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The length-biased power hazard rate distribution: Some properties and applications

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ABSTRACT

In this article, the length-biased power hazard rate distribution has introduced and investigated several statistical properties. This distribution reports an extension of several probability distributions, namely: exponential, Rayleigh, Weibull, and linear hazard rate. The procedure of maximum likelihood estimation is taken for parameters. Finally, the applicability of the model is explored by three real data sets. To examine, the performance of the technique, a simulation study is extracted.

Key words: length-biased, power hazard rate distribution, maximum likelihood estimation.

1. Introduction

Importance of the statistical distributions in different fields of studies, researchers have shown their curiosity to suggest a new distribution via numerous methods. In pioneering work, Cox (1962) proposed a model dealing with the unequal probability of sample observation termed as length-biased technique. This concept has many applications in biomedical sciences, Lawless (2003).

Several papers have been arisen to investigate the performance of length-biased distributions. For instance see Gupta and Keating (1985), Khattree (1989), Gupta and Tripathi (1990), Oluyede (1999), Das and Roy (2011a,b), Ratnaparkhi and Nimbalkar (2012), Al-Khadim and Hussain (2014), Nanuwong and Bodhisuwan (2014), Seenoi et al. (2014) , Modi (2015), Saghir et al. (2016), Saghir et al. (2017), Mudasir and Ahmad (2018) and Parveen and Ahmad (2018), among others.

The lifetime distributions are always characterized by selecting a specific hazard rate function (HRF). The power HRF is one of them. The HRF is used in many fields of study (reliability analysis, actuarial sciences, demography, and economics). The inference on hazard function for lifetime data has become a prevalent tool for researchers.

The power hazard function (PHF) was introduced by Mugdadi (2005).

$$h(x) = \lambda x^{\nu}, \quad x > 0, \lambda > 0, \nu > -1. \tag{1}$$

In view of (1) cumulative distribution function (cdf) is given

$$F(x) = 1 - e^{-\frac{\lambda}{\nu+1} x^{\nu+1}}, \quad x > 0, \lambda > 0, \nu > -1, \tag{2}$$

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and the probability density function (pdf) is

$$f(x) = \lambda x^{\nu} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}}, \quad x > 0,\ \lambda > 0,\ \nu > -1. \tag{3}$$

If X has pdf (3), we denote it by $X \sim \text{PHRD}(\lambda, \nu)$.

The PHF is very simple, and it could be increasing, decreasing, or constant. Therefore, the PHR distribution contributes a better fit over two-parameter distributions when modelling monotone hazard rates. More explorationon the PHR distribution can be seen in Ismail (2014), Mugdadi and Min (2009), Tarvirdizade and Nematollahi (2016) and Tarvirdizade and Nematollahi (2020). It is important to note that some familiar distributions are special case of (3) reported in Section 2.1.

The paper is organized as follows. The formulation of length-biased PHR distribution (LBPHRD) and its structured properties are discussed in Section 2. Section 3 is devoted to estimating the parameters via the maximum likelihood method. Section 4 reveals the usefulness of the new model and, also simulation study is evaluated to examine the performance of MLEs. The conclusion is presented in Section 5.

2. Length-Biased Power Hazard Rate Distribution

The LBPHR distribution is proposed in this section. The shape of the pdf, hazard rate and some sub-models are established also.

Definition 1. If the random variable X has a pdf $f(x)$ and expected value $E(X) < 0$ then the pdf of the length- biased distribution of X can be formulated as

$$g(x) = \frac{xf(x)}{E(X)}. \tag{4}$$

From (3) and (4), the LBPHR distribution with two parameters λ (scale) and ν (shape) can be obtained as follows

$$g(x) = \frac{\lambda x^{\nu+1} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}}}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)}, \quad x > 0, \tag{5}$$

where $\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} du$ is gamma function.

The graph of the pdf of LBPHRD is shown in Figure 1, for various values of λ and ν.

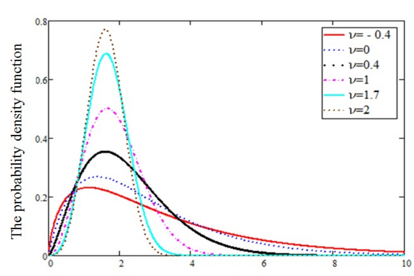

Figure 1. The plot of $g_{LBPHR}(x)$ for $\lambda = 0.73$ and various values of ν.

From Figure 1, the pdf of LBPHRD has one peak, so there is one mode.

The cdf of LBPHR distribution has the form

$$G(x) = \frac{\gamma\left(\frac{\nu+2}{\nu+1}, \frac{\lambda}{\nu+1}x^{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}, \quad x > 0, \tag{6}$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ is an upper incomplete gamma function.

The survival (reliability) function of LBPHRD is given as

$$\bar{G}(x) = \frac{\Gamma\left(\frac{\nu+2}{\nu+1}, \frac{\lambda}{\nu+1}x^{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}, \quad x > 0, \tag{7}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is an incomplete gamma function.

The hazard rate of LBPHRD takes the form

$$h(x) = \frac{\lambda x^{\nu+1} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}}}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)}, \quad x > 0. \tag{8}$$

Derivative the $h(x)$, w.r.t. x,

$$h'(x) = \frac{1}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)} \left[\lambda(\nu+1) - \lambda^2 x^{\nu+1}\right] x^\nu e^{-\frac{\lambda}{\nu+1}x^{\nu+1}},$$

by equating $h'(x)$ by zero, we find $x = 0$ and $x = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}$ are the critical points for $h(x)$.

