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The odd power generalized Weibull-G power series class of distributions: properties and applications

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ABSTRACT

We develop a new class of distributions, namely, the odd power generalized Weibull-G power series (OPGW-GPS) class of distributions. We present some special classes of the proposed distribution. Structural properties, have also been derived. We conducted a simulation study to evaluate the consistency of the maximum likelihood estimates. Moreover, two real data examples on selected data sets, to illustrate the usefulness of the new class of distributions. The proposed model outperforms several non-nested models on selected data sets.

Key words: Weibull-g distribution, power series, Poisson distribution, logarithmic distribution, maximum likelihood estimation.

1. Introduction

Existing distributions or a family of distributions cannot model all real lifetime data. Thus, there is a need to modify them by adding one or more parameters to gain flexibility. Some families of distributions available in the literature include the Weibull-G distribution by Bourguignon et al. (2014), the odd generalized half-logistic Weibull-G family of distributions by Chipepa et al. (2020a), the exponentiated generalized (EG) class of distributions by Cordeiro et al. (2013), beta-G family by Eugene et al. (2002), new power generalized Weibull-G family by Oluyede et al. (2021), the odd exponentiated half-logistic-G family of distributions by Afify et al. (2017), to mention a few.

Several generalized distributions proposed in the literature involving the power series include the exponentiated generalized power series class of distributions by Oluyede et al. (2020c), a new generalized Lindley-Weibull class of distributions by Makubate et al. (2020), the exponentiated power generalized Weibull power series family of distributions by Aldahlan et al. (2019), Weibull-power series distributions by Morais and Barreto-Souza (2011), complementary exponential power series by Flores et al. (2013), complementary extended Weibull-power series by Cordeiro and Silva (2014), Burr XII power series by Silva and Silva and Cordeiro (2015), extended Weibull-power series (EWPS) distribution by Silva et al. (2013).

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In this paper, we propose a new class of distributions, namely the odd power generalized Weibull-G power series (OPGW-GPS) class of distributions. An attractive feature about the model is that the extra parameter introduced have the capability to control both the weights at the tails of the density function. Also, the new class of distributions can model different types of failure rate functions that are available in different areas like reliability, engineering and biological studies. The new proposed distribution offers more flexibility in data modelling since the special cases exhibit more non-monotonic shapes for the hazard rate function compared to the other power series reviewed in this paper. Furthermore, the new class of distributions gives birth to more families of distributions by choosing any continuous probability distribution as a baseline distribution $G(x;\underline{\psi})$.

In a recent note, Moakofi et al. (2021) developed the odd power generalized Weibull-G (OPGW-G) family of distributions. The cumulative distribution function (cdf) and probability density function (pdf) of the OPGW-G distribution are given by

$$F(x;\alpha,\beta,\underline{\psi}) \quad = \quad 1-\exp\{1-(1+t)^{\beta}\} \tag{1}$$

and

$$f(x;\alpha,\beta,\underline{\psi}) \quad = \quad \frac{\alpha\beta(1+t)^{\beta-1}\exp\{1-(1+t)^{\beta}\}g(x;\underline{\psi})}{\left(1-G(x;\underline{\psi})\right)^{2}}\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha-1}, \tag{2}$$

respectively, where $t=\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}$, for $\alpha,\beta>0$ and parameter vector $\underline{\psi}$. In this note, we extend the OPGW-G family of distributions by compounding it with the power series distribution.

Let N be a zero truncated discrete random variable having a power series distribution, whose probability mass function (pmf) is given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)}, n=1,2,3,..., \tag{3}$$

where $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite, $\theta>0$ and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. If we consider $X_{(1)}=\min(X_1,X_2,...,X_N)$, then the cumulative distribution function (cdf) and probability density function (pdf) of $X_{(1)}|N=n$ are defined by

$$F_{X_{(1)}|N=n}(x)=1-\frac{C(\theta S(x;\underline{\psi}))}{C(\theta)}, \tag{4}$$

and

$$f_{X_{(1)}|N=n}(x)=\frac{\theta g(x;\underline{\psi})C'(\theta S(x;\underline{\psi}))}{C(\theta)}, \tag{5}$$

where $S(x;\underline{\psi})$ is the survival function of the baseline distribution and $\underline{\psi}$ is a vector of parameters from the baseline distribution $g(x;\underline{\psi})$. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions Johnson et al. (1994).

The rest of the paper is organized as follows: In Section 2, we present the new model and some of the statistical properties. We present some special cases of the proposed class of distributions in Section 3. A simulation study is presented in Section 4 and applications in Section 5, followed by concluding remarks.

2. The Model, Sub-Classes and Properties

In this section, we develop the new model, referred to as the odd power generalized Weibull-G power series (OPGW-GPS) class of distributions. Some statistical properties, including expansion of the density function, hazard rate function, quantile function, sub-classes, moments, moment generating function and maximum likelihood estimation of model parameters are derived. Details on the derivations of other statistical properties are given in the Web-Appendix.

2.1. The Model

Using equation (4), the odd power generalized Weibull-G power series (OPGW-GPS) class of distributions denoted by OPGW-GPS($\alpha,\beta,\,\theta,\,\psi$) has cdf and pdf given by

$$F_{OPGW-GPS}(x) \quad = \quad 1-\frac{C(\theta(\exp\{1-(1+t)^{\beta}\}))}{C(\theta)}, \tag{6}$$

and

$$\begin{aligned} f_{OPGW-GPS}(x) \quad &= \quad \frac{\theta\alpha\beta(1+t)^{\beta-1}g(x;\underline{\psi})}{\left(1-G(x;\underline{\psi})\right)^{2}}\exp\{1-(1+t)^{\beta}\}\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha-1} \\ &\times \quad \frac{C'\left(\theta[\exp\{1-(1+t)^{\beta}\}]\right)}{C(\theta)}, \end{aligned} \tag{7}$$

respectively, where $t=\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}$, for $\alpha,\beta,\,\theta,\,x>0$ and parameter vector $\underline{\psi}$.

Table 1 below presents the special families of OPGW-GPS distribution when $C(\theta)$ is specified in equation (6).

