ARTICLES

CENTRAL EUROPEAN REVIEW OF ECONOMICS & FINANCE
Vol. 43. No 2 (2023) pp. 5-16
DOI https://doi.org/10.24136/ceref.2023.006

Ewa Falkiewicz[1]

DIFFERENTIAL GEOMETRY APPROACH IN OPTIMIZATION OF THE BUSINESS PRODUCTION PROCESS

Abstract
In this work we show a new approach to the optimization of the production process - from a differential geometry point of view. It is known ([2]) analytical conditions of profit maximization and minimization of the cost in an enterprise. In the first part of this work, we show such a classical approach. In the second part of the work, we use geometrical methods to obtain a new geometrical approach to the production process.
Key words: production function, profit function, cost function, extremes of function, differential space, differential manifold

JEL codes: D20, D24, D29

1. Production process optimization

Let consider the long-term optimization issues of the business production process. The company wants to maximize profits or minimize production costs. Our purpose in this section is to find the answer to the question – which conditions should be filled?

1.1. Production function

We will denote by $x=(x_1,...,x_n)$ n - dimensional vector of inputs, where $x_i \geq 0$ for $i=1,...,n,$ by $y=(y_1,...,y_n)$ n - dimensional vector of products, where $y_i \geq 0$ for $i=1,...,n$, $n \in \mathbf{N}$. By the production process

[1] PhD, assistant professor, Mathematics Department, University of Technology and Humanities in Radom, e-mail: e.falkiewicz@uthrad.pl

we understand such set of activities as a result of which a given bundle of inputs is transformed into a specific bundle of products. The production process is described by not negative, 2n - dimensional vector $(x, y) \geq 0$, in which x is n - dimensional vector of inputs needed to produce n - dimensional vector of products y, $n \in \mathbf{N}$. Vectors x and y form a technologically acceptable production process. A set $Z \subset \mathbf{R}^{2n}$ of all technologically acceptable production processes with the norm defined by the formula:

$$\|z\| = \max\{x_1, ..., x_n, y_1, ..., y_n\} \text{ for } z \in Z$$

we call ***p* – production space**.

The production process $(x, y) \in Z$ we call **technologically effective**, if there is not exists another production process $(x, y') \in Z$ such that $y' > y$. ([1], s. 74)

With the technologically effective production process a vector – valued production function is associated. We define it in the following way:

Definition 1. *If there exists a vector – valued function $f: \boldsymbol{R}_+^n \to \boldsymbol{R}_+^n$, such that $y = f(x)$, if and only if the production process $(x, y) \in Z$ is technologically effective, then the function f is called* ***vector – valued production function associated with p – production space Z.***

We will be concerned only with the case of production process in which manufacturer produce only one product using to this *k* inputs, $k \leq n$. In such situation vector – valued production function defined above reduces to **the inner, *k* - arguments production function** $f: \boldsymbol{R}_+^k \to \boldsymbol{R}$, $k \leq n$, which is associated with the *p* – production space $Z \subset \boldsymbol{R}^{k+1}$. It is a function which maps each not negative vector of inputs $x = (x_1, \ldots, x_k)$ to such result of production $y = f(x)$, that the pair (x, y) makes technologically effective production process.

We will make the following assumptions about the inner, *k* - arguments production function $f: \boldsymbol{R}_+^k \to \boldsymbol{R}$:

- (F1) *f* is continue and second order differentiable on the interior of its domain int $\boldsymbol{R}_+^k$;
- (F2) $f(0, \ldots, 0) = 0$ – zero vector of inputs give zero production result;
- (F3) *f* is increasing function on int $\boldsymbol{R}_+^k$, i.e. each any small increase of inputs induces production increase;

(F4) *f* is concave function on int $\boldsymbol{R}_{+}^{k}$, i.e.

$\forall x^1, x^2 \in R_{+}^{k}, \forall \alpha, \beta \geq 0, \alpha + \beta = 1: (f(\alpha x^1 + \beta x^2) \geq \alpha f(x^1) + \beta f(x^2)),$

that is $\left(\frac{\partial^2 f}{\partial x^2}\right)_{k \times k}$ is not positive matrix on $\boldsymbol{R}_{+}^{k}$;

(F5) function *f* is positive homogenous of rank $\theta > 0$, i.e.

$$\forall x \in R_{+}^{k}, \forall \lambda \geq 0: (f(\lambda x) = \lambda^{\theta} f(x)).$$

1.2. Profit maximization problem

In this section, we will be concerned with the problem of profit maximization in the enterprise. Suppose a company operates in the long run (long–term strategy). This postulate implies that it has unlimited freedom to determine the size and structure of the factors of production involved.

