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Relationships for moments of the progressively Type-II right censored order statistics from the power Lomax distribution and the associated inference

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ABSTRACT

In this paper, we establish several recurrence relations between single and product moments of progressively Type-II right censored order statistics from the power Lomax distribution. The relations enable the computation of all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \dots, R_m), m \leq n$, in a simple recursive manner. The maximum likelihood approach is used for the estimation of the parameters and the reliability characteristic. A Monte Carlo simulation study has been conducted to compare the performance of the estimates for different censoring schemes.

Key words: progressively Type-II right censored order statistics, single moments, product moments, recurrence relations, power Lomax distribution, maximum likelihood estimation.
Mathematics Subject Classification: 62G30; 62G05

1. Introduction

The Lomax distribution, proposed by Lomax (1954) was introduced originally for modelling business data and has been widely applied in a variety of contexts. In lifetime models, it is considered as an important model and belongs to the family of decreasing failure rate. Bryson (1974) found that this distribution can be used as heavy tailed alternative to the exponential, Weibull and gamma distributions.

Many authors constructed generalizations of the Lomax distribution. For example, Ghitany et al. (2007) introduced the Marshall-Olkin extended Lomax distribution, Abdul-Moniem and Abdel-Hameed (2012) introduced the exponentiated Lomax distribution, Tahir et al. (2015) introduced the Weibull Lomax distribution, Al-Zahrani and Sagor (2014) studied the Poisson Lomax distribution. Recently, Tahir et al. (2016)

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and Afify et al. (2016) introduced the Gumbel-Lomax distribution and the Transmuted Weibull Lomax distribution, respectively, and studied their mathematical and statistical properties.

A new extension of the Lomax distribution was proposed by Rady et al. (2016) as three parameter power Lomax distribution, by considering the power transformation $X = T^{1/\beta}$, where the random variable $(r.v.)$ T follows the Lomax distribution with parameters α and λ . Then the distribution of $r.v.$ X with three parameters α, β and λ is referred to as "power Lomax distribution", where α and β are the shape parameters and λ is the scale parameter of the distribution.

The probability density function (p.d.f.) of $r.v.$ X following the power Lomax distribution is given as

$$f(x) = \frac{\alpha\beta}{\lambda} x^{\beta-1}\left(1 + \frac{x^\beta}{\lambda}\right)^{-(\alpha+1)}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.1}$$

The corresponding cumulative distribution function (c.d.f.) is given by

$$F(x) = 1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.2}$$

The reliability (survival) function $R(x)$ of the power Lomax distribution is given as

$$R(x) = \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha}, \quad x > 0, \alpha, \beta, \lambda > 0, \tag{1.3}$$

and the failure rate function (hazard function) of the power Lomax distribution is given by

$$h(x) = \frac{f(x)}{R(x)} = \frac{\alpha\beta x^{\beta-1}}{\lambda + x^\beta}, \quad x > 0, \alpha, \beta, \lambda > 0. \tag{1.4}$$

From Eqs. (1.1) and (1.2), one can observe that the characterizing differential equation for the power Lomax distribution is given as

$$\alpha\beta(1 - F(x)) = \left(x + \lambda x^{1-\beta}\right) f(x). \tag{1.5}$$

Note: For $\beta = 1$ in Eq. (1.1), the p.d.f. reduces to that of the Lomax distribution.

2. Progressively Type-II right censored order statistics

The progressive Type-II right censoring scheme is quite useful in reliability and life-testing experiments because it allows the experimenter for items to be removed before the termination of the experiment to save time and cost. The progressive censoring scheme and associated inferential procedures have been discussed by several authors including Aggarwala and Balakrishnan (1996, 1998), Balakrishnan and Aggarwala (2000), Cohen (1963, 1976, 1991), Cohen and Whitten (1988), Balakrishnan and Sandhu (1995), Athar et al. (2014), Saran and Pushkarna (2001, 2014), Saran and Pande (2012), Pushkarna et al. (2015) and Saran et al. (2018). The progressive censoring scheme can be described as follows:

Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables representing failure times of n identical units placed on a life-test. Under the progressively Type-II right censoring scheme, at the time of i^{th} failure ($i = 1,2,\dots,m,$ where $m \leq n$), R_i surviving items are removed at random from the experiment, where $R_1, R_2, \dots, R_m$ are fixed integers. In other words, if a censoring scheme $(R_1, R_2, \dots, R_m)$ is fixed such that immediately following the first failure, R_1 surviving items are removed from the experiment at random; immediately following the first failure after that point; i.e. after second observed failure, R_2 surviving items are removed from the experiment at random; this process continues until, at the m^{th} observed failure, R_m items are removed from the experiment.

Thus, in this type of sampling, m failures are observed and

$\sum_{i=1}^{m} R_i$ items are progressively censored so that $n = m + \sum_{i=1}^{m} R_i$. The withdrawal of

items may be seen as a model describing drop-outs of units due to failures, which have causes other than the specific one under study. In this sense, progressive censoring schemes are applied in clinical trials as well. The drop-outs of patients may be caused, e.g. by personal or ethical decisions, and they are regarded as random withdrawals.

Let $X_{1:m:n}^{(R_1,R_2,\dots,R_m)} < X_{2:m:n}^{(R_1,R_2,\dots,R_m)} < \dots < X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$, be the m ordered observed failure times in a sample of size n from the Power Lomax distribution as defined by (1.1), under the progressively Type-II right censoring scheme $(R_1, R_2, \dots, R_m)$, $m \leq n$.

Then, the joint p.d.f. of $X_{1:m:n}^{(R_1,R_2,\dots,R_m)}, X_{2:m:n}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$ is given by (Balakrishnan and Sandhu (1995))

$$f_{1,2,\dots,m:m:n}(x_1, x_2, \dots, x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$

$$0 < x_1 < x_2 < \dots < x_m < \infty, \qquad (2.1)$$

where

$$A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1), \qquad (2.2)$$

$f(x)$ and $F(x)$ are given by (1.1) and (1.2), respectively. Here, note that all the factors in $A(n, m-1)$ are positive integers. Also, it may be observed that the different factors in
$A(n, m-1)$ represent the number of units still on test immediately preceding the $1^{st}, 2^{nd}, \ldots, m^{th}$ observed failures, respectively.

Similarly, for convenience in notation, let us define for $q = 0,1, \ldots, (p-1)$,

$$A(p,q) = p(p - R_1 - 1)(p - R_1 - R_2 - 2) \ldots \big(p - R_1 - R_2 - \cdots - R_q - q\big),$$

with all the factors being positive integers.