By using the second derivetives test, we can find

$$h''(x) = \frac{\lambda}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)} \left[\nu(\nu+1) - \lambda(3\nu+2)x^{\nu+1} + \lambda^2 x^{2(\nu+1)}\right] x^{\nu-1} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}}.$$

- At $x = 0$, $h''(x) = 0$, then $x = 0$ is the inflection point.

- At $x = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}$,

$$h''(x) = -\frac{\lambda(\nu+1)^2 x^{\nu-1}}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}} < 0,$$

  then $h(x)$ has a local maximum at $x_0 = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}$.

Some hazard rate plots of the LBPHR distribution with specific parameter values are given in Figure 2.

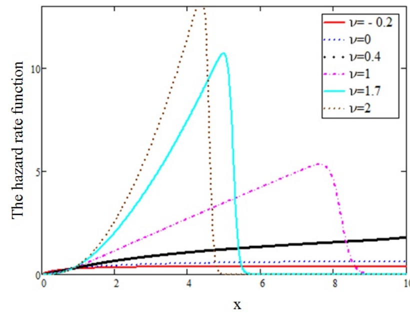

Figure 2. The HRF of the LBPHRD for $\lambda = 0.73$ and various values of ν.

Therefore, the function $h(x) \uparrow$ on the interval $(0, x_0)$ and $h(x) \downarrow$ on the interval (x_0, ∞). From the Figure 2, the hazard function exhibits that proposed model becomes a major tool to fit many lifetime data in (reliability, survival analysis, finance and economics).

2.1. Special cases of LBPHRD

The LBPHRD is very versatile distribution. It covers many noted distribution as follows.

1. Setting $\nu = \lambda - 1$, we obtain the length-biased Weibull (LBW) distribution as obtained by Shaban and Boudrissa (2007).
2. Setting $\nu = 1$, we obtain the length-biased Rayleigh (LBR) distribution with parameter $\frac{1}{\lambda}$ as obtained by Ajami and Jahanshahi (2017).
3. Setting $\nu = 0$, we obtain the length-biased exponential (LBE) distribution as obtained by Mir et al. (2013).
4. Setting $\nu = 1$, we obtain the length-biased linear failure rate (LBLFR) distribution.

The results obtained in this paper can be valid for these distributions and the other distributions which have a power hazard function.

2.2. Statistical properties

Some statistical properties of the LBPHRD are discussed in this section.
Theorem 1. If $X \sim$ LBPHRD(λ, ν) then the rth moment is given as

$$E(X^r) = \frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{r}{\nu+1}} \Gamma\left(\frac{r+\nu+2}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}. \tag{9}$$

Proof. The rth moments of LBPHRD can be attained by

$$E(X^r) = \int_0^\infty x^r g(x) dx,$$

from (5), then

$$E(X^r) = \int_0^\infty \frac{\lambda x^{r+\nu+1} e^{-\frac{\lambda}{\nu+1}x^{\nu+1}}}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)} dx. \tag{10}$$

Let $u = \frac{\lambda}{\nu+1}x^{\nu+1}$, $du = \lambda x^\nu dx$. Upon simplification, (10) leads to

$$E(X^r) = \frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{r}{\nu+1}} \Gamma\left(\frac{r+\nu+2}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}. \tag{11}$$

The mean and variance for LBPHRD can be calculated from (11) as follows. Setting $r = 1$, in (11),

$$E(X) = \frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+3}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}. \tag{12}$$

Putting $r = 2$, in (11),

$$E(X^2) = \frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{2}{\nu+1}} \Gamma\left(\frac{\nu+4}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}. \tag{13}$$

Therefore, variance of LBPHRD is

$$Var(X) = \frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{2}{\nu+1}} \Gamma\left(\frac{\nu+4}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)} - \left[\frac{\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+3}{\nu+1}\right)}{\Gamma\left(\frac{\nu+2}{\nu+1}\right)}\right]^2. \tag{14}$$

The shape characteristics of the probability distribution, skewness and kurtosis play an important role.These can be derived from Theorem 1, using the following relations.

$$S_k = \frac{\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'})^{3/2}}, \qquad K_u = \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6\mu_1^{'2}\mu_2^{'} - 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'})^2},$$

where $\mu_r^{'} = E(X^r)$.

The mode of the LBPHRD:

Taking the logarithm of (5), we have

$$\ln g(x) = \ln(\lambda) + (\nu+1)\ln(x) - \frac{\lambda}{\nu+1}x^{\nu+1} - \ln\left[\left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}} \Gamma\left(\frac{\nu+2}{\nu+1}\right)\right]. \tag{15}$$

Differentiate (15) w.r.t. x and equating it zero,

$$\frac{d}{dx}\ln g(x) = \frac{\nu+1}{x} - \lambda x^\nu = 0, \tag{16}$$

therefore

$$x = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}.$$

Again differentiate (16),

$$\frac{d^2}{dx^2}\ln g(x) = -\frac{\nu+1}{x^2} - \lambda\nu x^{\nu-1} = -\frac{(\nu+1)+\lambda\nu x^{\nu+1}}{x^2},$$

at $x = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}$, then

$$\frac{d^2}{dx^2}\ln g(x) = -\frac{(\nu+1)^2}{x^2} < 0.$$

Therefore, the mode is $x = \left(\frac{\nu+1}{\lambda}\right)^{\frac{1}{\nu+1}}$.