Let $x = (x_1, \dots, x_k)$ be inputs vector of productive factors expressed in physical units, where each productive factor satisfies the condition $x_i \geq 0$ for $i = 1, \dots, k$. Let further $w = (w_1, \dots, w_k)$ be cost vector of productive factors. We will denote by $\langle \mathrm{w}, \mathrm{x} \rangle$ the inner product of vectors *x* and *w*:

$$\langle w, x \rangle = w \circ x = \sum_{\mathrm{i}=1}^{\mathrm{k}} w_{\mathrm{i}} x_{\mathrm{i}} = w_1 x_1 + \ldots + w_k x_k .$$

Let us consider the function $f: \boldsymbol{R}_{+}^{k} \to \boldsymbol{R}_{+}$ satisfying the conditions (F1)-(F5). We will denote by $f(x)$ quantity of fabricated product (in physical units), by *p* – the price of this product.

Define profit function $Z: \boldsymbol{R}_{+}^{k} \to \boldsymbol{R}_{+}$ of the company by the formula:

$$Z(x) = pf(x) - \langle \mathrm{w}, \mathrm{x} \rangle, \tag{1}$$

where $pf(x)$ is product sale income, $\langle \mathrm{w}, \mathrm{x} \rangle$ denotes the cost of manufactured product.

Aur purpose is to maximize the company's profit when the production process is described by inner, *k* - arguments production function $f: \boldsymbol{R}_{+}^{k} \to \boldsymbol{R}$ determined in the definition 1. We are looking for such a vector of inputs of production factors $x^* \geq 0$, for which profit function *Z* of the company, given by formula (1), riches its maximum.

A necessary and sufficient condition for the existence of an optimal solution $x = x^*$ is given by the following theorem:

Theorem 1. *Let $f: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ be inner, k - arguments production function satisfying the conditions (F1)-(F5). Let further the product price p>0 and the production factor price vector w>0 satisfy the condition*

$$\lim_{x\to\infty} p\frac{\partial f(x)}{\partial x} < w < p\frac{\partial f(x)}{\partial x}\bigg|_{x=0}.$$

Then the optimal solution to the problem of maximizing company's profit is the positive input vector of production factors $x^* > 0$, and a necessary and sufficient condition for the existence of an optimal solution is

$$p\frac{\partial f(x)}{\partial x}\bigg|_{x=x^*} = w,$$

or otherwise

$$\frac{\partial Z(x)}{\partial x}\bigg|_{x=x^*} = 0$$

(see [1]).

Remark 1. *The assumption of a strong concavity of the production function f does not guarantee the existence of an optimal solution to the problem of maximizing the company's profit. For if the production factor price vector satisfies the system of inequalities:*

$$\frac{\partial Z(x)}{\partial x}\bigg|_{x=x^*} \leq 0,$$

then the optimal solution to our task is the zero input vector $x^* = 0$. On the other hand, when the system of inequalities is satisfied

$$0 \leq \lim_{x\to\infty} \frac{\partial Z(x)}{\partial x}\bigg|_{x=x^*},$$

the problem of maximizing the company's profit does not have a finite optimal solution.

1.3. Problem of minimizing production costs

Let $K: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ be linear function of enterprise costs given by the formula

$$K(x) = \langle \mathrm{w}, \mathrm{x} \rangle = w \circ x = w_1x_1 + \ldots + w_kx_k, \tag{2}$$

where $x = (x_1, \ldots, x_k)$ is the input vector of the factors of production of the enterprise, $w = (w_1, \ldots, w_k)$ is the production factor price vector. We are interested in minimizing production costs. Approaching the problem analytically we want to determine the minimum of the cost function *K* with the existing constraints:

$$f(x) = \text{y=const.}>0, \qquad (3)$$
$$x \geq 0,$$

where *y*=const.>*0* is fixed production level, *f* is production function.