We shall denote the k^{th} single moment of the i^{th} progressively Type-II right censored order statistics, from (2.1), as

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)(k)} = E\left[X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}\right]^k$$

$$= A(n, m-1) \underset{0<x_1<\ldots<x_m<\infty}{\int\int \ldots \int} x_i^k \prod_{t=1}^{m} f(x_t)[1 - F(x_t)]^{R_t}\, dx_t\,,\ 1 \le i \le m \le n\,, k \ge 0, \quad (2.3)$$

and the $(r,s)^{th}$ product moment of the i^{th} and j^{th} progressively Type-II right censored order statistics from (2.1), as

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)(r,s)} = E\left[\left\{X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^r \left\{X_{j:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^s\right]$$

$$= A(n, m-1) \underset{0<x_1<\ldots<x_m<\infty}{\int\int \ldots \int} x_i^r x_j^s \prod_{t=1}^{m} f(x_t)[1 - F(x_t)]^{R_t}\, dx_t;$$

$$1 \le i < j \le m \le n, r, s \ge 0, \quad (2.4)$$

where $A(n, m-1)$ is defined before.

In Sections 3 and 4, utilizing the characterizing differential Eq. (1.5), we have derived recurrence relations for the single and the product moments of progressively Type-II right censored order statistics from the power Lomax distribution. These relations along with the recursive algorithm presented in Section 5 would enable one to compute all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \ldots, R_m)$, $m \le n$, in a simple recursive manner. In Section 6, for the estimation of the parameters and the reliability characteristics, maximum likelihood approach is used. In Section 7, Monte Carlo simulation study is conducted to compare the performance of the estimates for different censoring schemes.

3. Recurrence relations for single moments

In this section, we shall exploit the relation (1.5) to establish recurrence relations for the single moments of progressively Type-II right censored order statistics from the power Lomax distribution. The results are presented in the form of the following theorems.

Theorem 3.1: For $2 \le m \le n$ and for $k \ge 0$,

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}}\left[1-\frac{\alpha\beta}{(k+\beta)}(R_1+1)\right] = \frac{\alpha\beta}{(k+\beta)}(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+\beta)}} - \lambda\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}, \quad (3.1)$$

and for $m=1, n=1,2,\dots$ and $k \ge 0$,

$$\mu_{1:1:n}^{(n-1)^{(k+\beta)}}\left(\frac{n\alpha\beta}{k+\beta}-1\right) = \left[\lambda\mu_{1:1:n}^{(n-1)^{(k)}}\right]. \quad (3.2)$$

Proof: From (2.3), we have

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}} + \lambda\,\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}$$

$$= A(n,m-1)\int\int_{0<x_2<x_3\dots<x_m<\infty}\dots\int\left\{\int_0^{x_2}\left(x_1^{k+\beta}+\lambda x_1^k\right)f(x_1)[1-F(x_1)]^{R_1}dx_1\right\}$$

$$\times f(x_2)[1-F(x_2)]^{R_2}\dots f(x_m)[1-F(x_m)]^{R_m}dx_2\,dx_3\dots dx_m$$

$$= A(n,m-1)\int\int_{0<x_2<x_3<\dots<x_m<\infty}\dots\int I(x_2)\prod_{t=2}^{m}f(x_t)[1-F(x_t)]^{R_t}\,dx_t\,, \quad (3.3)$$

where

$$I(x_2) = \int_0^{x_2}\left(x_1^{k+\beta}+\lambda x_1^k\right)f(x_1)[1-F(x_1)]^{R_1}dx_1$$

$$= \int_0^{x_2} x_1^{k+\beta-1}\left(x_1+\lambda x_1^{1-\beta}\right)f(x_1)[1-F(x_1)]^{R_1}dx_1.$$

Making use of the relation (1.5), we have

$$I(x_2) = \alpha\beta\int_0^{x_2} x_1^{k+\beta-1}[1-F(x_1)]^{R_1+1}dx_1.$$

Upon integrating by parts by treating $x_1^{k+\beta-1}$ for integration and $[1-F(x_1)]^{R_1+1}$ for differentiation we have

$$I(x_2) = \frac{\alpha\beta}{(k+\beta)}\left[x_2^{k+\beta}[1-F(x_2)]^{R_1+1} + (R_1+1)\int_0^{x_2}x_1^{k+\beta}[1-F(x_1)]^{R_1}f(x_1)dx_1\right]. \quad (3.4)$$

Substituting the resultant expression of $I(x_2)$ from (3.4) in (3.3), we get

$$\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}} + \lambda\, \mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = \frac{\alpha\beta}{(k+\beta)}\Big[(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+\beta)}} + (R_1+1)\mu_{1:m:n}^{(R_1,R_2,R_3,\dots,R_m)^{(k+\beta)}}\Big],$$

which upon rearrangement yields the relation in (3.1).

Next, for m=1, n=1,2,… and k≥0,

$$\begin{aligned}\mu_{1:1:n}^{(R_1)^{(k+\beta)}} + \lambda\, \mu_{1:1:n}^{(R_1)^{(k)}} &= A(n,0)\int_0^\infty \left(x_1^{k+\beta} + \lambda x_1^k\right) f(x_1)[1-F(x_1)]^{R_1} dx_1 \\ &= n\int_0^\infty x_1^{k+\beta-1}\left(x_1 + \lambda x_1^{1-\beta}\right) f(x_1)[1-F(x_1)]^{R_1} dx_1 \\ &= n\alpha\beta\int_0^\infty x_1^{k+\beta-1}[1-F(x_1)]^{R_1+1} dx_1 \\ &= n\frac{\alpha\beta}{(k+\beta)}\mu_{1:1:n}^{(n-1)^{(k+\beta)}},\end{aligned}$$

which, upon rearrangements, yields the relation in (3.2).

Remark 3.1: It may be noted that the first progressively Type-II right censored order statistic

$X_{1:m:n}^{(R_1,R_2,\dots,R_m)}$ is the same as the first usual order statistic from a sample of size n, regardless of the censoring scheme employed. This is because no censoring has taken place before this time.