Table 1: Special Families of the OPGW-GPS Distribution

Distribution	$C(\theta)$	a_n	cdf
OPGW-G Poisson	$e^{\theta}-1$	$(n!)^{-1}$	1-$\frac{\exp(\theta[\exp\{1-(1+t)^{\beta}\}])-1}{\exp(\theta)-1}$
OPGW-G Geometric	$\theta(1-\theta)^{-1}$	1	1-$\frac{(1-\theta)(\exp\{1-(1+t)^{\beta}\})}{(1-\theta[\exp\{1-(1+t)^{\beta}\}])}$
OPGW-G Logarithmic	$-\log(1-\theta)$	n^{-1}	1-$\frac{\log(\exp\{1-(1+t)^{\beta}\})}{\log(1-\theta)}$
OPGW-G Binomial	$(1+\theta)^{m}-1$	$\binom{m}{n}$	1-$\frac{(1+\theta[\exp\{1-(1+t)^{\beta}\}])^{m}-1}{(1+\theta)^{m}-1}$

2.2. Regularity Condition

We use the Kullback-Leibler distance between densities f_{α}, for $\alpha_1 \neq \alpha_2$ $D(f_1,f_2)=\int f(x|\alpha_1)\log\left(\frac{f(x|\alpha_1)}{f(x|\alpha_2)}\right)dx>0$. Hence, we obtain

$$\begin{aligned} D(f_1,f_2) &= \int f(x|\alpha_1)\left(\log\left[\frac{\alpha_1}{\alpha_2}\right]+(\alpha_1-\alpha_2)\log\left[\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right]\right)dx \\ &= \int f(x|\alpha_1)\left(\log\left[\frac{\alpha_1[1-G(x;\underline{\psi})]^{(\alpha_1-\alpha_2)}}{\alpha_2[G(x;\underline{\psi})]^{(\alpha_1-\alpha_2)}}\right]\right)dx, \end{aligned} \tag{8}$$

therefore, $D(f_1,f_2)>0$, for $\alpha_1 \neq \alpha_2$ since $\log\left[\frac{\alpha_1[1-G(x;\underline{\psi})]^{(\alpha_1-\alpha_2)}}{\alpha_2[G(x;\underline{\psi})]^{(\alpha_1-\alpha_2)}}\right]>0.$

2.3. Quantile Function

Let X be a random variable with cdf defined by equation (6). The quantile function $Q_{OPGW-GPS}(u)$ is defined by $F_{OPGW-GPS}(Q_{OPGW-GPS}(u))=u, 0\leq u\leq 1$ so that the quantile function of the OPGW-GPS class of distributions is given by

$$Q_{OPGW-GPS}(u)=G^{-1}\left[\left(\left[\left(1-\log\left(\frac{C^{-1}\left[C(\theta)(1-u)\right]}{\theta}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{-1}{\alpha}}+1\right)^{-1}\right]. \tag{9}$$

2.4. Expansion of Density

The pdf of the OPGW-GPS class of distributions is an infinite linear combination of exponentiated-G distribution expressed as

$$f_{OPGW-GPS}(x) = \sum_{m=0}^{\infty} w_{m+1}g_{m+1}(x;\underline{\psi}), \tag{10}$$

where $g_{m+1}(x;\underline{\psi}) = (m+1)\left(G(x;\underline{\psi})\right)^m g(x;\underline{\psi})$ is the Exp-G distribution with power parameter $(m+1)$ and

$$\begin{aligned} w_{m+1} &= \sum_{j,k,i,l=0}^{\infty}\sum_{n=1}^{\infty}\binom{j}{k}\binom{\beta(k+1)-1}{i}\binom{\alpha(i+1)-1}{l}\binom{-(\alpha(i+1)+1)+l}{m} \\ &\times \frac{\alpha\beta(-1)^{k+l+m}n^{j+1}a_n\theta^n}{C(\theta)j!}\frac{1}{m+1}. \end{aligned} \tag{11}$$

2.5. Moments and Generating Function

If X follows the OPGW-GPS distribution and $Y \sim Exp-G(m+1)$, then using equation (10) the p^{th} raw moment, μ'_p of the OPGW-GPS class of distributions is obtained as

$$\begin{aligned} \mu'_p = E(X^p) &= \int_{-\infty}^{\infty} x^p f(x)dx \\ &= \sum_{m=0}^{\infty} w_{m+1}E(Y^p), \end{aligned}$$

where w_{m+1} is given by equation (11). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{m=0}^{\infty} w_{m+1}M_Y(t),$$

where $M_Y(t)$ is the mgf of Y and w_{m+1} is given by equation (11).

2.6. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from OPGW-GPS class of distributions and suppose $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{m=0}^{\infty}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l h_{m+1}g_{m+1}(x;\underline{\psi}), \tag{12}$$

where $g_{m+1}(x;\underline{\psi}) = (m+1)g(x;\underline{\psi})G^m(x;\underline{\psi})$ is an Exp-G with power parameter $m+1$ and the linear component

$$\begin{aligned} h_{m+1} &= \sum_{p,j,k,i,v=0}^{\infty}\sum_{n,z=1}^{\infty}\frac{na_nd_{z,p}\theta^{z+n}}{C^{z+1}(\theta)}\frac{(n+z)^j}{j!}\binom{k+l-1}{p}\binom{j}{k}\binom{\beta(k+1)-1}{i} \\ &\times \binom{\alpha(i+1)-1}{v}\binom{-(\alpha(i+1)+1)+v}{m}(-1)^{p+k+m}\alpha\beta\frac{1}{m+1}. \end{aligned} \tag{13}$$

2.7. Rényi Entropy

In this subsection, Rényi entropy for OPGW-GPS class of distributions is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy by Rényi (1961) is a generalization of Shannon entropy by Shannon (1951). Rényi entropy for OPGW-GPS class of distributions is given by

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{m=0}^{\infty} w^* e^{(1-v)I_{REG}}\right), \tag{14}$$

where $I_{REG} = \int_0^\infty [(1+m/v)g(x;\underline{\psi})G^{m/v}(x;\underline{\psi})]^v dx$ is Rényi entropy for an Exp-G distribution with power parameter $(m/v+1)$ and

$$\begin{aligned} w^* &= \sum_{j,k,i,l,m=0}^{\infty}\sum_{n=1}^{\infty}\frac{d_{v,n}\theta^{v+n-1}}{\left(C(\theta)\right)^v}(v+(n-1))^j(\alpha\beta)^v\binom{j}{k}\binom{\beta(k+v)-v}{i}\frac{(-1)^{k+l+m}}{j!} \\ &\times \binom{\alpha(i+v)-1}{l}\binom{-(\alpha(i+v)+v)+l}{m}\frac{1}{(1+m/v)^v}. \end{aligned} \tag{15}$$

Consequently, Rényi entropy for OPGW-GPS class of distributions can be obtained from Rényi entropy of the Exp-G distribution.

