh parameters λ and μ, we have computed the BEs, under the three different loss functions; LF_1, LF_2 and LF_3 for some values of a^* $(0.1, 1.0, 5)$. To compute the different BEs we have assumed $\pi_1(\lambda)$, the prior of λ, has gamma density function with the shape and scale parameters c and d, respectively. Regarding the computations of BEs, we consider two types of prior for both λ and μ; Prior 0: the non-informative prior $(i.e.\ a = b = c = d = 0)$ and Prior 1: the informative prior $(i.e.\ a = b = 1, c = 2, d = 1)$. We have computed the mean squared errors (MSEs) for BEs and MLEs based on 1000 replications to compare their performances under different number of observed records. The CIs for the Rayleigh parameters λ and μ are also computed. In the prediction problem and based on observed sequences of record data, we compute the point predictors and 95% PIs for the future n^{th} record $X_{U(n)}$ for Prior 1 under the suggested loss functions; LF_1, LF_2 and LF_3 and for the values of a^* $(0.1, 1.0, 5.0)$. The prediction computations are conducted for the following cases of sample sizes $n = 3, n = 5$, and $n = 7$.

In Table 1, we present the MLEs as well as the BEs of the scale and location parameters of the Rayleigh distribution λ and μ, under the different loss functions used in this paper, when Prior 0 is used. Also, we present the MSEs of MLEs and BEs for the scale and location parameters λ and μ. MCMC samples are used to compute the MSEs based on $M = 1000$ replications. In Table 2, we present the BEs of λ and μ, under the different loss functions, when Prior 1 is used. In Table 3, we show numerical comparisons between the average lengths of the credible intervals of λ and μ when Prior 0 and 1 are used for all of the considered cases.

Table 1: MLEs and BEs (Bayes Estimates) with respect to different loss functions when Prior 0 is used, for $\lambda = 2$ and $\mu = 1$.

Cases		MLE	Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
$n = 3$	λ	3.5966	1.9841	1.9761	1.9784	1.9738	1.9537
		(0.9098)	(0.0206)	(0.0208)	(0.0206)	(0.0207)	(0.0215)
	μ	1.2771	0.9165	0.9069	0.9069	0.8991	0.8668
		(0.0210)	(0.0163)	(0.0170)	(0.0164)	(0.0166)	(0.0191)
$n = 5$	λ	2.1131	1.9695	1.9566	1.9639	1.9594	1.9400
		(0.0971)	(0.0202)	(0.0206)	(0.0203)	(0.0203)	(0.0211)
	μ	1.2794	0.9488	0.9496	0.9389	0.9307	0.8953
		(0.0186)	(0.0169)	(0.0178)	(0.0170)	(0.0173)	(0.0203)
$n = 7$	λ	1.5667	1.9402	1.9160	1.9351	1.9310	1.9140
		(0.0521)	(0.0183)	(0.0190)	(0.0183)	(0.0184)	(0.0190)
	μ	1.2753	0.9629	0.9652	0.9530	0.9448	0.9089
		(0.0195)	(0.0171)	(0.0180)	(0.0172)	(0.0175)	(0.0205)

Note: The first entry represents the average estimate and the second entry is the MSE.

Table 2: BEs with respect to different loss functions when Prior 1 is used, for $\lambda = 2$ and $\mu = 1$.

Cases		Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
$n = 3$	λ	1.9775	1.9687	1.9719	1.9674	1.9479
		(0.0202)	(0.0204)	(0.0202)	(0.0203)	(0.0210)
	μ	0.9521	0.9512	0.9428	0.9352	0.9020
		(0.0161)	(0.0170)	(0.0162)	(0.0164)	(0.0190)
$n = 5$	λ	1.9620	1.9464	1.9567	1.9523	1.9339
		(0.0194)	(0.0198)	(0.0195)	(0.0195)	(0.0202)
	μ	0.9614	0.9695	0.9526	0.9452	0.9123
		(0.0153)	(0.0159)	(0.0154)	(0.0156)	(0.0180)
$n = 7$	λ	1.9352	1.9092	1.9302	1.9262	1.9097
		(0.0179)	(0.0188)	(0.0180)	(0.0180)	(0.019)
	μ	0.9853	0.9879	0.9783	0.9726	0.9468
		(0.0123)	(0.0131)	(0.0124)	(0.0125)	(0.0141)

Table 3: Average CI lengths (AL) and coverage percentage (CP).