Theorem 3.2: For $2 \le i \le m-1, m \le n$ and $k \ge 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}}\left[1 - \frac{\alpha\beta(R_i+1)}{(k+\beta)}\right] &= \frac{\alpha\beta}{(k+\beta)}\Big[(n-R_1-R_2-\dots-R_i \\ &- i)\mu_{i:m-1:n}^{(R_1,R_2,\dots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\dots,R_m)^{(k+\beta)}} \\ &- (n-R_1-R_2-\dots-R_{i-1}-i \\ &+ 1)\mu_{i-1:m-1:n}^{(R_1,R_2,\dots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\dots,R_m)^{(k+\beta)}}\Big] - \lambda\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}}.\end{aligned} \quad (3.5)$$

Proof: From (2.3), we have

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k+\beta)}} + \lambda\, \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = A(n,m-1) \underset{0<x_1<\dots<x_{i-1}<x_{i+1}<\dots<x_m}{\int \dots \int \dots \int} J(x_{i-1},x_{i+1}) \prod_{\substack{t=1\\t\ne i}}^m f(x_t)[1-F(x_t)]^{R_t}\, dx_t, \quad (3.6)$$

where

$$J(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta-1}\left(x_i + \lambda x_i^{1-\beta}\right) f(x_i)[1-F(x_i)]^{R_i} dx_i . \tag{3.7}$$

Making use of the relation in (1.5), we have

$$J(x_{i-1},x_{i+1}) = \alpha\beta \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta-1}[1-F(x_i)]^{R_i+1} dx_i . \tag{3.8}$$

Integrating by parts by treating $x_i^{k+\beta-1}$ for integration and $[1-F(x_i)]^{R_i+1}$ for differentiation, we have

$$J(x_{i-1},x_{i+1}) = \frac{\alpha\beta}{(k+\beta)}\Big[x_{i+1}^{k+\beta}[1-F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+\beta}[1-F(x_{i-1})]^{R_i+1} + (R_i+1)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta}[1-F(x_i)]^{R_i} f(x_i) dx_i \Big]. \tag{3.9}$$

Substituting the resultant expression of $J(x_{i-1},x_{i+1})$ from (3.9) in (3.6) and simplifying, leads to (3.5).

Likewise, the following recurrence relation can also be established.

Theorem 3.3: For $m \leq n$ and $k \geq 0$,

$$\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)(k+\beta)}\left[\frac{\alpha\beta(R_m+1)}{(k+\beta)} - 1\right] = \frac{\alpha\beta}{(k+\beta)}\left[(n-R_1-R_2-\dots-R_{m-1}-m+1)\times\mu_{m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},R_{m-1}+R_m+1)(k+\beta)}\right] + \lambda\mu_{m:m:n}^{(R_1,R_2,\dots,R_m)(k)}. \tag{3.10}$$

4. Recurrence relations for product moments

In this section, we shall exploit the relation (1.5) to establish recurrence relations for the product moments, defined in Eq. (2.4), of progressively Type-II right censored order statistics from the power Lomax distribution. The results are presented in the form of the following theorems.

Theorem 4.1: For $1 \leq i < j < m, m \leq n,$ and $r,s \geq 0$,

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)(r,s+\beta)}\left[1-\frac{\alpha\beta(R_j+1)}{(s+\beta)}\right] = \frac{\alpha\beta}{(s+\beta)}\Big[(n-R_1-R_2-\dots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\dots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\dots,R_m)(r,s+\beta)} - \left(n-R_1-R_2-\dots-R_{j-1}-(j-1)\right) \times \mu_{i,j-1:m-1:n}^{(R_1,R_2,\dots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\dots,R_m)(r,s+\beta)}\Big] - \lambda\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)(r,s)}. \tag{4.1}$$

Proof: From (2.4), we have

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+\beta)}} + \lambda\, \mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}$$

$$= A(n,m-1)\underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \ldots \int \ldots \int} x_i^r J(x_{j-1},x_{j+1}) \prod_{\substack{t=1\\ t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t, \tag{4.2}$$

where

$$J(x_{j-1},x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta-1}\left(x_j + \lambda x_j^{1-\beta}\right) f(x_j)[1-F(x_j)]^{R_j} dx_j\,. \tag{4.3}$$

Using (3.9), we get

$$J(x_{j-1},x_{j+1}) = \frac{\alpha\beta}{(s+\beta)}\begin{bmatrix} x_{j+1}^{s+\beta}[1-F(x_{j+1})]^{R_j+1} - x_{j-1}^{s+\beta}[1-F(x_{j-1})]^{R_j+1} \\ +(R_j+1)\displaystyle\int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta}[1-F(x_j)]^{R_j} f(x_j) dx_j \end{bmatrix}. \tag{4.4}$$

Substituting the resultant expression for $J(x_{j-1},x_{j+1})$ from (4.4) in (4.2) and simplifying, on using (2.4), we get

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+\beta)}} + \lambda\, \mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}$$

$$= \frac{\alpha\beta A(n,m-1)}{(s+\beta)}\underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \ldots \int \ldots \int} x_i^r x_{j+1}^{s+\beta}[1-F(x_{j+1})]^{R_j+1} \prod_{\substack{t=1\\ t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$

$$- \frac{\alpha\beta A(n,m-1)}{(s+\beta)}\underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \ldots \int \ldots \int} x_i^r x_{j-1}^{s+\beta}[1-F(x_{j-1})]^{R_j+1} \prod_{\substack{t=1\\ t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$

$$+ \frac{\alpha\beta(R_j+1)}{(s+\beta)}\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+\beta)}}$$

$$= \frac{\alpha\beta A(n,m-1)}{(s+\beta)}$$

$$\times \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int \ldots \int \ldots \int} x_i^r x_{j+1}^{s+\beta}[1-F(x_{j+1})]^{R_j+R_{j+1}+1} f(x_{j+1}) dx_{j+1} \prod_{\substack{t=1\\ t\neq j,j+1}}^{m} f(x_t)[1-F(x_t)]^{R_t}\, dx_t$$

$$-\frac{\alpha\beta A(n,m-1)}{(s+\beta)}$$
$$\times \int \dots \int \dots \int x_i^r x_{j-1}^{s+\beta}[1-F(x_{j-1})]^{R_{j-1}+R_{j}+1} f(x_{j-1}) dx_{j+1} \prod_{\substack{t=1 \\ t\neq j-1,j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t$$
$$0< x_1 < \dots < x_{j-1} < x_{j+1} < \dots < x_m < \infty$$

$$+\frac{\alpha\beta(R_j+1)}{(s+\beta)} \mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)(r,s+\beta)}$$
$$= \frac{\alpha\beta}{(s+\beta)}\Big[(n-R_1-R_2-\dots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\dots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\dots,R_m)(r,s+\beta)}$$
$$-\Big(n-R_1-R_2-\dots-R_{j-1}$$
$$-(j-1)\Big)\mu_{i,j-1:m-1:n}^{(R_1,R_2,\dots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\dots,R_m)(r,s+\beta)}$$
$$+(R_j+1)\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)(r,s+\beta)}\Big],$$

which on rearranging the terms leads to (4.1).