The condition for the existence of a minimum of the cost function *K* under constraints (3) is given by the following Theorem:

Theorem 2. *Vector* $x^* > 0$ *is the optimal solution for the task of minimizing the costs of the company if and only if there exists a constant* $\lambda^* > 0$ *such that the pair* (x^*, λ^*) *satisfies the system of equations:*

$$\left.\frac{\partial f(x)}{\partial x_i}\right|_{x=x^*} = \lambda^* w_i, \qquad where\ \ i = 1, \ldots, k.$$

The above Theorem implies that cost minimization of the company takes place when the marginal productivity of *i* – the factor of production x_i, expressed in the form of partial derivative $\frac{\partial f(x)}{\partial x_i}$ of production function *f*, is proportional to the price w_i of this factor.

2. Geometrical aspects of function extremes

Our starting point are Sikorski differential spaces [4, 5, 6], which are subsets of Cartesian space $\boldsymbol{R}^n$ and differential manifolds, that is topological spaces locally homeomorphic to open subsets of $\boldsymbol{R}^n$. On such spaces we will develop the theory of geometrization of production process of the enterprise.

Sikorski differential space (Sikorski, 1972) is a ringed space (M, C), where M is set of points and C is **differential structure** on *M*, i.e. a set of real valued functions on M, $f: M \to \boldsymbol{R}$ satisfying the following conditions:

1. C is closed with respect to localization, $C_M = C$, where C_M is the set of all local C - functions in the weakest topology τ_C in which all functionf from C are continous;
2. C is closed with respect to superpositions with smooth functions, $scC = C$, where

$$\mathrm{sc}C \coloneqq \{\omega \circ (f_1, \ldots, f_n): n \in \boldsymbol{N}, \omega \in C^\infty(\boldsymbol{R}^n), f_1, \ldots, f_n \in C\}.$$

A mapping $f:(M,C)\to(N,D)$ is **smooth**, if for any function $\alpha\in D$,

$$\alpha\circ f\in D,$$

where (M,C) and (N,D) are Sikorski differential spaces.

Functions belonging to a differential structure C are by definition smooth (C^∞ class) on M.

Differential manifolds of dimension *n* we call Hausdorff space M locally homeomorphic with $\boldsymbol{R}^n$, i.e. for any point $p\in M$ there exists surrounding $U\ni p$ and homeomorphism $x:U\to x(U)$ on the open subset of $\boldsymbol{R}^n$.

For any point $p\in M$, where *M* is differential space or differential manifold, by T_pM we will denote the set of all vectors tangent to *M* at a point *p*. We say that any mapping $v:C\to\boldsymbol{R}$ is **vector tangent to *M*** if *v* is ***R*** – linear and satisfies the Leibniz rule:

$$v(f\cdot g)=v(f)\cdot g(p)+f(p)\cdot v(g)\ \text{ for } f,g\in C. \tag{4}$$

2.1. The problem of the extreme on a differential manifold

Let *M* be a differential manifold, $C^\infty(M)$ – let be an algebra of all smooth functions (i.e. functions having derivatives of any order, that are continuous functions) on *M*. Let's take a function $f\in C^\infty(M)$, $f:M\to\boldsymbol{R}$ and a point $p\in M$. By the symbol $(df)_p$ we will denote the differential of function f at a point $p\in M$:

$$(df)_p:T_pM\to\boldsymbol{R},$$

which is given by the formula:

$$(df)_p(v)=v(f),$$

where $v\in T_pM$ is a vector tangent to differentiable manifold *M* at a point *p,* i.e. ***R*** – linear mapping which satisfies the Leibniz rule in formula (4).

Tangent vectors and differentials are related to a certain map. A collection of maps is called an atlas. Let $x=(x^1,\dots,x^n)$ be a map in a surrounding of a point $p\in M$ and $x^1,\dots,x^n$ let be coordinates of this map. Then the following equalities are true:

$$\left.\frac{\partial}{\partial x^i}\right|_p\in T_pM\ \text{for } i=1,\dots,n$$

And

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = (f \circ x^{-1})'_i(x(p)) = \partial_i(f \circ x^{-1})(x(p)) = \frac{\partial f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}\bigg|_p.$$

The differentials $(dx^i)\big|_p$ are basis covectors for $i = 1,...,n$. Any differential of a function *f* a point *p* can be expressed by basis covectors:

$$(df)_p = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\bigg|_p \cdot (dx^i)\big|_p. \tag{5}$$

The coordinates $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ form a gradient of a function *f* at a point *p*, i.e. vector of partial derivatives of function *f* at a point *p:*

$$\text{grad}\, f_p = (\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p).$$

Theorem 3. *If a function* $f \in C^\infty(M)$ *reaches an extreme at a point* $p \in M$*, then*

$$(df)_p = 0, \tag{6}$$

i.e. the differential of a function f disappears at that point.