Using the following relation, pth percentile can be obtained

$$G(x_p;\lambda,\nu) = p. \tag{17}$$

Substituting from (6) into (17), x_p satisfies the equation

$$\Gamma\left(\frac{\nu+2}{\nu+1}, \frac{\lambda}{\nu+1}x^{\nu+1}\right) - p\Gamma\left(\frac{\nu+2}{\nu+1}\right) = 0. \tag{18}$$

The pth percentile can be calculated numerically by using Equation (18).

The median can be calculated from Equation (18), at $p = 0.5$.

For $\lambda = 0.5$, $\nu \in (0,5)$, the values of $E(X)$, mode, $Var(X)$, s_k, k_u and CV for LBPHRD and PHRD, respectively are presented in Table 1.

Table 1. Some statistical measures for $\lambda = 0.5, \nu \in (0,5)$.

	LBPHRD						PHRD					
ν	$E(X)$	Mode	$Var(X)$	S_k	K_u	CV	$E(X)$	Mode	$Var(X)$	S_k	K_u	CV
0.0	4.000	2.000	8.000	1.414	6.000	70.71	2.000	0.000	4.000	2.000	9.000	100.00
0.5	2.743	2.080	2.059	0.813	3.780	52.31	1.878	1.000	1.626	1.072	4.390	67.90
1.0	2.257	2.000	0.907	0.486	3.108	42.20	1.772	1.414	0.858	0.631	3.245	52.27
1.5	1.998	1.904	0.507	0.269	2.864	35.64	1.689	1.552	0.522	0.359	2.857	42.78
2.0	1.837	1.817	0.323	0.111	2.786	30.94	1.623	1.587	0.348	0.168	2.729	36.35
2.5	1.726	1.744	0.224	-0.012	2.784	27.42	1.569	1.584	0.246	0.025	2.713	31.61
3.0	1.644	1.682	0.164	-0.110	2.819	24.63	1.524	1.565	0.183	-0.087	2.748	28.07
3.5	1.582	1.629	0.125	-0.191	2.874	22.35	1.487	1.541	0.141	-0.178	2.808	25.25
4.0	1.532	1.585	0.099	-0.259	2.938	20.54	1.455	1.516	0.111	-0.254	2.880	22.90
4.5	1.491	1.546	0.080	-0.318	3.006	18.97	1.428	1.491	0.09	-0.318	2.957	21.01
5.0	1.456	1.513	0.066	-0.369	3.076	17.64	1.404	1.468	0.074	-0.373	3.035	19.38

From Table 1, we can conclude that:

1. the LBPHRD is positive skewed, for $\nu < 2.5$, while PHRD is positive skewed, for $\nu \leq 2.5$.

2. the LBPHRD is negative skewed, for $\nu \geq 2.5$, while PHRD is negative skewed, for $\nu > 2.5$

3. when $\nu = 0.0$, the LBPHRD and PHR are highly skewed, ($S_k > 1$).

4. when $\nu = 0.5$, the LBPHRD is moderately skewed, $(0.5 < S_k < 1)$, while PHRD is highly skewed.
5. when $\nu = 1$, the LBPHRD is approximately symmetric, $(-0.5 < S_k < 0.5)$, while PHRD is moderately skewed $(0.5 < s_k < 1)$.
6. when $1.5 \leq \nu \leq 5$, the LBPHRD and PHRD are approximately symmetric.
7. the dispersion for the distributions are decreasing for ν increasing.
8. for $0.0 \leq \nu \leq 1.0$ the LBPHRD and PHRD are leptokurtic $(S_k > 3)$.
9. for $1.5 \leq \nu \leq 4$, the LBRHRD and PHRD are platykurtic $(S_k < 3)$.
10. for $4.5 \leq \nu \leq 5$, the LBPHRD and PHRD are mesokurtic $(S_k \cong 3)$.
11. Since the coefficient of variation $(Cv = \frac{\sqrt{Var(X)}}{mean} \times 100)$ is larger for PHRD, the PHRD are more variable than the LBPHRD, for all values of ν.

Therefore, the LBPHR model is more flexible than PHR model.

3. Estimation of Parameters

Consider $X_1, X_2, \cdots, X_n$ be a random sample from LBPHRD, the Maximum likelihood estimation (MLE) can be applied to estimate the parameters as follows. The likelihood function is given by

$$L(\lambda, \nu; x) = \frac{\lambda^n \left(\prod_{i=1}^n x_i^{\nu+1}\right) e^{-\frac{\lambda}{\nu+1}\sum_{i=1}^n x_i^{\nu+1}}}{\left(\frac{\nu+1}{\lambda}\right)^{\frac{n}{\nu+1}} \left(\frac{1}{\nu+1}\right)^n \Gamma^n\left(\frac{1}{\nu+1}\right)}, \quad x > 0. \tag{19}$$

The log-likelihood function is

$$\begin{aligned}\mathscr{L} &= n\ln(\lambda) + (\nu+1)\sum_{i=1}^n \ln(x_i) - \frac{\lambda}{\nu+1}\sum_{i=1}^n x_i^{\nu+1} - \frac{n}{\nu+1}\ln\left(\frac{\nu+1}{\lambda}\right) \\ &\quad + n\ln(\nu+1) - n\ln\left[\Gamma\left(\frac{1}{\nu+1}\right)\right]. \end{aligned} \tag{20}$$