Theorem 4.2: For $1 \leq i \leq m-1$ and $m \leq n$ and $r,s \geq 0$,

$$\mu_{i,m:m:n}^{(R_1,R_2,\dots,R_m)(r,s+\beta)}\left[\frac{\alpha\beta(R_m+1)}{(s+\beta)}-1\right]$$
$$= \frac{\alpha\beta}{(s+\beta)}\left[(n-R_1-R_2-\dots-R_{m-1}-(m-1))\mu_{i,m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},R_{m-1}+R_m+1)(r,s+\beta)}\right]$$
$$+\lambda\mu_{i,m:m:n}^{(R_1,R_2,\dots,R_m)(r,s)}. \tag{4.5}$$

Proof: The relation in (4.5) may be proved by following exactly the same steps as those used in proving Theorem 4.1.

Remark 4.1: It may be noted that Theorem 4.1 holds even for $j=i+1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,R_2,\dots,R_m)(r,s)} = \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)(r+s)}$.

Remark 4.2: For the special case $R_1 = R_2 = \dots = R_m = 0$ so that $m=n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, whose single moments are denoted by $\mu_{i:n}^{(k)}$ for $1 \leq i \leq n$ and product moments are denoted by $\mu_{i,j:n}^{(r,s)}$ for $1 \leq i < j \leq n$, the recurrence relations established in Sections 3 and 4 reduce to that of usual order statistics from Power Lomax distribution.

5. Recursive computational algorithm

Thomas and Wilson (1972) gave a computational method for obtaining single and product moments of progressively Type-II right censored order statistics from an arbitrary continuous distribution through a mixture form that expresses them in terms of those of the usual order statistics from a sample of size n. Utilizing the knowledge of recurrence relations obtained in Sections 3 and 4 in a systematic manner, along with the mixture formula for missing moments, one can evaluate the moments of progressively Type-II right censored order statistics from the power Lomax distribution for all sample sizes and all censoring schemes $(R_1, R_2, \dots, R_m)$ in a simple recursive way. The same has been demonstrated in the Sub- Sections 5.2 and 5.3.
First, we will derive the exact explicit forms for the single and product moments of order statistics from a given random sample $X_1, X_2, \dots, X_n$ from the power Lomax distribution.

5.1. Exact expressions for single and product moments of order statistics from power Lomax distribution

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the power Lomax distribution defined in (1.1) and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Then the probability density function (p.d.f.) of $X_{i:n}$ $(1 \le i \le n)$ is given by:

$$f_{i:n}(x) = C_{i:n}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x), \quad 0 < x < \infty, \tag{5.1}$$

and the joint density function of $X_{i:n}$ and $X_{j:n}(1 \le i < j \le n)$ is given by

$$f_{i,j:n}(x,y) = C_{i,j:n}[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}\,[1-F(y)]^{n-j}f(y)f(x), \quad 0 < x < y < \infty, \tag{5.2}$$

where $f(x)$ and $F(x)$ are given by (1.1) and (1.2), respectively, and

$$C_{i:n} = \frac{n!}{(i-1)!\,(n-i)!} \text{ and } C_{i,j:n} = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}.$$

Then, the single moments of order statistics $X_{i:n}(1 \le i \le n)$ are given by

$$\mu_{i:n}^{(k)} = E\left(X_{i:n}^{k}\right) = \int_0^{\infty} x^k f_{i:n}(x)dx, \quad k = 1,2, \dots . \tag{5.3}$$

Similarly, the product moments of $X_{i:n}$ and $X_{j:n}(1 \le i < j \le n)$ are given by

$$\mu_{i,j:n}^{(r,s)} = E\left(X_{i:n}^{r}X_{j:n}^{s}\right) = \int_0^{\infty}\int_x^{\infty} x^r y^s f_{i,j:n}(x,y)dydx, \quad r,s = 1,2, \dots . \tag{5.4}$$

Theorem 5.1: For the power Lomax distribution as given in (1.1) and for $1 \leq i \leq n$, and $k = 1,2,3,\ldots$, we have

$$\mu_{i:n}^{(k)} = E\left(X_{i:n}^{k}\right) = \alpha\lambda^{k/\beta} C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \frac{\Gamma\left(1+\frac{k}{\beta}\right)\Gamma\left(\alpha+\alpha n+l\alpha-\frac{k}{\beta}-\alpha i\right)}{\Gamma(\alpha+\alpha n+l\alpha+1-\alpha i)}, \tag{5.5}$$

exists for the choice of α and β such that $\alpha > \frac{k}{n\beta}$.

Proof: Using (5.1) and binomial expansion of $[1-(1-F(x))]^{i-1}$, Eq. (5.3) can be rewritten as

$$\mu_{i:n}^{(k)} = 1C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \int_{0}^{\infty} x^{k}[1-F(x)]^{n-i+l} f(x)dx\,.$$

Substituting the values of $f(x)$ and $F(x)$ as given by (1.1) and (1.2), in the above equation we get

$$\mu_{i:n}^{(k)} = C_{i:n} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l} \frac{\alpha\beta}{\lambda} \int_{0}^{\infty} x^{k+\beta-1} \left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+\alpha n+l\alpha+1-\alpha i)} dx\,.$$

Simplifying the above integral we get the desired result as given by Eq.(5.5).

Theorem 5.2: For the power Lomax distribution as given in (1.1) and for $1 \leq i < j \leq n$, and $r, s = 1,2,3,\ldots$, and $\frac{s}{\beta} \in Z^{+}$, we have

$$\mu_{i,j:n}^{(r,s)} = \alpha^{2}\lambda^{\frac{(r+s)}{\beta}} C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \sum_{u=0}^{\frac{s}{\beta}} \binom{i-1}{t} \binom{j-i-1}{m} \binom{s/\beta}{u} (-1)^{t+m+u}$$
$$\times \frac{1}{c} \frac{\Gamma\left(1+\frac{r}{\beta}\right)\Gamma\left(\alpha(a+1)+c-\frac{r}{\beta}\right)}{\Gamma(\alpha(a+1)+c+1)}, \tag{5.6}$$

where

$$a = t+j-i-m-1,\, b = m+n-j \text{ and } c = \alpha(1+b)+u-\frac{s}{\beta}.$$

Proof: Using (5.2) and binomial expansion of $[F(x)]^{i-1}$ in the powers of $[1-F(x)]$, and binomial expansion of $[F(y)-F(x)]^{j-i-1}$ in the powers of $[1-F(x)]$ and $[1-F(y)]$, Eq. (5.4) can be rewritten as