2.8. Maximum Likelihood Estimation

We obtain the maximum likelihood estimates of the parameters of the OPGW-GPS class of distributions in this section. Let $X_i \sim OPGW-GPS(\alpha,\beta,\theta,\underline{\psi})$ and $\Delta = (\alpha,\beta,\theta,\underline{\psi})^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell(\Delta) &= n\ln[\theta\alpha\beta] + (\beta-1)\sum_{i=1}^{n}\ln[1+t] - n\ln[C(\theta)] + \sum_{i=1}^{n}(1-(1+t)^{\beta}) \\ &+ (\alpha-1)\sum_{i=1}^{n}\ln\left[\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right] + \sum_{i=1}^{n}\ln\left[C'\left(\theta\left[\exp\left(1-[1+t]^{\beta}\right)\right]\right)\right] \\ &+ \sum_{i=1}^{n}\ln\left[g(x;\underline{\psi})\right] - 2\sum_{i=1}^{n}\ln\left[\left(1-G(x;\underline{\psi})\right)^2\right], \end{aligned}$$

where $t = \left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}$. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\theta},\frac{\partial\ell_n}{\partial\underline{\psi}_k})^T = \mathbf{0}$, using a numerical method such as the Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,\underline{0})^T$ and $J(\hat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\Delta}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3. Some Special Classes of the OPGW-GPS Class of Distributions

In this section, special classes of OPGW-GPS class of distributions are presented by specifying the baseline distribution to be Weibull and log-logistic distributions, respectively. We considered the power series distributions Poisson and Logarithmic for each selected baseline distribution. The cdf and pdf of the Weibull distribution are given by $G(x;\lambda) = 1-\exp\left(-x^{\lambda}\right)$ and $g(x;\lambda) = \lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)$, for $\lambda > 0$, and $x > 0$. Furthermore, the log-logistic distribution has cdf and pdf given by $G(x;\lambda) = 1-\left(1+x^{\lambda}\right)^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{-2}$, for $\lambda > 0$, and $x > 0$.

3.1. Odd Power Generalized Weibull-Weibull Poisson (OPGW-WP) Distribution

The cdf and pdf of the OPGW-WP distribution are given by

$$F_{OPGW-WP}(x) = 1-\frac{\exp(\theta[\exp\{1-(1+z)^{\beta}\}]-1)}{\exp(\theta)-1},$$

and

$$\begin{aligned} f_{OPGW-WP}(x) &= \theta\alpha\beta(1+z)^{\beta-1}\left(\frac{(1-\exp\{-x^{\lambda}\})}{\exp\{-x^{\lambda}\}}\right)^{\alpha-1}\exp\{1-(1+z)^{\beta}\} \\ &\times \frac{\lambda x^{\lambda-1}\exp\{-x^{\lambda}\}}{\exp\{-x^{\lambda}\}^{2}}\frac{\exp\{\theta[\exp\{1-(1+z)^{\beta}\}]\}}{(\exp(\theta)-1)}, \end{aligned}$$

respectively, where $z = \left(\frac{1-\exp\{-x^{\lambda}\}}{\exp\{-x^{\lambda}\}}\right)^{\alpha}$, for α, β, λ and $\theta > 0$.

Figure 1 shows the plots of pdfs and hrfs of the OPGW-WP distribution. The pdf can take various shapes that include uni-modal, reverse-J, left skewed and right-skewed. Furthermore, the hazard rate functions (hrfs) for the OPGW-WP distribution exhibit increasing, reverse-J, bathtub, and upside bathtub shapes.

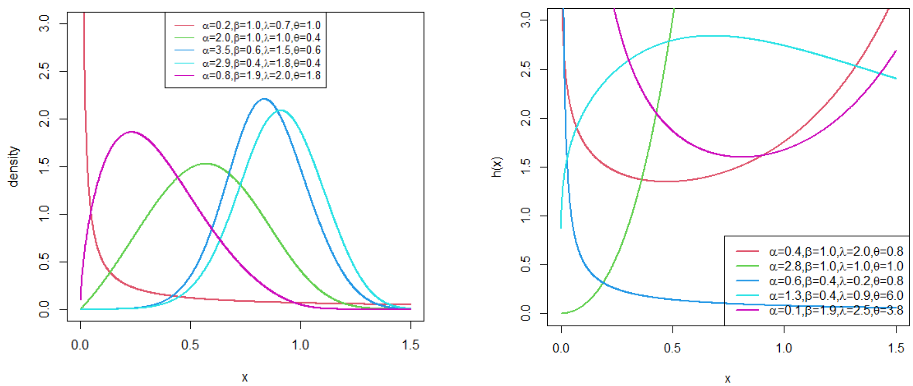

Figure 1: Plots of the pdf and hrf for the OPGW-WP distribution

3.2. Odd Power Generalized Weibull-Weibull Logarithmic (OPGW-WLoG) Distribution

The cdf and pdf of the OPGW-WLoG distribution are given by

$$F_{OPGW-WLoG}(x) = 1 - \frac{\log[1-\theta(\exp\{1-(1+z)^{\beta}\})]}{\log[1-\theta]},$$

and

$$\begin{aligned} f_{OPGW-WLoG}(x) &= \theta\alpha\beta(1+z)^{\beta-1}\left(\frac{1-\exp\{-x^{\lambda}\}}{\exp\{-x^{\lambda}\}}\right)^{\alpha-1}\exp\{1-[1+z]^{\beta}\} \\ &\times \frac{\lambda x^{\lambda-1}\exp\{-x^{\lambda}\}}{\exp\{-x^{\lambda}\}^{2}}\,\frac{(1-\theta[\exp\{1-(1+z)^{\beta}\}])^{-1}}{-\log[1-\theta]}, \end{aligned}$$

respectively, for $\alpha, \beta, \lambda > 0$ and $0 < \theta < 1$.

Figure 2 shows the plots of pdfs and hrfs of the OPGW-WLoG distribution. The pdf can take various shapes that include uni-modal, reverse-J, left or right-skewed. Furthermore, the hazard rate functions (hrfs) for the OPGW-WLoG distribution exhibit increasing, reverse-J, bathtub, upside-down bathtub, and upside-down bathtub follwed by bathtub shapes.