Differentiate Equation (20) w.r.t. λ and ν. Equating the derivatives to zero, we get the normal equations as follows.

$$\frac{\partial \mathscr{L}}{\partial \lambda} = \frac{n}{\lambda}\left(1 + \frac{1}{\nu+1}\right) - \frac{1}{\nu+1}\sum_{i=1}^n x_i^{\nu+1} = 0, \tag{21}$$

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial \nu} &= \sum_{i=1}^n \ln(x_i) + \frac{\lambda}{(\nu+1)^2}\sum_{i=1}^n x_i^{\nu+1}\Big[1 - (\nu+1)\ln(x_i)\Big] + \frac{n}{(\nu+1)^2}\ln\left(\frac{\nu+1}{\lambda}\right) \\ &\quad + \frac{n\nu}{(\nu+1)^2} - n\psi\left(\frac{1}{\nu+1}\right) = 0, \end{aligned} \tag{22}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ be a digamma function.

The asymptotic normality of the MLEs can be applied to compute the confidence interval (C.I.) for the parameters. The observed variance and covariance matrix of $\Theta = (\lambda, \nu)$ is

$$I^{-1}(\Theta) = \begin{bmatrix} -\frac{\partial^2 \mathscr{L}}{\partial \lambda^2} & -\frac{\partial^2 \mathscr{L}}{\partial \lambda \partial \nu} \\ -\frac{\partial^2 \mathscr{L}}{\partial \nu \partial \lambda} & -\frac{\partial^2 \mathscr{L}}{\partial \nu^2} \end{bmatrix}^{-1} = \begin{bmatrix} -I_{11} & -I_{12} \\ -I_{21} & -I_{22} \end{bmatrix}^{-1},$$

where

$$I_{11} = -\frac{n}{\lambda^2}\left(1 + \frac{1}{\nu+1}\right), \tag{23}$$

$$I_{12} = -\frac{n}{\lambda(\nu+1)^2} + \frac{1}{(\nu+1)^2}\sum_{i=1}^{n} x_i^{\nu+1} - \frac{1}{\nu+1}\sum_{i=1}^{n} x_i^{\nu+1}\ln(x_i), \tag{24}$$

$$I_{21} = I_{12}, \tag{25}$$

$$\begin{aligned} I_{22} = & -\frac{\lambda}{(\nu+1)^3}\sum_{i=1}^{n} x_i^{\nu+1}\Big[1-(\nu+1)\ln(x_i)\Big]\Big[2-(\nu+1)\ln(x_i)\Big] - \frac{\lambda}{(\nu+1)^2} \times \\ & \sum_{i=1}^{n} x_i^{\nu+1}\ln(x_i) - \frac{2n}{(\nu+1)^3}\ln\left(\frac{\nu+1}{\lambda}\right) + \frac{(2-\nu)n}{(\nu+1)^3} - n\psi'\left(\frac{1}{\nu+1}\right), \end{aligned} \tag{26}$$

and $\psi'(x) = \frac{d^2}{dx^2} \ln \Gamma(x)$.

Asymptotic confidence interval can be derived by using observed variance and covariance matrix. A $100(1-\alpha)\%$ C.I.s of $\Theta = (\lambda, \nu)$ have the form $\hat{\lambda} \pm z_{\alpha/2}\sqrt{Var(\hat{\lambda})}$ and $\hat{\nu} \pm z_{\alpha/2}\sqrt{Var(\hat{\nu})}$. The $z_{\alpha/2}$ is upper $(\alpha/2)$th percentile of the standard normal distribution.

4. Applications

4.1. Real data

In this section, an application of LBPHR distribution using three real data sets to illustrate that it provides significant improvements over its sub-model.

Example 4.1. The data of fatigue cycle of 6061–T6 aluminum coupons cut in the horizontal direction of rolling, which is oscillated 18 rounds per second reported by Birnbaum and Saunders (1969). The data set includes 100 observations having an optional stress per round 31×10^3 psi which is reported after reducing 65 as follows.

5	25	31	32	34	35	38	39	39	40	42	43	43	43	44	44
47	47	48	49	49	49	51	54	55	55	55	56	56	56	58	59
59	59	59	59	63	63	64	64	65	65	65	66	66	66	66	66
67	67	67	68	69	69	69	69	71	71	72	73	73	73	74	74
76	76	77	77	77	77	77	77	79	79	80	81	83	83	84	86
86	87	90	91	92	92	92	92	93	94	97	98	98	99	101	103
105	109	136	147												

In Table 2, MLEs of the unknown parameters of LBR, LBW, PHR and LBPHR distributions are given along with criterion log-likelihood, AIC (Akaike's information criterion) and BIC (Bayesian information criterion).

Table 2. MLEs, $\mathscr{L}$, AIC and BIC.

Model	θ	λ	ν	$\mathscr{L}$	AIC	BIC
LBR	1.722×10^3	–	–	-874.485	1.751×10^3	1.754×10^3
LBW	–	0.342	–	-553.06	1.108×10^3	1.111×10^3
PHR	–	1.303×10^{-5}	1.85	-475.692	955.384	960.594
LBPHR	–	1.028×10^{-4}	1.425	-454.493	912.986	918.197

Table 2 indicates that the LBPHR is best than LBR, LBW and PHR distributions in terms of model fitting for this data.

