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ARFURIMA models: simulations of their properties and application

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ABSTRACT

This article defines the Autoregressive Fractional Unit Root Integrated Moving Average (ARFURIMA) model for modelling ILM time series with fractional difference value in the interval of $1 < d < 2$. The performance of the ARFURIMA model is examined through a Monte Carlo simulation. Also, some applications were presented using the energy series, bitcoin exchange rates and some financial data to compare the performance of the ARFURIMA and the Semiparametric Fractional Autoregressive Moving Average (SEMIFARMA) models. Findings showed that the ARFURIMA outperformed the SEMIFARMA model. The study's conclusion provides another perspective in analysing large time series data for modelling and forecasting, and the findings suggest that the ARFURIMA model should be applied if the studied data show a type of ILM process with a degree of fractional difference in the interval of $1 < d < 2$.

Key words: interminable long memory, autocorrelation, fractional unit root integrated series, fractional unit root differencing, ARFURIMA model.

1. Introduction

Long Memory (LM) is a statistical property that may arise in time series data. The information of its occurrence in financial and economic variables can be exploited by investors and policy makers to predict equity prices, quantify market's risk, inflation and economic growth of the country. There are past and recent works that have discovered degree of long memory also called fractional differencing value in the interval of $0 < d < 1$ (see Granger and Joyeux (1980), Hosking (1981), Ballie et al., (2014), Boubaker et al. (2016) and Pumi et al., (2019)).

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Furthermore, according to Rahman and Jibrin (2018), when the ACF of time series exhibits decays more slowly and have fractional difference value in the interval of $1 < d < 2$, the series are said to be having an Interminable LM (ILM) process. In view of this, we need a family of models that can simulate a very strong dependent relationship (autocorrelation) between distance observations, at the same time being flexible enough to model both the integrated, $I(1)$ and Fractional Unit Root Integrated (FURI), I$(1 < d < 2)$ process.

The Autoregressive Fractional Unit Root Integrated Moving Average (ARFURIMA) model, suggested by Rahman and Jibrin (2019), provides a method for modelling FURI time series with fractional difference value in the interval of $1 < d < 2$. Therefore, in this paper, some of the basic properties of the ARFURIMA model were derived and presented as follows. In Section 2 we state the basic properties of the ARFURIMA*(0,d,0)* process. This is followed by the general ARFURIMA(*p,d,q*) family, its properties, and some special cases of ARFURIMA(*p,d,q*) in Section 3. A short introduction of SEMIFARMA(*p,d,q*) model is given in Section 4. Then some simulations by using the ARFURIMA are carried out in Section 5 to assess its properties. Finally, its applications by using the energy series, Bitcoin exchange rate and financial data in are presented in Section 6.

2. Methodology

Dolado and Marmol (1997) named the Data Generating Process (DGP) of y_t to be Nonstationary Fractionally Integrated (NFI) process and defined it as:

$$(1 - L)^d y_t = \varepsilon_t, \tag{1}$$

where $d \geq \frac{1}{2}$, $\varepsilon_t \sim iid(0, \sigma^2)$ and d is decomposed as $d = \propto + \delta$, where $\propto = 1,2,3, \dots \dots \dots$ and $|\delta| < \frac{1}{2}$. In this case, d can be in the range of $1 < d < \infty$ while most time series usually have a fractional difference value, d, in the interval of $1 < d < 2$ (see Gil-Alana et al., (2018) and Sabzikar et al. (2019)). However, Hurvich and Chen (2000) and Erfani and Samimi (2009) have highlighted the repercussion of over-differencing including loss of information, negative values of differenced series, $d \leq -0.5$ and estimation of complex models. In view of these, Rahman and Jibrin (2019) resolved that the possible highest value of fractional difference is in the interval of $1 < d < 2$. Also, this type of time series exhibits very slow decaying ACF, which is slower than usual decay seen in the literature of time series and LM analysis. Having said this, Rahman and Jibrin (2019) named the DGP of y_t to be the FURI process and defined its operator as:

$$\{(1 - L)(1 - d^*)(1 + L)\} y_t = \varepsilon_t, \tag{2}$$

where $d^* = d - 1$, $0 < d^* < 1$ and $1 < d < 2$.

Details of the derivation and its R algorithm can be found in Rahman and Jibrin (2019).