Cases		Prior 0		Prior 1	
		AL	CP	AL	CP
$n = 3$	λ	0.7468	0.96	0.7249	0.95
	μ	0.6377	0.96	0.5697	0.96
$n = 5$	λ	0.7265	0.94	0.6789	0.93
	μ	0.6343	0.95	0.5628	0.93
$n = 7$	λ	0.6972	0.95	0.6610	0.95
	μ	0.6273	0.96	0.5585	0.92

From Tables 1 and 2, it is clear that as n increases the performances of MLEs of λ and μ become better in terms of the MSEs. Also, we observe that the Bayes estimates of λ and μ obtained by using Prior 0 and with respect to different loss functions LF_1, LF_2 and LF_3, are quite close to each other and are much better than the MLEs of λ and μ in terms of the MSEs for all considered cases. It can also noticed that the Bayes estimators of λ and μ obtained by using Prior 1 (informative prior) are much better than the Bayes estimators of λ and μ obtained by using Prior 0 (non-informative prior) in terms of the MSEs in most of the considered cases. Moreover, it is worth noting here that the BEs based on the squared error loss function (LF_1) are much better than other BEs that depends on other loss functions based on the reported MSEs values. In Table 3, we observe that the average lengths of the credible intervals for λ and μ, when Prior 1 is used, become smaller as expected, and decrease as n increases. For both Prior 0 and 1, the simulated probabilities for 0.95 are quite close to 0.95.

In Table 4, we present the point predictors and the corresponding 95% PIs for the future n^{th} record $X_{U(n)}, 1 \leq m < n$, based on set of observed records of size m, for all considered cases and for all used loss functions LF_1, LF_2 and LF_3 with many choices of $a^* : 0.1, 1.0, 5.0$. The simulated point predictors and 95% PIs are computed based on MCMC samples $\{(\lambda_i, \mu_i), i = 1, 2, ..., M\}$ and $M = 1000$. In this table the first three future n^{th} records after the last observed record are computed. It is observed from Table 4 that the predicted values for the future records $X_{U(n)}$ (unobserved record) under different loss functions, are quite close to each other and fall in their corresponding 95% PIs. It can be also noticed that the PIs computed based on LINEX loss function (LF_3) when $a^* = 0.5$ are better than other PIs in terms of the length of the PIs reported in the table. Also, as expected, the PIs lengths increase as n increases for given values of m.

5.2. Data analysis

Example 1 (real data):

In this example, we analyze the data of thirty successive March precipitation (in inches) observations. These data are presented in Hinkley (1977), pp. 67-69. The data set is:

```
0.77  1.74  0.81  1.20  1.95  1.20  0.47  1.43  3.37
2.2   3     3.09  1.51  2.1   0.52  1.62  1.31  0.32
0.59  0.81  2.81  1.87  1.18  1.35  4.75  2.48  0.96
1.89  0.90  2.05
```

The Kolmogorov-Smirnov (KS) distance is found to be 0.0770 and the corresponding p-value is 0.9900. Therefore, KS indicates that the Rayleigh distribution can be used to analyze these data. Moreover, graphical tools of empirical and theoretical CDFs and the Q-Q plot given in Figures 1 and 2, give a very good evidence that the Rayleigh distribution fits the data very well. From these data, we have $n = 5$ observed upper record values: 0.77, 1.74, 1.95, 3.37 and 4.75.

Table 4: Point predictors and the corresponding PIs for future records $X_{U(n)}, 1 \leq m < n$ based on some observed records.