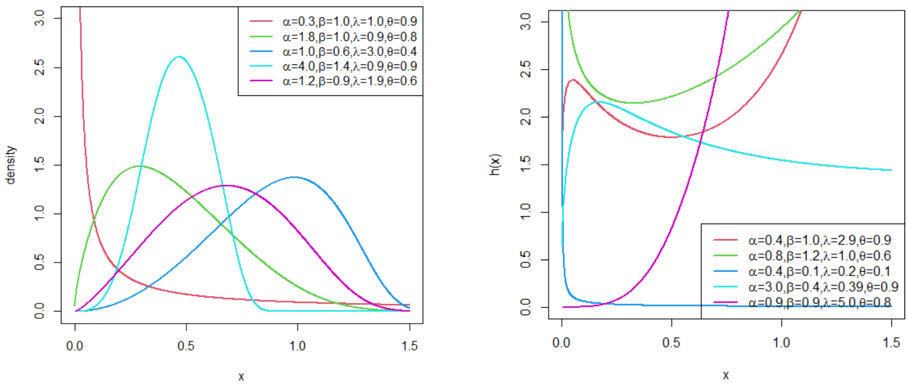

Figure 2: Plots of the pdf and hrf for the OPGW-WLoG distribution

3.3. Odd Power Generalized Weibull-Log-Logistic Poisson (OPGW-WLLoGP) Distribution

The cdf and pdf of the OPGW-LLoGP distribution are given by

$$F_{OPGW-LLoGP}(x) = 1 - \frac{\exp\{\theta[\exp\{1-(1+w)^{\beta}\}]\}-1}{\exp(\theta)-1},$$

and

$$f_{OPGW-LLoGP}(x) = \theta\alpha\beta(1+w)^{\beta-1}\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{\alpha-1}\exp\{1-(1+w)^{\beta}\} \times \frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{(1+x^{\lambda})^{-2}}\frac{\exp\{\theta[\exp\{1-(1+w)^{\beta}\}]\}}{\exp\{\theta\}-1},$$

respectively, where $w = \left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{\alpha}$, for α, β, λ and $\theta > 0$.

Figure 3 shows the plots of the pdfs and hrfs of the OPGW-LLoGP distribution. The pdf can take various shapes that include almost-symmetric, reverse-J, left or right-skewed. The hazard rate functions (hrfs) for the OPGW-LLoGP distribution exhibit increasing, reverse-J, bathtub and upside-down bathtub shapes.

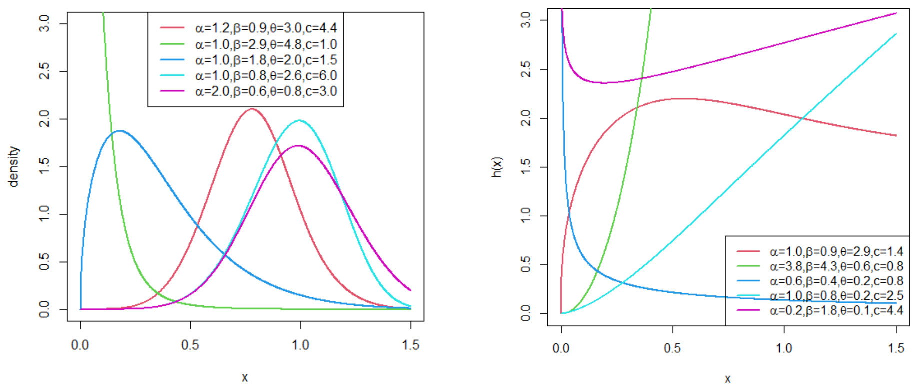

Figure 3: Plots of the pdf and hrf for the OPGW-LLoGP distribution

3.4. Odd Power Generalized Weibull-Log-Logistic Logarithmic (OPGW-LLoGLoG) Distribution

The cdf and pdf of the OPGW-LLoGLoG distribution are given by

$$F_{OPGW-LLoGLoG}(x) \quad = \quad 1-\frac{\log[1-\theta(\exp\{1-(1+w)^{\beta}\})]}{\log[1-\theta]},$$

and

$$\begin{aligned} f_{OPGW-LLoGLoG}(x) \quad &= \quad \theta\alpha\beta(1+w)^{\beta-1}\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{\alpha-1}\frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{(1+x^{\lambda})^{-2}} \\ &\times \quad \exp\{1-(1+w)^{\beta}\}\frac{(1-\theta[\exp\{1-(1+w)^{\beta}\}])^{-1}}{-\log[1-\theta]}, \end{aligned}$$

respectively, for $\alpha, \beta, \lambda > 0$ and $0 < \theta < 1$.

Figure 4 shows the pdfs of the OPGW-LLoGLoG distribution. The pdf can take various shapes that include unimodal, reverse-J, left or right-skewed. Furthermore, the hazard rate functions (hrfs) for the OPGW-LLoGLoG distribution exhibit increasing, reverse-J, bathtub, upside-down bathtub, and upside-down bathtub follwed by bathtub shapes.

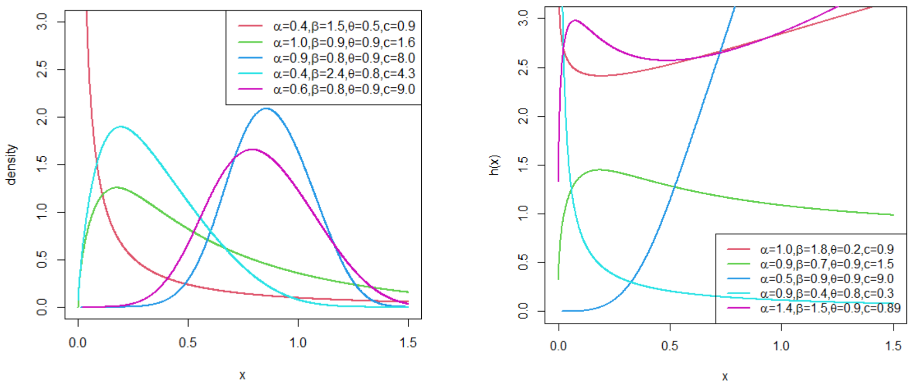

Figure 4: Plots of the pdf and hrf for the OPGW-LLoGLoG distribution

4. Simulation Study

In this section, the performance of the OPGW-WP distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800 and 1000) via the R package. We simulate $N = 1000$ samples for the true parameters values given in Table 2. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE) and average bias.