Proof: Let's take the manifold $M = \boldsymbol{R}^n$, on which the atlas in a surrounding of a point $p \in \boldsymbol{R}^n$ is in the form of single map $x = (x^1, \dots, x^n) = \mathrm{id}_{\boldsymbol{R}^n}$. Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a smooth function on $\boldsymbol{R}^n$, i.e. $f \in C^\infty(R^n)$. If a function *f* possesses a local extreme at a point *p*, then it is well known that the necessary condition for the existence of an extremum of a function *f* at a point *p* is that the partial derivatives of a function *f* at this point are equal to zero:

$$\frac{\partial f}{\partial x^i}(p) = 0,\ i = 1, \dots, n. \tag{7}$$

The condition (7) implies

$$\sum_{i=1}^{n} \frac{\partial}{\partial x^i}\bigg|_p \cdot (dx^i)\big|_p = 0, \tag{8}$$

From the formulas (8) and (5) we obtain

$$(df)_p = 0.$$

We have proved that the Theorem 3 is true for the manifold being a Cartesian space $M = \boldsymbol{R}^n$.

Now let's consider the case where M is any differential manifold. Let $f: M \to \boldsymbol{R}$ be smooth function on M, $f \in C^\infty(M)$. If $x = (x^1, \dots, x^n)$ is a map on M, then $f \circ x^{-1}$ is smooth function on the open submanifold $(x(U), \{\mathrm{id}_{x(U)}\})$. From this fact we obtain that

$$\partial_i(f \circ x^{-1})\big(x(p)\big) = 0 \ \text{ for } \ i = 1, \dots, n. \tag{9}$$

The condition (9) is equivalent to the fact that for $= 1, \dots, n$:

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = 0. \tag{10}$$

From the equality (10) finally we have:

$$(df)_p = 0,$$

which means that the differential of the function *f* at a point *p* disappears. ■

Corollary 1. *The vanishing of the differential of the function f at a point $p \in M$ means the perpendicularity of the gradient of the function f at a point $p \in M$ to the manifold M on the Cartesian space $\boldsymbol{R}^n$, and thus the perpendicularity of the gradient to the tangent vector $v \in T_pM$, what can be expressed:*

$$(\mathrm{grad}\, f)\ (p) \perp v. \tag{11}$$

In other words, the formula (11) means that the scalar product of gradient of the function f at a point $p \in M$ and tangent vector $v \in T_pM$ is equal to zero:

$$(\mathrm{grad}\, f)\ (p) \circ v = 0.$$

2.2. Maximization the company's profit in geometric terms

Let $x = (x_1, \dots, x_k)$ be an input vector of the factors of production of the enterprise, $w = (w_1, \dots, w_k)$ let be a vector of factor prices. Let $f: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ be the production function which satisfies the assumptions (F1)-(F5).

Our aim is to find a geometric condition for a company to maximize its profit. The Theorem 1 gives the condition which must be fulfilled for the profit function *Z* defined by the formula (1) to reach a maximum. However, this theorem is formulated in the language of mathematical analysis. Our task is to solve the profit maximization problem in the language of differential geometry. This is described by the following theorem:

Theorem 4. *If $f: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ is inner, k – arguments production function, continuous, twice differentiable and strongly concave, $Z: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ is inner, k – arguments profit function given by* (1)*, that is continuous, twice differentiable and strongly concave, p is a price of a product, p>0, $w = (w_1, \dots, w_k)$ is positive vector of factor prices, then the optimal solution of the task of profit maximization of a company is positive input vector of the factors of production $x^* > 0$, and the necessary and sufficient condition of the existence of optimal solution is that the gradient of the function f is parallel to the vector of factor prices:*

$$\operatorname{grad} f \,\| w.$$

The condition of parallelism of the gradient of the production function to the vector of factor prices also means that the gradient of the profit function *Z* is zero:

$$\operatorname{grad} Z = 0,$$

which follows directly from the form of the profit function (1). The profit function *Z* is strongly concave (because the production function *f* is strongly concave and the costs function $\langle w, x \rangle$ is strongly concave). Therefore, the profit function *Z* has exactly one global maximum at a point $x^* > 0$.