Cases	$X_{U(n)}$	Loss function	Predicted	95% PIs
$m=3$	$X_{U(4)}$	LF_1	2.4043	(2.2988, 3.2135)
		LF_2	2.4699	(2.3009, 3.4610)
		$LF_3(a^*=0.1)$	2.5331	(2.2987, 3.0233)
		$LF_3(a^*=1.0)$	2.5241	(2.2987, 2.9080)
		$LF_3(a^*=5.0)$	2.4921	(2.2987, 2.7335)
	$X_{U(5)}$	LF_1	2.6253	(2.3511, 3.6133)
		LF_2	2.6903	(2.3628, 3.5598)
		$LF_3(a^*=0.1)$	2.7460	(2.3509, 3.2943)
		$LF_3(a^*=1.0)$	2.7308	(2.3507, 3.1122)
		$LF_3(a^*=5.0)$	2.6749	(2.3500, 2.8980)
	$X_{U(6)}$	LF_1	2.8236	(2.4329, 3.9355)
		LF_2	2.8870	(2.4643, 3.8435)
		$LF_3(a^*=0.1)$	2.9381	(2.4322, 3.5032)
		$LF_3(a^*=1.0)$	2.9181	(2.4317, 3.2669)
		$LF_3(a^*=5.0)$	2.8434	(2.4291, 3.0304)
$m=5$	$X_{U(6)}$	LF_1	3.1282	(3.0323, 4.0164)
		LF_2	3.3455	(3.0327, 3.7852)
		$LF_3(a^*=0.1)$	3.3799	(3.0321, 3.8923)
		$LF_3(a^*=1.0)$	3.3715	(3.0319, 3.8292)
		$LF_3(a^*=5.0)$	3.3378	(3.0311, 3.6714)
	$X_{U(7)}$	LF_1	3.3592	(3.1533, 4.3148)
		LF_2	3.5844	(3.2271, 4.3505)
		$LF_3(a^*=0.1)$	3.6084	(3.1516, 4.1179)
		$LF_3(a^*=1.0)$	3.5980	(3.1501, 4.0403)
		$LF_3(a^*=5.0)$	3.5558	(3.1432, 3.8750)
	$X_{U(8)}$	LF_1	3.5373	(3.2850, 4.5489)
		LF_2	3.7616	(3.4005, 4.5491)
		$LF_3(a^*=0.1)$	3.7851	(3.2822, 4.2852)
		$LF_3(a^*=1.0)$	3.7736	(3.2797, 4.1954)
		$LF_3(a^*=5.0)$	3.7269	(3.2680, 4.0173)
$m=7$	$X_{U(8)}$	LF_1	3.7850	(3.6146, 4.3022)
		LF_2	3.7377	(3.6166, 4.4392)
		$LF_3(a^*=0.1)$	3.7909	(3.6146, 4.2066)
		$LF_3(a^*=1.0)$	3.7872	(3.6146, 4.1380)
		$LF_3(a^*=5.0)$	3.7728	(3.6146, 4.0217)
	$X_{U(9)}$	LF_1	3.9596	(3.6527, 4.6313)
		LF_2	3.9096	(3.6504, 4.4205)
		$LF_3(a^*=0.1)$	3.9616	(3.6526, 4.4628)
		$LF_3(a^*=1.0)$	3.9547	(3.6526, 4.3459)
		$LF_3(a^*=5.0)$	3.9278	(3.6522, 4.1901)
	$X_{U(10)}$	LF_1	4.1248	(3.7153, 4.9056)
		LF_2	4.0737	(3.7120, 4.6262)
		$LF_3(a^*=0.1)$	4.1233	(3.7150, 4.6696)
		$LF_3(a^*=1.0)$	4.1137	(3.7147, 4.5100)
		$LF_3(a^*=5.0)$	4.0758	(3.7135, 4.3282)

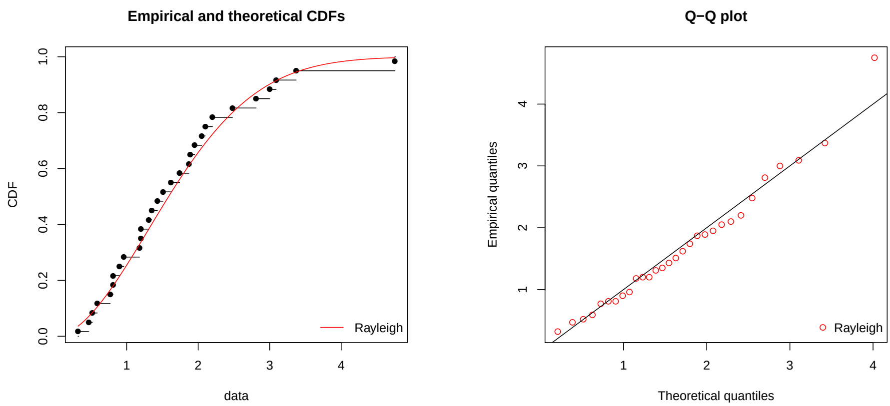

Figure 1: Empirical and fitted distribution functions and Q-Q Plots for data set of example 1.

Example 2 (real data):

In this example, we analyze the survival times in (days) of a group of size $m = 16$ lung cancer patients (Lawless [1982, p. 319]) were considered as follows:

6.96	9.30	6.96	7.24	9.30	4.90	8.42	6.05
10.18	6.82	8.58	7.77	11.94	11.25	12.94	12.94

From these data, we have $n = 5$ observed upper record values: 6.96, 9.30, 10.18, 11.94 and 12.94. Soliman and Al-Aboud (2008) showed that the Rayleigh distribution fits the observed record values well. Seo and Kim (2018) used this real example to apply an objective Bayesian method under the observed upper record values.

For the above mentioned examples, we compute the BEs based on different loss functions: LF_1, LF_2 and LF_3 and for the values a^* $(0.1, 1.0, 5.0)$. The results are presented in Tables 5 and 6. We can simply see from Tables 5 and 6 that all the estimates are quite close to each other. Furthermore, we obtain the 95% credible intervals for λ and μ, respectively for both examples. For example 1, the 95% CI are given by $(1.5457, 2.4826)$ and $(0.5693, 1.4614)$. For example 2, they are $(1.5473, 2.4828)$ and $(0.6092, 1.4683)$. It can be noticed that the BEs of λ and μ are falling in their credible intervals.