The variance and covariance matrix is given as

$$I^{-1} = \begin{bmatrix} 8.494\times10^{-9} & -1.991\times10^{-5} \\ -1.991\times10^{-5} & 0.047 \end{bmatrix}$$

Then the 95% C.I. for λ and ν for LBPHRD are $(0, 2.83481\times10^{-4})$ and $(0.9993, 1.84998)$, respectively.

Figure 3 shows that the likelihood function has unique solution.

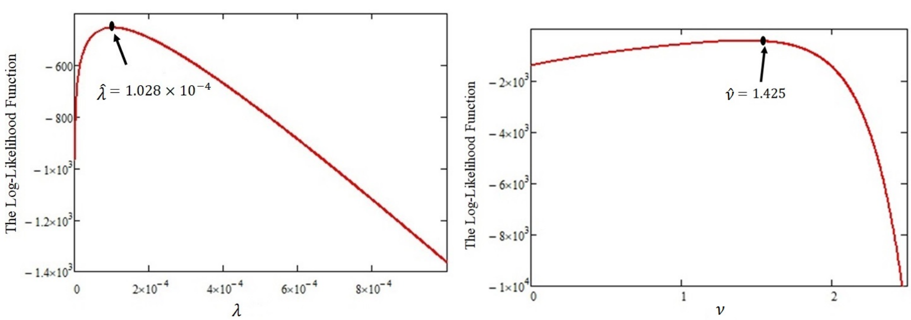

Figure 3. The outline of the $\mathscr{L}$ of λ and ν.

For $\hat{\lambda} = 1.028\times10^{-4}$ and $\hat{\nu} = 1.425$, some statistical measures can be calculated, see Table 3.

Table 3. Some statistical measures for LBPHR at $\hat{\lambda}$ and $\hat{\nu}$.

Mean	Mode	Variance	Skewness	Kurtosis
67.283	63.585	602.188	0.297	2.887

From Table 3, the LBPHR distribution has,

1. the distribution is right skewed ($S_k > 0$) and it is approximately symmetric($-0.5 < S_k < 0.5$).
2. the distribution is platykurtic ($K_u < 3$).

Example 4.2. We use data collected by Balakrishnan et al. (l2010). The behavioral and emotional issues of children are scaled by GRASP (general rating of affective symptoms for preschoolers). The data (with frequency in parenthesis is the score of GRASP measurement of children) are:

19(16)	20(15)	21(14)	22(9)	23(12)	24(10)	25(6)	26(9)	27(8)	28(5)	29(6)
30(4)	31(3)	32(4)	33	34	35(4)	36(2)	37(2)	39	42	44

The MLEs and $\mathscr{L}$, AIC and BIC are reported in Table 4.

Table 4. MLEs and $\mathscr{L}$, AIC and BIC.

Model	θ	λ	ν	$\mathscr{L}$	AIC	BIC
LBR	217.216	–	–	-884.464	1.771×10^3	1.774×10^3
LBW	–	0.411	–	-594.469	1.191×10^3	1.194×10^3
PHR	–	8.275×10^{-5}	2.234	-436.482	876.963	882.759
LBPHR	–	1.216×10^{-5}	2.929	-420.866	845.731	851.527

Table 4 indicates that the LBPHR is best than LBR, LBW and PHR distributions in terms of model fitting for this data.

The variance and covariance matrix is given as

$$I^{-1} = \begin{bmatrix} 1.091\times10^{-10} & -2.797\times10^{-6} \\ -2.797\times10^{-6} & 0.072 \end{bmatrix}$$

Then the 95% C.I. for λ and ν are $(0, 3.26336\times10^{-5})$ and $(2.40217, 3.45613)$, respectively.

Figure 4 shows that the likelihood function has unique solution.

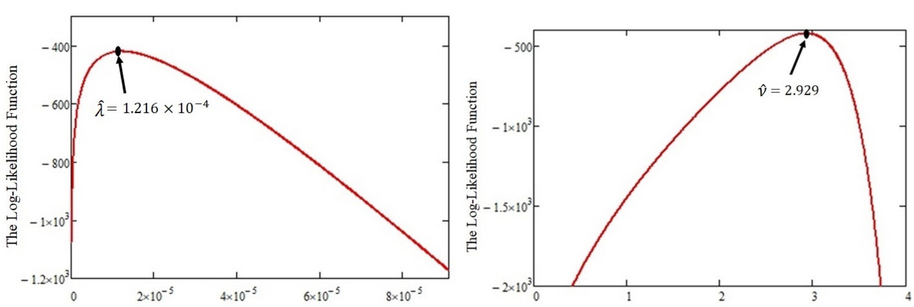

Figure 4. The sketch of the log-likelihood function of λ and ν.

For $\hat{\lambda} = 1.216\times10^{-5}$ and $\hat{\nu} = 2.929$, some statistical measures can be calculated, see Table 5.

Table 5. Some statistical measures, for LBPHR at $\hat{\lambda}$ and $\hat{\nu}$.

Mean	Mode	Variance	Skewness	Kurtosis
24.716	25.244	38.138	-0.097	2.813

From Table 5,

1. the distribution is left skewed ($S_k < 0$) and it is approximately symmetric ($-0.5 < S_k < 0.5$).