2.2.1. Minimization of production costs of a company in geometric terms

The theorem 2 gives an analytical condition for minimizing costs in the enterprise.

We now turn on to the formulation of an analogous condition in the language of differential geometry.

Let $K: \boldsymbol{R}_+^k \to \boldsymbol{R}_+$ be linear function of enterprise costs given by the formula (2). We want to find the minimum of the function *K* defined on a differential space M, which is subspace of Cartesian space, $M \subseteq \boldsymbol{R}_+^k$ and is given by constraint:

$$M: f(x) = y = \text{const.} > 0, \tag{12}$$

where $x \geq 0$ is not negative input vector, y=const.>0 is fixed production level and *f* is production function.

The following theorem gives geometric condition for minimizing the company's production costs under constraint (12):

Theorem 5. *The cost function K given by the formula* (2) *has a minimum at point* $x = p \geq 0$ *on differential space* M *defined by condition (12), if the gradient of f is perpendicular to the tangent vector v at point p, i.e.:*

$$\operatorname{grad} f \perp v \ \text{ for } \ v \in T_p M,$$

where $T_p M$ *is space tangent to differential space* M *at point* $p \in M$ *.*

Proof: Differential space given by condition (12) can be expressed as:

$$M = f^{-1}(y).$$

From the condition (12) we know that *f* is constant function, so the gradient of this function is zero vector:

$$\text{grad } f = 0.$$

From the above also differential of gradient of function f is equal to zero:

$$d\,(\text{grad } f) = 0$$

and differential of function *f* at point $p \in M$ is equal to zero:

$$d_p f = 0.$$

From the above we obtain that the value of the differential of function *f* at point $p \in M$ at vector *v* tangent to *M* is also the zero:

$$\left(d_p f\right)(v) = 0 \ \text{ dla } \ v \in T_p M,$$

what can be expressed as follows:

$$\sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}\Big|_p \cdot dx_i|_p\right)(v) = 0. \qquad (13)$$

On the other hand, we have:

$$(dx_i|_p)(v) = v(x_i),$$

So, from equation (13) we get:

$$\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}\bigg|_p \cdot v(x_i) = 0. \tag{14}$$

Equation (13) denotes that the inner product of the vector of partial derivatives of the function *f* at point *p* (i.e. the gradient of the function *f* at point *p*) and the vector tangent to the differential space M at point *p* is zero:

$$\left(\frac{\partial f}{\partial x_1}\bigg|_p, \dots, \frac{\partial f}{\partial x_k}\bigg|_p\right) \circ (v_1, \dots, v_k) = (\mathrm{grad}\, f\,)(p) \circ v = 0,$$

which is equivalent to the fact that the gradient of the function *f* at point *p* is perpendicular to the vector tangent to *M* at this point:

$$\mathrm{grad}\, f \perp v \quad \text{for} \quad v \in T_pM. \ \blacksquare$$

3. Conclusions

The considerations in the paper show that the production function theory can be presented in the language of differential geometry. Moreover, it turns out that the geometric approach simplifies the proofs of profit maximization and costs minimization theorems. Replacing analitical concepts with geoemetric ones makes that it is possible to prove the conditions sufficient for existence of the extremum of the production function in a simple and brief way.

References

1) Panek E., (2000). Ekonomia matematyczna. Wydawnictwo Akadamii Ekonomicznej w Poznaniu, Poznań.
2) Sasin W., Żekanowski Z., (1987). On locally finitely generated differential spaces. Demonstratio Mathematica 20, 477-487.
3) Sikorski R., (1969). Rachunek różniczkowy i całkowy. Funkcje wielu zmiennych. PWN, Warszawa.
4) Sikorski R., (1967). Abstract Covariant Derivativ*,* Colloquium Mathematicum 18, 252-272.

5) Sikorski R., (1971). Differential Modules. Colloquium Mathematicum 24, 46-79.
6) Sikorski R., (1972). Wstęp do geometrii różniczkowej. PWN, Warszawa.