$$\mu_{i,j:n}^{(r,s)} = C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \binom{i-1}{t} \binom{j-i-1}{m} (-1)^{t+m}$$
$$\times \int_{0}^{\infty} \int_{x}^{\infty} x^{r} y^{s} [1-F(x)]^{t+j-i-m-1} [1-F(y)]^{m+n-j} f(x) f(y) dy dx$$
$$= C_{i,j:n} \sum_{t=0}^{i-1} \sum_{m=0}^{j-i-1} \binom{i-1}{t} \binom{j-i-1}{m} (-1)^{t+m} \int_{0}^{\infty} x^{r} [1-F(x)]^{a} f(x) I_{1}(x) dx, \tag{5.7}$$

where $a = t + j - i - m - 1, b = m + n - j$ and

$$I_1(x) = \int_x^\infty y^s \,[1 - F(y)]^b f(y) dy. \tag{5.8}$$

Substituting the values of *f(y)* and *F(y)* from (1.1) and (1.2),respectively in Eq.(5.8) and simplifying we get

$$I_1(x) = \alpha\lambda^{s/\beta} \sum_{u=0}^{\frac{s}{\beta}} \binom{s/\beta}{u} (-1)^u \frac{\left(1+\frac{x^\beta}{\lambda}\right)^{-c}}{c} \tag{5.9}$$

where $c = \alpha(1 + b) + u - \frac{s}{\beta}$.

Substituting the value of $I_1(x)$ from Eq. (5.9) in (5.7) and simplifying the expression by putting the values of $f(x)$ and $F(x)$ as given by (1.1) and (1.2), we get the desired result (5.6).

5.2. Recursive algorithm for single moments

Case I: When $n = 1$, then $m = 1$

In this case, we have only one progressive censoring scheme $R_1 = 0$. Thus, from Eq. (2.3) and using Eq. (5.5), we have for $\alpha > \frac{k}{\beta}$ and $k = 1,2, \dots$,

$$E\left(X_{1:1:1}^{(0)}\right)^k = \mu_{1:1:1}^{(0)^{(k)}} = \mu_{1:1}^{(k)} = E\left(X^k\right) = \alpha\lambda^{k/\beta} \frac{\Gamma(1+\frac{k}{\beta})\Gamma(\alpha-\frac{k}{\beta})}{\Gamma(\alpha+1)}. \tag{5.10}$$

Using (5.10), $\mu_{1:1:1}^{(0)^{(k)}}$ $\forall\, k = 1,2, \dots$, can be calculated.

Alternatively, these moments can also be obtained by using the recurrence relation given in

Eq. (3.2) on putting n=1, i.e. by using the relation

$$\mu_{1:1:1}^{(0)^{(k+\beta)}} = \mu_{1:1}^{(k+\beta)} = \frac{\lambda(k+\beta)}{\beta(\alpha-1)-k}\mu_{1:1}^{(k)}.$$

Case II: When $n = 2$, then $m = 1$ or 2

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 1$, and in this case we have from Eq. (3.2), on putting $n = 2$,

$$\mu_{1:1:2}^{(1)^{(k+\beta)}} \left(\frac{2\alpha\beta}{k+\beta} - 1\right) = \lambda\, \mu_{1:1:2}^{(1)^{(k)}}, \tag{5.11}$$

where

$$\mu_{1:1:2}^{(1)^{(k)}} = \mu_{1:1:2}^{(k)} = \mu_{1:2}^{(k)} = 2\alpha\lambda^{k/\beta} \frac{\Gamma(1+\frac{k}{\beta})\Gamma(2\alpha-\frac{k}{\beta})}{\Gamma(2\alpha+1)}. \tag{5.12}$$

(Obtained on putting $i = 1$ and $n = 2$ in Eq. (5.5))

Using Eq. (5.12) and the recurrence relation given by Eq. (5.11) (for values of $\beta \in Z^{+}$), $\forall\ k = 1,2,\ldots$, and $\alpha > \frac{k}{2\beta}$, $\mu_{1:1:2}^{(1)^{(k)}}$ can be calculated.

Subcase (ii): $m = 2$

We have only one progressive censoring scheme $R_1 = R_2 = 0$, In this case we have

$$E\left(X_{1:2:2}^{(0,0)}\right) = \mu_{1:2:2}^{(0,0)} = \mu_{1:2} \text{ and } E\left(X_{2:2:2}^{(0,0)}\right) = \mu_{2:2:2}^{(0,0)} = \mu_{2:2}.$$

Also, $E\left(X_{1:2:2}^{(0,0)}\right)^2 = \mu_{1:2:2}^{(0,0)^{(2)}} = \mu_{1:2}^{(2)}$ and $E\left(X_{2:2:2}^{(0,0)}\right)^2 = \mu_{2:2:2}^{(0,0)^{(2)}} = \mu_{2:2}^{(2)}$

and these values concerning ordinary order statistics can be evaluated using Eq. (5.5).

Case III: When $n = 3$, then $m = 1$ or 2 or 3

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 2$, and in this case we have from Eq. (3.2), on putting $n = 3$, we get

$$\mu_{1:1:3}^{(2)^{(k+\beta)}}\left(\frac{3\alpha\beta}{k+\beta} - 1\right) = \lambda\, \mu_{1:1:3}^{(2)^{(k)}}, \tag{5.13}$$

where

$$\mu_{1:1:3}^{(2)^{(k)}} = \mu_{1:1:3}^{(k)} = \mu_{1:3}^{(k)} = 3\alpha\lambda^{k/\beta}\frac{\Gamma(1+\frac{k}{\beta})\Gamma(3\alpha-\frac{k}{\beta})}{\Gamma(3\alpha+1)}. \tag{5.14}$$

(Obtained on putting $i = 1$ and $n = 3$ in Eq. (5.5))

Using Eq. (5.14) and the recurrence relation given by Eq. (5.13) (for values of $\beta \in Z^{+}$), $\forall\ k = 1,2,\ldots$, and $\alpha > \frac{k}{3\beta}$, $\mu_{1:1:3}^{(2)^{(k)}}$ can be calculated.

Subcase (ii): $m = 2$

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

On putting $n = 3, m = 2,\ R_1 = 1$ and $R_2 = 0$ in (3.1), we get

$$\mu_{1:2:3}^{(1,0)^{(k+\beta)}}\left(1 - \frac{2\alpha\beta}{k+\beta}\right) = \frac{\alpha\beta}{k+\beta}\,\mu_{1:1:3}^{(2)^{(k+\beta)}} - \lambda\mu_{1:2:3}^{(1,0)^{(k)}}, \tag{5.15}$$

where $\mu_{1:1:3}^{(2)^{(k+\beta)}}$ can be calculated using (5.14), and $\mu_{1:2:3}^{(1,0)^{(k)}} = \mu_{1:3}^{(k)}$.