Also, we consider the prediction of the 6^{th}, 7^{th} and 8^{th} future records. The predicted values and the 95% PIs for the 6^{th}, 7^{th} and 8^{th} future records are presented in Tables 7 and 8, for examples 1 and 2, respectively. It is observed that all predicted values, under different loss functions, are all ordered and fall in their corresponding prediction intervals.

Table 5: BEs based on different loss functions for example 1.

	LF_1	LF_2	$LF_3(a^*=0.1)$	$LF_3(a^*=1.0)$	$LF_3(a^*=5.0)$
λ	2.0848	2.1239	2.0636	2.0455	1.9614
μ	1.0215	1.0263	0.9867	0.9568	0.8284

Table 6: BEs based on different loss functions for example 2.

	LF_1	LF_2	$LF_3(a^*=0.1)$	$LF_3(a^*=1.0)$	$LF_3(a^*=5.0)$
λ	2.0841	2.1229	2.0628	2.0446	1.9608
μ	1.0774	1.0899	1.0457	1.0177	0.8874

Table 7: Point predictors and PIs for the 6^{th}, 7^{th} and 8^{th} future records for example 1.

Number of observed records	$X_{U(n)}$	Loss function	Predicted values	95% PIs
$m=5$	$X_{U(6)}$	LF_1	4.8320	$(4.7548, 5.5207)$
		LF_2	4.8840	$(4.7559, 5.4676)$
		$LF_3(a^*=0.1)$	4.9453	$(4.7548, 5.3315)$
		$LF_3(a^*=0.5)$	4.9418	$(4.7548, 5.2064)$
		$LF_3(a^*=1.0)$	4.9278	$(4.7547, 5.0566)$
	$X_{U(7)}$	LF_1	4.9840	$(4.7928, 5.8867)$
		LF_2	5.0593	$(4.8058, 5.7628)$
		$LF_3(a^*=0.1)$	5.1235	$(4.7926, 5.5638)$
		$LF_3(a^*=0.5)$	5.1169	$(4.7925, 5.3605)$
		$LF_3(a^*=1.0)$	5.0908	$(4.7918, 5.1797)$
	$X_{U(8)}$	LF_1	5.1589	$(4.8531, 6.1875)$
		LF_2	5.2227	$(4.8893, 5.9981)$
		$LF_3(a^*=0.1)$	5.2881	$(4.8523, 5.7464)$
		$LF_3(a^*=0.5)$	5.2788	$(4.8518, 5.4798)$
		$LF_3(a^*=1.0)$	5.2418	$(4.8490, 5.2816)$

Table 8: Point predictors and PIs for the 6^{th}, 7^{th} and 8^{th} future records for example 2.

Number of observed records	$X_{U(n)}$	Loss function	Predicted values	95% PIs
$m = 5$	$X_{U(6)}$	LF_1	13.4808	$(12.9604, 15.6118)$
		LF_2	13.4851	$(12.9617, 15.6530)$
		$LF_3(a^* = 0.1)$	13.6791	$(12.9604, 15.5010)$
		$LF_3(a^* = 0.5)$	13.6638	$(12.9603, 15.4030)$
		$LF_3(a^* = 1.0)$	13.6036	$(12.9603, 15.0070)$
	$X_{U(7)}$	LF_1	14.1850	$(13.1308, 16.7771)$
		LF_2	14.2062	$(13.1448, 16.8060)$
		$LF_3(a^* = 0.1)$	14.3694	$(13.1307, 16.5900)$
		$LF_3(a^* = 0.5)$	14.3428	$(13.1306, 16.4170)$
		$LF_3(a^* = 1.0)$	14.2361	$(13.1301, 16.6830)$
	$X_{U(8)}$	LF_1	14.8853	$(13.4155, 17.7224)$
		LF_2	14.8836	$(13.4533, 17.7340)$
		$LF_3(a^* = 0.1)$	15.0188	$(13.4150, 17.4670)$
		$LF_3(a^* = 0.5)$	14.9836	$(13.4146, 17.2250)$
		$LF_3(a^* = 1.0)$	14.8406	$(13.4128, 16.6900)$

6. Conclusion

In this work, we have considered the problem of classical and Bayesian estimations of the Rayleigh record model. We have also developed a Bayesian approach to compute the point future records as well as their corresponding prediction intervals. For comparison purposes, we have employed Markov Chain Monte Carlo (MCMC) samples generated from the Rayleigh record model to compute the Bayesian estimators and the point predictors of the future data. Monte Carlo simulations are used to study the behaviour of the obtained methods. Also, two real data examples have been analyzed to illustrate the procedures developed in this article.

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