2. the distribution is platykurtic ($K_u < 3$).

Example 4.3. The following uncensored data is taken from Mahmoud and Mandouh (2013), which comprises 100 observations(breaking the stress of carbon fibers in Gba) are:

0.92	0.928	0.997	0.9971	1.061	1.117	1.162	1.183	1.187	1.192	1.196
1.213	1.215	1.2199	1.22	1.224	1.225	1.228	1.237	1.24	1.244	1.259
1.261	1.263	1.276	1.31	1.321	1.329	1.331	1.337	1.351	1.359	1.388
1.408	1.449	1.4497	1.45	1.459	1.471	1.475	1.477	1.48	1.489	1.501
1.507	1.515	1.53	1.5304	1.533	1.544	1.5443	1.552	1.556	1.562	1.566
1.585	1.586	1.599	1.602	1.614	1.616	1.617	1.628	1.684	1.711	1.718
1.733	1.738	1.743	1.759	1.777	1.794	1.799	1.806	1.814	1.816	1.828
1.830	1.884	1.892	1.944	1.972	1.984	1.987	2.020	2.0304	2.029	2.035
2.037	2.043	2.046	2.059	2.111	2.165	2.686	2.778	2.972	3.504	3.863
5.306										

The MLES, $\mathscr{L}$, AIC and BIC are given in Table 6.

Table 6. MLEs of the parameters and $\mathscr{L}$, AIC and BIC.

Model	θ	λ	ν	$\mathscr{L}$	AIC	BIC
LBR	1.035	–	–	-131.653	265.306	267.911
LBW	–	1.406	–	-101.918	205.835	208.44
PHR	–	0.521	1.632	-90.149	184.298	189.509
LBPHR	–	0.877	1.237	-84.566	173.132	178.342

Table 6 indicates that the LBPHR is best than LBR, LBW and PHR distributions in terms of model fitting for this data.

The variance and covariance matrix is given as

$$I^{-1} = \begin{bmatrix} 9.448 \times 10^{-3} & -0.011 \\ -0.011 & 0.027 \end{bmatrix}$$

Then the 95% C.I. for λ and ν are $(0.68651, 1.06753)$ and $(0.91431, 1.55924)$, respectively.

Figure 5 shows that the likelihood function has unique solution.

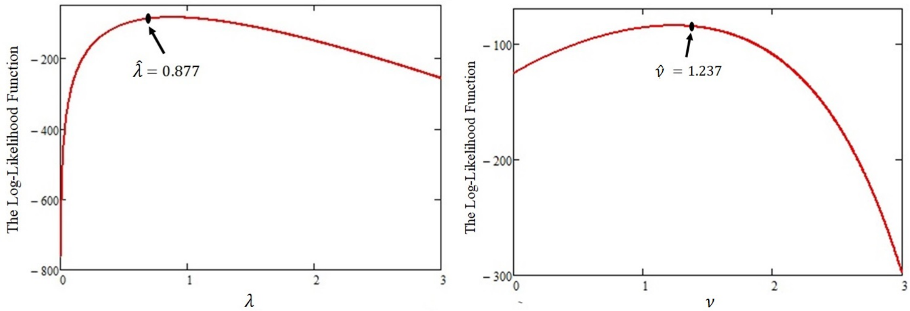

Figure 5. The shape of the log-likelihood function of λ and ν.

For $\hat{\lambda} = 0.877$ and $\hat{\nu} = 1.237$, some statistical measures can be calculated, see Table 7.

Table 7. Some statistical measures, for LBPHRD at $\hat{\lambda}$ and $\hat{\nu}$.

Mean	Mode	Variance	Skewness	Kurtosis
1.647	1.52	0.408	0.374	2.962

From Table 7, we observe that

1. the distribution is right skewed ($S_k > 0$) and it approximately symmetric skewed ($-0.5 < S_k < 0.5$).

2. the distribution is platykurtic. ($K_u < 3$).

4.2. Simulation study

We evaluate the performance of MLE of the model through Monte-Carlo simulation. The simulation's steps are as follows.

1. Fix the vector of parameters $\Theta = (\lambda, \nu)$, and sample of size n.

2. From LBPHR(λ, ν) distribution generate random observation with size n. Since CDF for LBPHR has no closed form, the random observation can be generated by using the Newton's Raphson method.

$$x_{i+1} = x_i - \frac{F(x_i, \Theta) - u_i}{f(x_i, \Theta)}, \quad i = 0, 1, \cdots, n-1, \tag{27}$$

   where, $u \sim uniform(0,1)$.

3. Using step 2, estimate $\hat{\Theta}$ through MLE scheme.

4. Steps 2 and 3, repeated N times.

5. To enumerate MREs (mean relative estimates) and MSEs (mean square errors) using $\hat{\Theta}$ and Θ through the following equations.

$$MRE = \frac{1}{N}\sum_{j=1}^{N}\frac{\hat{\Theta}_{i,j}}{\Theta_i}, \qquad MSE = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{\Theta}_{i,j}-\Theta_i\right)^2,$$

$$Bias = \frac{1}{N}\sum_{j=1}^{N}\hat{\Theta}_{ij}-\Theta_i, \quad i=1,2.$$

Simulation results are obtained via MATHCAD 2007. The selected parameter values are $\Theta=(0.5,2)$, $N=10000$ and $n=(10,20,30,40,50,75,100,150,200,250,300,400,500)$.