Using the recurrence relation given by Eq. (5.15) (for values of $\beta \in Z^{+}$), $\mu_{1:2:3}^{(1,0)^{(k+\beta)}}$ $\forall\ k = 1,2,\ldots$, can be calculated.

Further, on using mixture formula, we have

$$\mu_{2:2:3}^{(1,0)} = \frac{1}{2}[\mu_{2:3} + \mu_{3:3}] \text{ and}$$

$$\mu_{2:2:3}^{(1,0)^{(2)}} = \frac{1}{2}\left[\mu_{2:3}^{(2)} + \mu_{3:3}^{(2)}\right].$$

Proceeding in a similar manner $\mu_{1:2:3}^{(1,0)^{(k)}}$ and $\mu_{2:2:3}^{(1,0)^{(k)}} \forall\, k = 1,2,\ldots.$ can be calculated.

When $R_1 = 0$ and $R_2 = 1$

In this case, we find that

$$E\left(X_{1:2:3}^{(0,1)}\right) = \mu_{1:2:3}^{(0,1)} = \mu_{1:3}, \qquad E\left(X_{2:2:3}^{(0,1)}\right) = \mu_{2:2:3}^{(0,1)} = \mu_{2:3},$$

$$E\left(X_{1:2:3}^{(0,1)}\right)^2 = \mu_{1:2:3}^{(0,1)^{(2)}} = \mu_{1:3}^{(2)} \text{ and } E\left(X_{2:2:3}^{(0,1)}\right)^2 = \mu_{2:2:3}^{(0,1)^{(2)}} = \mu_{2:3}^{(2)}.$$

Other moments can be obtained similarly.

Subcase (iii): $m = 3$

We have only one progressive censoring scheme $R_1 = 0,\ R_2 = 0$ and $R_3 = 0$. In this case

$$E\left(X_{1:3:3}^{(0,0,0)}\right) = \mu_{1:3:3}^{(0,0,0)} = \mu_{1:3}, \qquad E\left(X_{2:3:3}^{(0,0,0)}\right) = \mu_{2:3:3}^{(0,0,0)} = \mu_{2:3},$$

$$E\left(X_{1:3:3}^{(0,0,0)}\right)^2 = \mu_{1:3:3}^{(0,0,0)^{(2)}} = \mu_{1:3}^{(2)}, \qquad E\left(X_{2:3:3}^{(0,0,0)}\right)^2 = \mu_{2:3:3}^{(0,0,0)^{(2)}} = \mu_{2:3}^{(2)},$$

$$\text{and} \qquad E\left(X_{3:3:3}^{(0,0,0)}\right)^2 = \mu_{3:3:3}^{(0,0,0)^{(2)}} = \mu_{3:3}^{(2)}.$$

All these values can be obtained by using the result given in Eq. (5.5) for ordinary order statistics.

5.3. Recursive algorithm for product moments

To evaluate the moments of progressively Type-II right censored order statistics from Power Lomax distribution, we have considered the case for $r = s = 1$.

Case I: When $n = 2$ and $m = 2$

In this case we have only one progressive censoring scheme, i.e. $R_1 = R_2 = 0$. Thus, from Eq. (2.4), we have for $\alpha > \frac{1}{\beta}$

$$E\left(X_{1:2:2}^{(0,0)} X_{2:2:2}^{(0,0)}\right) = \mu_{1,2:2:2}^{(0,0)} = \mu_{1:2:2} = (\mu_{1:1})^2 = \left(\alpha\lambda^{1/\beta}\frac{\Gamma(1+\frac{1}{\beta})\Gamma(\alpha-\frac{1}{\beta})}{\Gamma(\alpha+1)}\right)^2,$$

(cf. Arnold et al. (1992), Eqn. (5.3.10)).

Case II: When $n = 3$ and $m = 2$

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

In this case we have

$$E\left(X_{1:2:3}^{(1,0)} X_{2:2:3}^{(1,0)}\right) = \mu_{1,2:2:3}^{(1,0)} = \frac{1}{2}\left(\mu_{1,2:3} + \mu_{1,3:3}\right), \text{ from the mixture formula.}$$

When $R_1 = 0$ and $R_2 = 1$

In this case we have $E\left(X_{1:2:3}^{(0,1)} X_{2:2:3}^{(0,1)}\right) = \mu_{1,2:2:3}^{(0,1)} = \mu_{1,2:3}$.

Case III: When $n = 3$ and $m = 3$

In this case we have only one progressive censoring scheme $R_1 = R_2 = R_3 = 0$ and

$$E\left(X_{1:3:3}^{(0,0,0)} X_{2:3:3}^{(0,0,0)}\right) = \mu_{1,2:3:3}^{(0,0,0)} = \mu_{1,2:3}, \; E\left(X_{1:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}\right) = \mu_{1,3:3:3}^{(0,0,0)} = \mu_{1,3:3}$$

and $E\left(X_{2:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}\right) = \mu_{2,3:3:3}^{(0,0,0)} = \mu_{2,3:3}.$

Likewise, using Eq. (5.6) and recurrence relations for product moments as derived in Section 4, one could proceed for higher values of n and all choices of m and $(R_1, R_2, \dots, R_m)$.

6. Maximum Likelihood Estimators (MLEs)

Based on the observed sample $x_1 < x_2 < \cdots < x_m$ from a progressively Type-II censoring scheme, $(R_1, R_2, \dots, R_m)$, the likelihood function can be written as

$$L(\alpha, \beta, \lambda) = A(n, m-1) \prod_{t=1}^{m} f(x_t, \alpha, \beta, \lambda)[1 - F(x_t, \alpha, \beta, \lambda)]^{R_t}; x > 0, \alpha, \beta, \lambda > 0, \tag{6.1}$$

where

$$A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1),$$

and $f(.)$ and $F(.)$ are same as defined in (1.1) and (1.2), respectively. Therefore, ignoring the additive constant the log-likelihood function is written as

$$log\big(L(\alpha, \beta, \lambda)\big) = mlog(\alpha) + mlog(\beta) - mlog(\lambda) + (\beta - 1)\sum_{t=1}^{m} log(x_t) - \sum_{t=1}^{m} (\alpha(R_t + 1) + 1)\, log\left(1 + \frac{x_t^\beta}{\lambda}\right). \tag{6.2}$$

To compute the MLEs of the unknown parameters α, β and λ, consider the three normal equations:

$$\frac{\partial log(L)}{\partial \alpha} = \frac{m}{\alpha} - \sum_{t=1}^{m} (1 + R_t)\, log\left(1 + \frac{{x_t}^{\beta}}{\lambda}\right) = 0,$$

$$\frac{\partial log(L)}{\partial \beta} = \frac{m}{\beta} + \sum_{t=1}^{m} log(x_t) - \sum_{t=1}^{m} \frac{(\alpha(1 + R_t) + 1){x_t}^{\beta}\, log(x_t)}{\lambda + {x_t}^{\beta}} = 0,$$

and

$$\frac{\partial log(L)}{\partial \lambda} = -\frac{m}{\lambda} + \frac{1}{\lambda}\sum_{t=1}^{m} \frac{{x_t}^{\beta}(\alpha(1 + R_t) + 1)}{\lambda + {x_t}^{\beta}} = 0,$$

whose solution provide the MLEs $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$.