The estimated parameter values in Table 2 indicate that the estimates are quite stable and, more importantly, are close to the true parameter values for these sample sizes. The simulation study shows that the maximum likelihood method is appropriate for estimating the OPGW-WP model parameters. In fact, the means of the parameters tend to be closer to the true parameter values when n increases. The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

5. Inference

We present two real data examples in this section, to illustrate the importance of the OPGW-WP distribution. We compared the OPGW-WP distribution to various models. We estimate model parameters using the maximum likelihood estimation technique via the **nlm** package in R Software (2014). Model performance was assessed using the **Adequacy-**

Table 2: Monte Carlo Simulation Results for OPGW-WP Distribution: Mean, RMSE and Average Bias

		$\alpha=0.5,\beta=1.1,\lambda=1.1,c=1.1$			$\alpha=1.0,\beta=1.0,\lambda=1.0,c=0.9$		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	0.5335	0.9724	0.0335	1.0068	1.4337	0.0068
	50	0.4575	0.4222	-0.0425	0.7489	0.7986	-0.2511
	100	0.4224	0.2242	-0.0776	0.7243	0.5732	-0.2757
α	200	0.4464	0.1718	-0.0536	0.7645	0.4457	-0.2355
	400	0.4505	0.1216	-0.0495	0.8235	0.3418	-0.1765
	800	0.4662	0.0888	-0.0338	0.8991	0.2600	-0.1009
	1000	0.4757	0.0779	-0.0243	0.9316	0.2332	-0.0684
	25	0.8481	0.3930	-0.2519	0.7820	0.3946	-0.2180
	50	0.8847	0.3324	-0.2153	0.8130	0.3100	-0.1870
	100	0.9469	0.2894	-0.1531	0.8612	0.2648	-0.1388
β	200	1.0077	0.2298	-0.0923	0.9268	0.2081	-0.0732
	400	1.0337	0.1653	-0.0663	0.9621	0.1488	-0.0379
	800	1.0636	0.1172	-0.0364	0.9873	0.0951	-0.0127
	1000	1.0737	0.0913	-0.0263	0.9907	0.0798	-0.0093
	25	2.8782	2.0091	1.3782	3.0075	2.9668	2.0075
	50	2.4305	1.4320	0.9305	2.5782	2.2024	1.5782
	100	2.2442	1.1478	0.7442	2.1886	1.7264	1.1886
λ	200	1.9503	0.8102	0.4503	1.7452	1.1217	0.7452
	400	1.8119	0.5772	0.3119	1.4616	0.7597	0.4616
	800	1.6766	0.3879	0.1766	1.2337	0.4634	0.2337
	1000	1.6274	0.3250	0.1274	1.1665	0.3893	0.1665
	25	2.6701	2.6410	1.5701	2.9833	3.9787	2.0833
	50	2.3715	1.8348	1.2715	2.5426	2.1073	1.6426
	100	2.0826	1.7178	0.9826	2.2745	2.0097	1.3745
c	200	1.6754	1.3790	0.5754	1.7600	1.5762	0.8600
	400	1.4741	0.9297	0.3741	1.4197	1.1446	0.5197
	800	1.3180	0.6384	0.2180	1.1350	0.7098	0.2350
	1000	1.2605	0.5092	0.1605	1.0595	0.5971	0.1595
		$\alpha=1.1,\beta=1.5,\lambda=0.9,c=1.1$			$\alpha=1.0,\beta=0.9,\lambda=1.0,c=0.9$		
	25	0.9414	1.3843	-0.1586	0.8803	1.0931	-0.1197
	50	0.8392	0.9418	-0.2608	0.7589	0.8029	-0.2411
	100	0.8497	0.8119	-0.2503	0.7232	0.5468	-0.2768
α	200	0.8751	0.6733	-0.2249	0.7517	0.4154	-0.2483
	400	0.9233	0.5315	-0.1767	0.8211	0.3367	-0.1789
	800	1.0137	0.4518	-0.0863	0.8871	0.2439	-0.1129
	1000	1.0340	0.4206	-0.0660	0.9209	0.2185	-0.0791
	25	1.8679	0.7760	0.3679	0.6926	0.3115	-0.2074
	50	1.7416	0.5930	0.2416	0.7050	0.2879	-0.1950
	100	1.6756	0.4820	0.1756	0.7525	0.2565	-0.1475
β	200	1.6521	0.3916	0.1521	0.7895	0.2225	-0.1105
	400	1.6211	0.3205	0.1211	0.8284	0.1683	-0.0716
	800	1.5767	0.2530	0.0767	0.8686	0.1063	-0.0314
	1000	1.5605	0.2270	0.0605	0.8756	0.0936	-0.0244
	25	2.7571	2.7198	1.8571	2.6099	2.3171	1.6099
	50	2.5146	2.3418	1.6146	2.3455	1.8754	1.3455
	100	2.2513	2.0824	1.3513	1.9934	1.3621	0.9934
λ	200	1.8883	1.6188	0.9883	1.6985	0.9996	0.6985
	400	1.5498	1.2004	0.6498	1.4514	0.7178	0.4514
	800	1.2673	0.8589	0.3673	1.2390	0.4494	0.2390
	1000	1.1892	0.7238	0.2892	1.1732	0.3883	0.1732
	25	2.0189	4.2126	0.9189	2.7413	3.1552	1.8413
	50	1.7098	2.4929	0.6098	2.6147	2.1555	1.7147
	100	1.4663	1.0845	0.3663	2.3388	2.0413	1.4388
c	200	1.2365	0.7964	0.1365	2.0201	1.8711	1.1201
	400	1.1102	0.7889	0.0102	1.6168	1.4018	0.7168
	800	1.0299	0.5098	-0.0701	1.2507	0.8697	0.3507
	1000	1.0471	0.4529	-0.0529	1.1567	0.7870	0.2567

Model package in R software R Software (2014) and the following goodness-of-fit statistics were considered: Cramer-von-Mises (W^*) and Andersen-Darling (A^*), -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic (and its p-value), and sum of squares (SS). The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