Table 8 contains the MLEs, Bias, MREs, and MSEs, for the estimators $\hat{\Theta}_i, i=1,2$, for different values of n.

Table 8. The MLEs, MREs and MSEs, for different values of n.

	λ				ν			
n	MLE	Bias	MRE	MSE	MLE	Bias	MRE	MSE
10	0.22357	-0.27643	0.44714	0.08764	3.56421	1.56421	1.78211	3.32282
20	0.49478	-0.00522	0.98956	0.00040	2.30885	0.30885	1.15443	0.15088
30	0.64100	0.14100	1.28200	0.02070	1.85722	-0.14278	0.92861	0.02979
40	0.49451	-0.00549	0.98902	0.00021	2.33661	0.33661	1.16830	0.17403
50	0.56267	0.06267	1.12535	0.00423	2.37851	0.37851	1.18926	0.14668
75	0.50153	0.00153	1.00306	0.00018	2.12308	0.12308	1.06154	0.01756
100	0.35535	-0.14465	0.71069	0.02103	2.42129	0.42129	1.21064	0.18009
150	0.39449	-0.10551	0.78898	0.01120	2.53190	0.53190	1.26595	0.30598
200	0.55981	0.05981	1.11963	0.00361	2.06537	0.06537	1.03269	0.00489
250	0.50780	0.00780	1.01559	0.00008	2.07485	0.07485	1.03743	0.00591
300	0.42144	-0.07856	0.84289	0.00618	2.16489	0.16489	1.08244	0.02738
400	0.54172	0.04172	1.08344	0.00175	1.90172	-0.09828	0.95086	0.00976
500	0.54186	0.04186	1.08372	0.00176	1.89278	-0.10722	0.94639	0.01154
Average	0.48004	-0.01996	0.96008	0.01223	2.27856	0.27856	1.13928	0.33749

MREs approximate to one when MSEs approaches to zero. Figures 6 – 9 display the estimated MLs,Bias, MREs and MSEs.

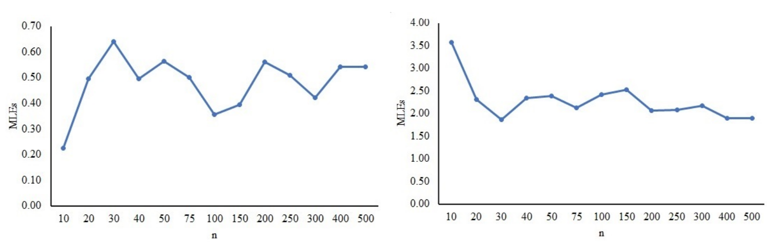

Figure 6. The MLEs for λ and ν.

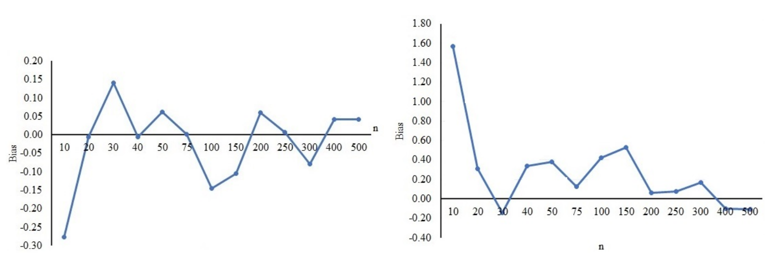

Figure 7. The Bias for λ and ν.

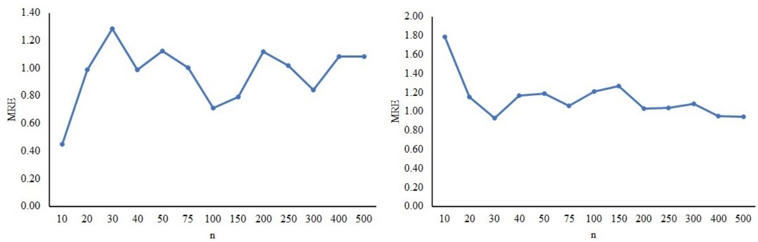

Figure 8. The MREs for λ and ν.

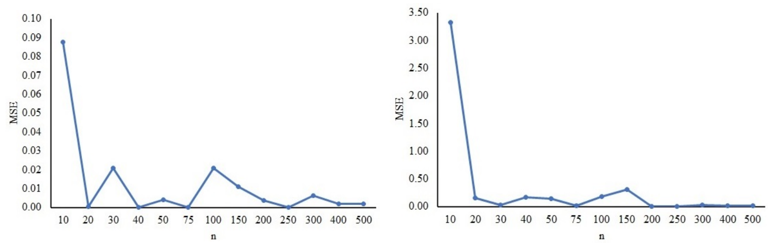

Figure 9. The MSEs for λ and ν,

We notice from Figures 6–9 as follows.

1. For large same size: (i) Estimate of MSE $\to$ 0, (ii) Expected (MRS) $\to$ 1, (iii) Biases of $(\lambda, \nu) \to 0$.

2. Biases of λ are positive/negative.

3. Biases of ν are approximately positive.

4. Estimates of parameters are asymptotically unbiased.

Therefore, the MLE is an suitable for estimating parameters of LBPHR distribution. Similar results can be obtained for different parameters.