Once MLEs of α, β and λ are obtained as $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$, the MLEs of $R(t)$ and $h(t)$ can be obtained using invariance property of MLEs as

$$\hat{R}(t) = \left(1 + \frac{t^{\hat{\beta}}}{\hat{\lambda}}\right)^{-\hat{\alpha}}, t > 0 \quad \text{and}$$

$$\hat{h}(t) = \frac{\hat{\alpha}\, \hat{\beta} t^{\hat{\beta}-1}}{\hat{\lambda} + t^{\hat{\beta}}}, t > 0.$$

7. Simulation study

In this Section, a simulation study is conducted to observe the behaviour of the proposed method for different sample sizes, different effective sample sizes and for different censoring schemes. We have considered different sample sizes; $n = 35,40,50$; different effective sample sizes; $m = 20,25,30,35,40,50$; different censoring schemes. In all the cases we have used $\alpha = 2, \beta = 1$ and $\lambda = 2$. For a given set of n, m and a censoring scheme, using the algorithm proposed by Balakrishnan and Sandhu (1995), a sample is generated. Using the sample, the MLEs of unknown parameters α, β and λ are computed based on the method proposed in Section 6. Finally, with 1000 replications, using a program in R, the MLEs of $\alpha, \beta, \lambda, R(t)$ and $h(t)$along with their average bias and mean square errors (MSEs) are obtained. The average bias is reported within brackets against each estimate and the results are presented in Tables 7.1, 7.2a and 7.2b.

Table 7.1. MLEs of α , β and λ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1$ and $\lambda = 2$

n	m	Censoring Scheme	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$MSE(\hat{\lambda})$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	1.94181 (-0.0582)	1.03903 (0.03903)	1.89249 (-0.1075)	0.35059	0.20826	0.31453
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1, 1,2*0)	1.95704 (-0.0429)	1.04216 (0.04216)	1.96635 (-0.0337)	0.36253	0.20725	0.34712
35	20	(15,19*0)	2.01236 (0.01236)	1.01958 (0.01958)	2.01264 (0.01264)	0.33291	0.13837	0.33011
35	25	(5*0,5,7*0,5,11*0)	1.95215 (-0.0478)	1.02663 (0.02663)	1.93115 (-0.0689)	0.32803	0.17725	0.30247
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1, 12*0)	1.96322 (-0.0368)	1.02506 (0.02506)	1.96717 (-0.0328)	0.32532	0.14379	0.31937
35	25	(10,24*0)	2.02428 (0.02428)	1.01749 (0.01749)	2.02533 (0.02533)	0.30106	0.12965	0.32033
35	35	(35*0)	1.99586 (-0.0041)	1.00524 (0.00524)	1.99661 (-0.0034)	0.19950	0.07522	0.20118
40	25	(5,4*0,5,7*0,5,11*0)	1.96951 (-0.0305)	1.01450 (0.0145)	1.94261 (-0.0574)	0.31210	0.19161	0.29743
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1, 1,7*0)	1.96329 (-0.0367)	1.02692 (0.02692)	1.97827 (-0.0217)	0.30661	0.13456	0.30967
40	25	(15,24*0)	2.02742 (0.02742)	1.02021 (0.02021)	2.04453 (0.04453)	0.28108	0.11671	0.29954
40	30	(3*0,5,10*0,5,15*0)	1.97058 (-0.0294)	1.02329 (0.02329)	1.95472 (-0.0453)	0.30219	0.18862	0.30098
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1, 19*0)	1.95811 (-0.0419)	1.02032 (0.02032)	1.97122 (-0.0288)	0.28149	0.12976	0.28352
40	30	(10,20*0)	1.99483 (-0.0052)	1.01458 (0.01458)	1.99475 (-0.0052)	0.27662	0.12839	0.29211
40	40	(40*0)	1.99802 (-0.0019)	1.00227 (0.00227)	1.99777 (-0.0023)	0.19019	0.05713	0.19717
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	2.02323 (0.02323)	0.99753 (-0.0025)	2.04855 (0.04855)	0.28921	0.13003	0.25912
50	25	(25,24*0)	2.00972 (0.00972)	1.01253 (0.01253)	2.03109 (0.03109)	0.25413	0.11080	0.24879
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	1.99786 (0.00214)	1.02274 (0.02274)	2.02981 (0.02981)	0.24142	0.12644	0.26907
50	30	(20,29*0)	2.01786 (0.01786)	0.99015 (-0.0099)	2.02153 (0.02153)	0.22832	0.12561	0.20991
50	40	(8*0,5,8*0,5,22*0)	1.95692 (-0.0431)	1.01711 (0.01711)	1.94165 (-0.0584)	0.21131	0.12675	0.25002
50	40	(10,39*0)	2.00875 (0.00875)	1.01251 (0.01251)	2.01141 (0.01141)	0.20149	0.10939	0.19972
50	50	(50*0)	2.00123 (0.00123)	1.00135 (0.00135)	2.00191 (0.00191)	0.12615	0.03927	0.16793