The OPGW-WP distribution was compared to the following models: odd Weibull-Topp-Leone-log-logistic Poisson (OW-TL-LLoGP), odd Weibull-Topp-Leone-log-logistic geometric (OW-TL-LLoGG) and odd Weibull-Topp-Leone-log-logistic logarithmic (OW-TL-LLoGL) by Oluyede et al. (2020b), exponentiated half-logistic-power generalized Weibull-log-logistic (EHL-PGW-LLoG) by Oluyede et al. (2020a), odd exponentiated half-logistic-Burr XII (OEHL-BXII) by Aldahlan and Afify (2018), exponentiated half-logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) by Chipepa et al. (2020a), odd generalized half-logistic Weibull-Weibull (OGHLW-W) by Chipepa et al. (2020b), odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. (2018), Kumaraswamy odd Lindley-log logistic (KOL-LLoG) by Chipepa et al. (2019) and Kumaraswamy-Weibull (Kw-W) by Cordeiro et al. (2010). The pdfs of the non-nested models are

$$\begin{aligned} f_{OW-TL-LLoGP}(x;\alpha,\lambda,\gamma,\theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}[1-(1+x^{\lambda})^{-2}]^{\gamma\alpha-1}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]^{\alpha+1}} \\ &\times \exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\} \\ &\times \frac{\exp\left(\theta\left(\exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\}\right)\right)}{\exp(\theta)-1}, \end{aligned}$$

for $\alpha,\lambda,\gamma,\theta>0$,

$$\begin{aligned} f_{OW-TL-LLoGG}(x;\alpha,\lambda,\gamma,\theta) &= \frac{2(1-\theta)\gamma\alpha\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}[1-(1+x^{\lambda})^{-2}]^{\gamma\alpha-1}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]^{\alpha+1}} \\ &\times \exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\} \\ &\times \left(1-\left(\theta\left(\exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\}\right)\right)\right)^{-2}, \end{aligned}$$

for $\alpha,\lambda,\gamma>0$ and $0<\theta<1$,

$$\begin{aligned} f_{OW-TL-LLoGL}(x;\alpha,\lambda,\gamma,\theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}[1-(1+x^{\lambda})^{-2}]^{\gamma\alpha-1}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]^{\alpha+1}} \\ &\times \exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\} \\ &\times \frac{\left(1-\left(\theta\left(\exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-(1-(1+x^{\lambda})^{-2})^{\gamma}]}\right]^{\alpha}\right\}\right)\right)\right)^{-1}}{-\log(1-\theta)}, \end{aligned}$$

for $\alpha,\lambda,\gamma>0$ and $0<\theta<1$,

$$\begin{aligned} f_{EHL-PGW-LLoG}(x;\alpha,\beta,\delta,c) &= 2\alpha\beta\delta\left[1+\left(\frac{1-(1+x^{c})^{-1}}{(1+x^{c})^{-1}}\right)^{\alpha}\right]^{\beta-1} e^{\left(1-\left[1+\left(\frac{1-(1+x^{c})^{-1}}{(1+x^{c})^{-1}}\right)^{\alpha}\right]^{\beta}\right)} \\ &\times \left((1+x^{c})^{-1}\right)^{-(\alpha+3)}\left(1+e^{\left(1-\left[1+\left(\frac{1-(1+x^{c})^{-1}}{(1+x^{c})^{-1}}\right)^{\alpha}\right]^{\beta}\right)}\right)^{-2} \\ &\times \left[\frac{1-e^{\left(1-\left[1+\left(\frac{1-(1+x^{c})^{-1}}{(1+x^{c})^{-1}}\right)^{\alpha}\right]^{\beta}\right)}}{1+e^{\left(1-\left[1+\left(\frac{1-(1+x^{c})^{-1}}{(1+x^{c})^{-1}}\right)^{\alpha}\right]^{\beta}\right)}}\right]^{\delta-1} cx^{c-1}\left(1-(1+x^{c})^{-1}\right)^{\alpha-1}, \end{aligned}$$

for $\alpha,\beta,\delta,c>0$,

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b)=\frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^{a})^{b}])(1-\exp(\lambda[1-(1+x^{a})^{b}]))^{\alpha-1}}{(1+x^{a})^{-b-1}(1+\exp(\lambda[1-(1+x^{a})^{b}]))^{\alpha+1}},$$

for $\alpha,\lambda,a,b>0$,

$$\begin{aligned} f_{EHLOW-TL-BXII}(x;\alpha,\beta,\delta,\lambda,\gamma) &= \frac{4\alpha\beta\delta\lambda\gamma x^{\lambda-1}(1+x^{\lambda})^{-2\gamma-1}[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha\beta-1}}{(1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha})^{\beta+1}} \\ &\times \exp(-t)\big(1+\exp(-t)\big)^{-2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta-1}, \end{aligned}$$

where $t=\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}$, for $\alpha,\beta,\delta,\lambda,\gamma>0$ (We obtain the EHLOW-TL-LLoG distribution from the EHLOW-TL-BXII distribution by setting $\gamma=1$),

$$f_{OGHLW-W}(x;\alpha,\beta,\lambda,\gamma)=\frac{2\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\beta-1}\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}}{e^{-(\beta+1)\lambda x^{\gamma}}\left(1+\exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}\right]^{\beta}\right\}\right)^{2}},$$

for $\alpha,\beta,\lambda,\gamma>0$,

$$f_{OLLEW}(x;\alpha,\beta,\gamma,\theta)=\frac{\theta\beta\gamma x^{\beta-1}e^{-(x/\alpha)^{\beta}}[1-e^{-(x/\alpha)^{\beta}}]^{\gamma\theta-1}(1-[1-e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta-1}}{\alpha\beta([1-e^{-(x/\alpha)^{\beta}}]^{\theta\gamma}+(1-[1-e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta})^{2}},$$

for $\alpha,\beta,\lambda,\gamma,\theta>0$,

$$\begin{aligned} f_{KOL-LLoG}(x;a,b,\lambda,c) &= ab\left[\frac{\lambda^2}{(1+\lambda)}\frac{cx^{c-1}}{(1+x^c)^{-1}}\exp(-\lambda z)\right] \\ &\times \left[1-\frac{\lambda+((1+x^c)^{-1})}{(1+\lambda)((1+x^c)^{-1})}\exp(-\lambda z)\right]^{a-1} \\ &\times \left(1-\left[1-\frac{\lambda+((1+x^c)^{-1})}{(1+\lambda)((1+x^c)^{-1})}\exp(-\lambda z)\right]^{a}\right)^{b-1}, \end{aligned}$$

where $z=\frac{(1-(1+x^c)^{-1})}{((1+x^c)^{-1})}$, $a,b,\lambda,c>0$, and

$$f_{Kw-W}(x;a,b,\alpha,\beta)=ab\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1}(1-(1-e^{-(\alpha x)^{\beta}})^{a})^{b-1},$$

for $a,b,\alpha,\beta>0$.