5. Conclusions

We propose the length-biased power hazard rate distribution and study its various characteristics. The maximum likelihood estimate for parameters is derived. The superiority of the new model has been exhibited by some real data sets. It has been seen that PHRD can adequately provide better fits over other models.

References

Ajami, M., Jahanshahi S. M. A., (2017). Parameter estimation in weighted Rayleigh distribution. Journal of Modern Applied Statistical Methods, 16(2), pp. 256–276.

Al-Khadim, A. K., Hussein A. N., (2014). New proposed length-biased weighted Exponential and Rayleigh distribution with application. *Mathematical Theory and Modeling*, 4, pp. 2224–5804.

Balakrishnan, N., Victor, L., Antonio, S., (2010). *A mixture model based on Birnhaum-Saunders Distributions, A study conducted by Authors regarding the Scores of the GRASP (General Rating of Affective Symptoms for Preschoolers), in a city located at South Part of the Chile*.

Birnbaum, Z. W., Saunders, S. C., (1969). Estimation for a family of life distribution with applications to fatigue. *Journal of Applied Probability*, 6(2), pp. 328–347.

Cox, D. R., (1962). *Renewal Theory*. New York, NY: Barnes & Noble.

Das, K. K., Roy, T. D., (2011). Applicability of length-biased generalized Rayleigh distribution. *Advances in Applied Science Research*, 2, pp. 320–327.

Das, K. K., Roy, T. D., (2011). On some length-biased weighted Weibull distribution. *Advances in Applied Science Research*, 2, pp. 465–475.

Gupta, P. L., Tripathi, R. C., (1990). Effect of length-biased sampling on the modeling error. *Communications Statistics - Theory and Methods*, 19, pp. 1483–1491.

Gupta, R. C., Keating, J. P., (1985). Relations for reliability measures under length-biased sampling. *Scandanavian Journal of Statistics*, 13, pp. 49–56.

Ismail, K., (2014). Estimation of for distribution having power hazard function. *Pakistan Journal of Statistics*, 30, pp. 57–70.

Lawless, J. F., (2003). Statistical Models and Methods for Lifetime Data, 2nd Edition, Weiley, Canada.

Mir, K. A., Ahmed, A., Reshi, J. A., (2013). On Size-biased Exponential Distribution. *Journal of Modern Mathematics and Statistics*, 7(2), pp. 21–25.

Khattree, R., (1989). Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions. *IEEE Transactions on Reliability*, 38, pp. 610–611.

Mahmoud, M. R., Mandouh, R. M., (2013). On the Transmuted Fréchet Distribution. *Journal of Applied Sciences Research*, 9(10), pp. 5553–5561.

Modi, K., (2015). Length-biased Weighted Maxwell distribution. *Pakistan Journal of Statistics and Operation Research*, 11(4), pp. 465–472.

Mudasir, S., Ahmad, S. P., (2018). Characterization and estimation of the length-biased Nakagami distribution. *Pakistan Journal of Statistics and Operation Research*, 14(3), pp. 697–715.

Mugdadi, A. R., (2005). The least squares type estimation of the parameters in the power hazard function. *Applied Mathematics Computation*, 169, pp. 737–748.

Mugdadi, A. R., Min, A., (2009). Bayes estimation of the power hazard function. *Journal of Interdisciplinary Mathematics*, 12, pp. 675–689.

Nanuwong, N., Bodhisuwan, W., (2014). Length-biased nu Pareto distribution and its structural properties with application. *Journal of Mathematics and Statistics*, 10, pp. 49–57.

Oluyede, B. O., (1999). On inequalities and selection of experiments for length-biased distributions. *Probability in the Engineering and Informational Sciences*, 13, pp. 169–185.

Praveen, Z., Ahmad, M., (2018). Some properties of size- biased weighted Weibull distribution. *International Journal of Advanced and Applied Sciences*, 5(5), pp. 92–98.

Ratnaparkhi, M. V., Naik-Nimbalkar, U. V., (2012). The length-biased lognormal distribution and its application in the analysis of data from oil field exploration studies. *Journal of Modern Applied Statistical Methods*, 11, pp. 225–260.

Saghir, A., Khadim, A., Lin, Z., (2017). The Maxwell length-biased distribution: Properties and Estimation. *Journal of Statistical Theory and Practice*, 11, pp. 26–40.

Saghir, A., Tazeem, S., Ahmad, I., (2016). The length-biased weighted exponentiated inverted Weibull distribution. *Cogent Mathematics*, 3(1), DOI: 10.1080/23311835.2016.1267299.

Seenoi, P., Supapa, K.T., Bodhisuwan, W., (2014). The length-biased exponentiated inverted Weibull Distribution. *International Journal of Pure and Applied Mathematics*, 92, pp. 191–206.

Shaban, S.A., Boudrissa, N. A., (2007). The Weibull length-biased distribution: Properties and estimation. *Interstat*,
http://interstat.statjournals.net/YEAR/2007/articles/0701002.pdf

Tarvirdizade, B., Nematollahi, N., (2016). Parameter estimation based on record data from power hazard rate distribution. *13th Iranian Statistical Conference*, Kerman, Iran.

Tarvirdizade, B., Nematollahi, N., (2020). Inference on $P(X > Y)$ based on record values from power hazard rate distribution. *Journal of Computational Statistics and Modelling*, 1(1), pp. 59–76.