Table 7.2a. MLEs of $R(t)$and $h(t)$ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1, \lambda = 2$ and $t = 0.5$

$$\boldsymbol{t = 0.5; R(t) = 0.64; h(t) = 0.8}$$

n	m	Censoring Scheme	$\hat{R}(t)$	$\hat{h}(t)$	$MSE(\hat{R}(t))$	$MSE(\hat{h}(t))$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	0.63804 (-0.002)	0.74962 (-0.0504)	0.01321	0.03465
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,2*0)	0.64294 (0.00294)	0.76023 (-0.0398)	0.01327	0.03022
35	20	(15,19*0)	0.63772 (-0.0023)	0.75152 (-0.0485)	0.01262	0.03349
35	25	(5*0,5,7*0,5,11*0)	0.64416 (0.00416)	0.76376 (-0.0362)	0.01329	0.02173
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1 ,12*0)	0.64215 (0.00215)	0.75891 (-0.0411)	0.00981	0.02020
35	25	(10,24*0)	0.64309 (0.00309)	0.76364 (-0.0364)	0.00976	0.01964
35	35	(35*0)	0.64043 (0.00043)	0.79646 (-0.0035)	0.00512	0.01582
40	25	(5,4*0,5,7*0,5,11*0)	0.63524 (-0.0048)	0.75699 (-0.043)	0.01236	0.03126
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,7*0)	0.64416 (0.00416)	0.75361 (-0.0464)	0.01483	0.03212
40	25	(15,24*0)	0.63783 (-0.0022)	0.76114 (-0.0389)	0.00933	0.02231
40	30	(3*0,5,10*0,5,15*0)	0.64176 (0.00176)	0.76083 (-0.0392)	0.00940	0.02153
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1 ,19*0)	0.63668 (-0.0033)	0.75993 (-0.0401)	0.00971	0.01987
40	30	(10,20*0)	0.64857 (0.00857)	0.76648 (-0.0335)	0.01050	0.01901
40	40	(40*0)	0.63928 (-0.0007)	0.79685 (-0.0032)	0.00425	0.01322
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	0.64424 (0.00424)	0.77405 (-0.026)	0.01393	0.01874
50	25	(25,24*0)	0.63967 (-0.0003)	0.78257 (-0.0174)	0.01009	0.01716
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	0.64205 (0.00205)	0.78017 (-0.0198)	0.01006	0.01718
50	30	(20,29*0)	0.63367 (-0.0063)	0.77948 (-0.0205)	0.00960	0.01872
50	40	(8*0,5,8*0,5,22*0)	0.64139 (0.00139)	0.78965 (-0.0104)	0.01110	0.01482
50	40	(10,39*0)	0.63778 (-0.0022)	0.78342 (-0.0166)	0.01211	0.01613
50	50	(50*0)	0.63994 (-0.00006)	0.79866 (-0.0013)	0.00303	0.01209

Table 7.2b. MLEs of $R(t)$and $h(t)$ along with their Average Bias and MSE for different censoring schemes, for $\alpha = 2, \beta = 1, \lambda = 2$ and t = 2

$$\boldsymbol{t = 2; R(t) = 0.25; h(t) = 0.5}$$

n	m	Censoring Scheme	$\hat{R}(t)$	$\hat{h}(t)$	$MSE(\hat{R}(t))$	$MSE(\hat{h}(t))$
35	20	(3*0,5,3*0,5,3*0,5,8*0)	0.25230 (0.0023)	0.51557 (0.01557)	0.00866	0.06067
35	20	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,2*0)	0.25098 (0.00098)	0.51532 (0.01532)	0.00834	0.05801
35	20	(15,19*0)	0.24831 (-0.0017)	0.48845 (-0.0116)	0.00902	0.04897
35	25	(5*0,5,7*0,5,11*0)	0.24792 (-0.0021)	0.50292 (0.00292)	0.00838	0.04923
35	25	(1, 2, 0, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1 ,12*0)	0.24803 (-0.002)	0.50371 (0.00371)	0.00692	0.04810
35	25	(10,24*0)	0.24672 (-0.0033)	0.50354 (0.00354)	0.00689	0.04799
35	35	(35*0)	0.24976 (-0.00024)	0.50102 (0.00102)	0.00451	0.02815
40	25	(5,4*0,5,7*0,5,11*0)	0.24562 (-0.0044)	0.49653 (-0.0035)	0.00767	0.03965
40	25	(1,2,1,2,1,2*0,1,1,0,1,0,1,0,2,0,1,1,7*0)	0.25093 (0.00093)	0.48032 (-0.0197)	0.00558	0.04365
40	25	(15,24*0)	0.24847 (-0.0015)	0.48964 (-0.0104)	0.00621	0.04221
40	30	(3*0,5,10*0,5,15*0)	0.24863 (-0.0014)	0.49876 (-0.0012)	0.00606	0.03932
40	30	(1, 2, 1, 2, 1, 0, 0, 1, 1, 0, 1 ,19*0)	0.24798 (-0.002)	0.49721 (-0.0028)	0.00755	0.03123
40	30	(10,20*0)	0.24832 (-0.0017)	0.50176 (0.00176)	0.00518	0.03078
40	40	(40*0)	0.24978 (-0.0002)	0.50090 (0.0009)	0.00317	0.02120
50	25	(5,4*0,5,4*0,5,4*0,5,4*0,5,4*0)	0.24882 (-0.0018)	0.50387 (0.00387)	0.00578	0.03821
50	25	(25,24*0)	0.25017 (0.00017)	0.49536 (-0.0046)	0.00592	0.03729
50	30	(5,4*0,5,5*0,5,5*0,5,12*0)	0.24932 (-0.0007)	0.49674 (-0.0033)	0.00503	0.03655
50	30	(20,29*0)	0.25102 (0.00102)	0.48991 (-0.0101)	0.00499	0.03812
50	40	(8*0,5,8*0,5,22*0)	0.25091 (0.00091)	0.50148 (0.00148)	0.00432	0.03556
50	40	(10,39*0)	0.24965 (-0.0004)	0.49839 (-0.0016)	0.00341	0.03233
50	50	(50*0)	0.25009 (0.00009)	0.50039 (0.00039)	0.00246	0.01109

From Table 7.1, we observe that for complete samples, MLEs of α, β and λ are very nearly unbiased and can be regarded as good estimators. It is also observed that for complete samples, as the sample size n increases the average MSE decreases. In addition, the MSE generally decreases as the failure information m increases, and for all the censoring schemes the MSE of the estimates is quite small and can be used in all practical situations. Here one has to make a trade-off between the precision of the estimation method and the cost of the experiment. Also, from Tables 7.2a and 7.2b, it is observed that for the MLEs of $R(t)$ and $h(t)$, the MSE generally decreases as the failure information m increases. In addition, for the complete samples, as the sample size n increases the average MSE decreases.

8. Conclusion

Some recurrence relations between the single and the product moments of progressively Type-II right censored order statistics from the power Lomax distribution have been derived, which would assist us to compute the moments of progressively Type-II right censored order statistics for every n and for different censoring arrangements $(R_1, R_2, \ldots, R_m)$, $m \leq n$. The recursive algorithm is presented with the help of which the single and product moments of progressively Type-II right censored order statistics from the power Lomax distribution can be easily obtained. Further, a maximum likelihood approach is used to estimate the parameters of the power Lomax distribution, which are further used to estimate the reliability characteristics. A Monte Carlo method is used to simulate the data and to compare the performance of the estimates for different censoring schemes.

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