Data analysis results are shown in Tables 3 and 4. A histogram of data, fitted densities and probability plots are shown in Figures 5 and 6.

5.1. Carbon Fibres Data

The data set consists of 66 observations on breaking stress of carbon fibres (Gba). The data set was reported by Nichols and Padgett (2006). The observations are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The estimated variance-covariance matrix is

$$\begin{bmatrix} 0.1944 & -1.0502\times10^{-3} & -0.0340 & -0.1122 \\ -0.0010 & 3.7606\times10^{-5} & -0.0008 & -0.0094 \\ -0.0340 & -8.5370\times10^{-4} & 0.0629 & 0.2165 \\ -0.1122 & -9.4433\times10^{-3} & 0.2165 & 4.2044 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha\in[1.1232\pm0.8643]$, $\beta\in[0.0096\pm0.0120]$, $\lambda\in[2.8341\pm0.4918]$ and $\theta\in[4.3616\pm4.0189]$.

Table 3: MLEs and goodness-of-fit statistics

Model	Estimates				Statistics							
	α	β	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
OPGW-WP	1.1232	0.0096	2.8341	4.3616	282.3	290.3	290.7	300.7	0.0629	0.3926	0.0609	0.8526
	(0.4409)	(0.0061)	(0.2509)	(2.0504)								
	α	λ	γ	θ								
OW-TL-LLoGP	6.3768	0.2383	4.3496	14.3196	282.6	290.6	291.0	301.0	0.0681	0.3966	0.0650	0.7924
	(8.7731)	(0.2573)	(3.4799)	(27.1783)								
OW-TL-LLoGG	2.3977	0.5925	5.4260	3.0075×10^{-13}	282.9	290.9	291.3	301.3	0.0725	0.4300	0.0636	0.8133
	(5.9558)	(1.0870)	(6.9440)	(2.0297)								
OW-TL-LLoGL	3.0041	0.4891	4.6641	1.0180×10^{-10}	282.8	290.8	291.2	301.2	0.0684	0.4186	0.0615	0.8438
	(3.0503)	(0.4251)	(2.8576)	(0.0010)								
	α	β	δ	c								
EHL-PGW-LLoG	1.2499	0.6264	2.6141	1.4037	286.8	294.8	295.2	305.2	0.1568	0.7964	0.1003	0.2664
	(63.3087)	(0.2445)	(0.9515)	(71.1000)								
	α	λ	θ	-								
OEHL-BXII	0.3078	0.0019	11.9671	0.4005	318.6	326.6	327.1	337.1	0.2041	1.4189	0.1301	0.0679
	(0.0616)	(0.0024)	(0.0016)	(0.0666)								
	b	β	δ	c								
EHLOW-TL-LLoG	3.8346	2.3341	1.3504	0.4819	282.4	290.4	290.8	300.8	0.0626	0.3766	0.0618	0.8392
	(5.5094)	(6.3418)	(0.8344)	(1.2822)								
	α	β	λ	γ								
OGHLW-W	2.4257×10^{-5}	0.4640	18.7820	0.2151	287.0	295.0	295.4	305.4	0.0699	0.5971	0.0635	0.8141
	(7.2507×10^{-6})	(4.5353×10^{-3})	(1.1192×10^{-4})	(0.0122)								
	α	β	γ	θ								
OLLEW	3.4848	2.5562	0.6938	1.4692	282.4	290.4	290.8	300.8	0.0659	0.3865	0.0631	0.8208
	(3.1200)	(1.3638)	(1.4331)	(1.5554)								
	a	b	λ	c								
KOL-LLoG	2.1807	8.9816	0.2946	1.1641	282.6	290.6	291.0	301.0	0.0684	0.3994	0.0646	0.7982
	(5.7138)	(75.5774)	(0.5355)	(2.5688)								
	a	b	α	β								
Kw-W	73.5730	3.6270×10^{3}	109.0600	0.1408	282.9	290.9	291.3	301.3	0.0804	0.4446	0.0688	0.7313
	(6.3506)	(0.0017)	(0.8936)	(0.0063)								

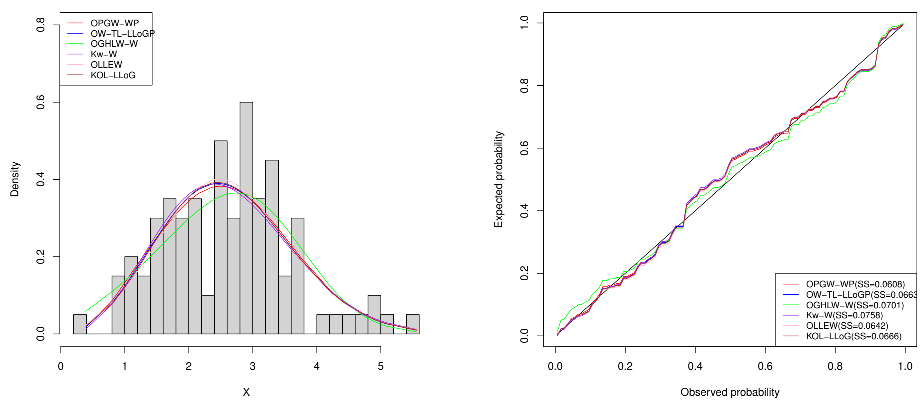

Figure 5: Fitted pdfs and probability plots for carbon fibres data set

Table 3 shows results for the various models fitted for carbon fibres data set. From the given results, we conclude that the OPGW-WP distribution is a good model compared to the selected models since it has the lowest values for the goodness-of-fit statistics: $-2\log L$, AIC, $AICC$, BIC, A^*, W^* and K-S (and the largest p-value for the K-S statistic). Also, from fitted densities and probability plots shown in Figure 5, we observe that the OPGW-WP model fit the data set better than the other models because it has the lowest value for the SS statistic.

5.2. Strengths of 1.5 cm Glass Fibres Data

The second data set represents strengths of 1.5 cm glass fibres. The data set was also analysed by Bourguignon et al. (2014) and Chipepa et al. (2020c). The data are 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

The estimated variance-covariance matrix is

$$\begin{bmatrix} 0.0318 & -5.4363\times10^{-4} & -0.0540 & -0.0432 \\ -0.0005 & 9.0349\times10^{-5} & -0.0029 & -0.0192 \\ -0.0540 & -2.9252\times10^{-3} & 0.4700 & 0.4033 \\ -0.0432 & -1.9221\times10^{-2} & 0.4033 & 7.5440 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [0.4327 \pm 0.3500]$, $\beta \in [0.0149 \pm 0.0186]$, $\lambda \in [6.6097 \pm 1.3438]$ and $\theta \in [6.02271 \pm 5.3834]$.

Table 4: MLEs and goodness-of-fit statistics

	Estimates				Statistics							
Model	α	β	λ	θ	$-2\log L$	*AIC*	*AICC*	*BIC*	W^*	A^*	K-S	p-value
OPGW-WP	0.43272	0.0149	6.6097	6.0271	25.3	33.3	34.0	41.9	0.1070	0.6061	0.1195	0.3298
	(0.1786)	(0.0095)	(0.6856)	(2.7466)								
	α	λ	γ	θ								
OW-TL-LLoGP	54.4878	0.0638	2.7087	488.4785	30.4	38.4	39.1	47.0	0.2382	1.3088	0.1527	0.1058
	(12.3996)	(0.0176)	(0.0756)	(0.5924)								
OW-TL-LLoGG	4.7219	0.6701	3.5777	1.8173×10^{-9}	31.1	39.1	39.8	47.7	0.2572	1.4103	0.1636	0.0686
	(2.4099)	(0.2953)	(0.6537)	(0.5338)								
OW-TL-LLoGL	4.1499	0.7465	3.7542	5.2222×10^{-8}	31.3	39.3	40.0	47.9	0.2634	1.4437	0.1642	0.0669
	(3.0503)	(0.4251)	(2.8576)	(0.0010)								
	α	β	δ	c								
EHL-PGW-LLoG	2.0377	0.6397	2.1273	1.7395	39.3	47.3	48.0	55.9	0.4178	2.2961	0.2077	0.0087
	(0.2532)	(0.1854)	(0.6324)	(0.2966)								
	α	λ	θ	-								
OEHL-BXII	0.3225	0.0030	11.8172	0.8356	50.3	58.3	59.0	66.9	0.2417	1.3747	0.1423	0.1558
	(0.0670)	(0.0036)	(0.0075)	(0.1347)								
	b	β	δ	c								
EHLOW-TL-LLoG	1.1293	0.1464	4.3716	7.8796	34.9	42.9	43.6	51.4	0.3373	1.8409	0.1868	0.0246
	(0.7335)	(0.0736)	(1.0252)	(3.8531)								
	α	β	λ	γ								
OGHLW-W	3.0734×10^{-5}	0.5007	16.9910	0.4785	27.1	35.1	35.8	43.6	0.1372	0.7816	0.1284	0.2500
	(3.2131×10^{-6})	(2.1967×10^{-9})	(6.4740×10^{-11})	(6.2517×10^{-10})								
	α	β	γ	θ								
OLLEW	1.9920	8.7485	0.3021	1.6872	28.0	36.0	36.7	44.6	0.1864	1.0314	0.1320	0.2223
	(0.2975)	(3.9396)	(0.2668)	(0.7436)								
	a	b	λ	c								
KOL-LLoG	0.5532	25.4210	0.0038	5.9116	27.4	35.4	36.1	44.0	0.1507	0.8450	0.1293	0.2429
	(0.0577)	(6.0680×10^{-6})	(0.0012)	(0.0066)								
	a	b	α	β								
Kw-W	1.04695	641.1439	0.2009	5.5116	30.4	38.4	39.1	47.0	0.2372	1.3038	0.1522	0.1082
	(3.9411)	(0.0539)	(0.0228)	(20.5514)								

Furthermore, from the results shown in Table 4, we conclude that the OPGW-WP distribution is indeed a better model compared to several selected models since it is associated with the lowest values for all the the goodness-of-fit statistics (and the largest p-value for the K-S statistic). We also observe from Figure 6 that the OPGW-WP model fit the data set better than the other models that were considered.

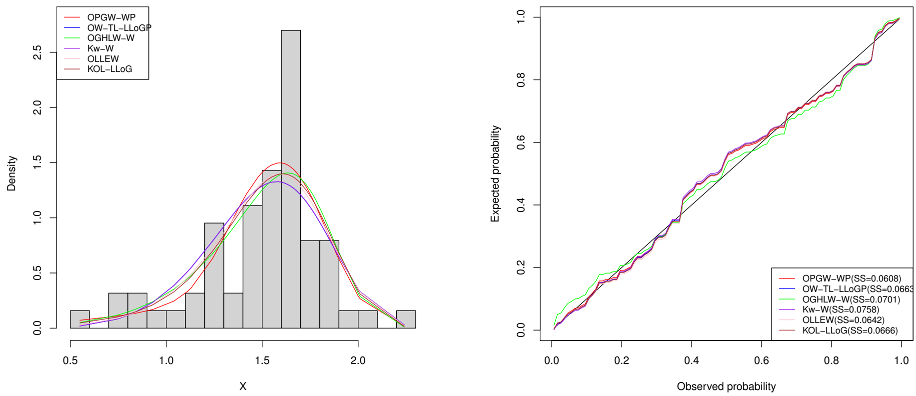

Figure 6: Fitted pdfs and probability plots for glass fibres data set

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Appendix

Useful expansions

$$\exp\left(n\left(1-\left[1+\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}\right]^{\beta}\right)\right) = \sum_{j=0}^{\infty}\frac{\left(n\left(1-\left[1+\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}\right]^{\beta}\right)\right)^{j}}{j!},$$

$$\left(1-\left[1+\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}\right]^{\beta}\right)^{j} = \sum_{k=0}^{\infty}\binom{j}{k}(-1)^{k}\left[1+\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}\right]^{\beta k},$$

and

$$\left[1+\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha}\right]^{\beta(k+1)-1} = \sum_{i=0}^{\infty}\binom{\beta(k+1)-1}{i}\left(\frac{G(x;\underline{\psi})}{1-G(x;\underline{\psi})}\right)^{\alpha i}.$$