2.1. The ARFURIMA(p,d,q) models

In a similar way on how Granger and Joyeux (1980) and Hosking (1981) introduced ARFIMA model due to FI(d) process, Porte-Hudak (1990) introduced SARFIMA model due to seasonal FI(d) process and ARTFIMA model of Meerschaert et al. (2014) was introduced due to tempered FI(d) process. Rahman and Jibrin (2019) introduced the ARFURIMA model due to the FURI(d) processes. In order to obtain the ARFURIMA model, the lag representation of the proposed non power operator is incorporated as:

$$\varphi(L)\{(1-L)\big(1-d^*(1+L)\big)\}Y_t=\theta(L)\varepsilon_t, \tag{3}$$

where $\varphi(L)$ and $\theta(L)$ are stationary and invertible. L is the backward shift operator, ε_t represents a white noise process and $\{(1-L)\big(1-d^*(1+L)\big)\}$ is the proposed non-power operator. The operator fractionally differenced is a process that exhibits a very slow decaying (unusual decay) ACF. Here, $d^*=d-1$ such that $0<d^*<1$ and $1<d<2$ and both d^* and d are the LM and ILM parameters respectively. The identification of the ARFURIMA(*p,d,q*) model followed the Box and Jenkins approach and was discussed in detail in Rahman and Jibrin (2019).

2.2. The ARFURIMA(*0,d,0*) process

The ARFURIMA(*0,d,0*) process was defined to be a discrete time series $\{Y_t\}$, which was presented as:

$$\big(\nabla-d^*\nabla(1+L)\big)Y_t=\varepsilon_t \tag{4}$$

where $\nabla=(1-L)$, L is the backward-shift operator and Y_t represents the FURI series. The fractional differencing parameter d was estimated by applying GPH (1983) semi-parametric method defined by:

$$\ln\{I(\varphi_j)\}=a-d\ln\left\{4\sin^2\left(\frac{\varphi_j}{2}\right)\right\}+\varepsilon_t, \tag{5}$$

where $j=1,\dots,n,$ and $I(\varphi_j)=\left(\frac{1}{2\pi}\right)\left|\sum_{i=1}^{T}y_t exp(i\varphi_j t)\right|^2$ was the periodogram at the frequency $\varphi_j=\frac{2\pi j}{T}$. The following theorems were some of the derived properties of ε_t.

Theorem 1

For $0<d^*<1$ such that $d^*=d-1$ and $1<d<2$, $\{Y_t\}$ is

a. stationary, causal and has infinite Moving Average (MA) function written as

$$Y_t=\psi(L)\varepsilon_t=\sum_{n=0}^{\infty}\psi_n L^n\varepsilon_t=\sum_{n=0}^{\infty}\psi_n\varepsilon_{t-n}. \tag{6}$$

The fractional differencing operator $(1-L)^d$, $0<d<1$ in Granger and Joyeux (1980), Hosking (1981), Dolado and Marmol (1997), Meerschaert et al., (2014), Boubaker et al., (2016) and Pumi et al., (2019), is defined as an infinite binomial series expansion in powers of the backward-shift operator

$$(1-L)^d Y_t = \sum_{i=0}^{\infty} \psi_i L^i Y_t = \sum_{i=0}^{\infty} \psi_i Y_{t-i}, \tag{7}$$

where the coefficient, ψ_i, is expanded by:

$$\psi_i = \prod_{k=1}^{i} \frac{k+d-1}{k} = \frac{\Gamma\mathrm{i}+\mathrm{d}}{\Gamma\mathrm{d}\;\Gamma\mathrm{i}+1}, i = 1,2,\ldots \tag{8}$$

However, ψ_n can be expanded as:

$$\psi_n = \left((L^n - L^{1+n}) - d^*(L^n - L^{2+n})\right). \tag{9}$$

So,

$$\psi_n \sim (1-L)(1-d^*(1+L), \tag{10}$$

and hence, (6) can be re-written as:

$$Y_t = \psi(L)\varepsilon_t = \sum_{n=0}^{\infty} \psi_n L^n \varepsilon_t = \sum_{n=0}^{\infty} \left((L^n - L^{1+n}) - d^*(L^n - L^{2+n})\right)\varepsilon_t, \tag{11}$$

where L, ε_t and $d^*$is as defined in (4). Also,

$$\textstyle\sum_{n=0}^{\infty} |\psi_n| < \infty \tag{12}$$

satisfied the causality condition, where it stated that $\{Y_t\}$ depended on past residuals ε_t and the dependency was gradually decreasing asymptotically for a long time.

Proof.

Using $Y_t = \psi(L)\varepsilon_t$, we have $\psi(L) = \left((L^n - L^{1+n}) - d^*(L^n - L^{2+n})\right)^{-1}$. When $1<d<2$, the expansion of $\psi(L)$ converged for $|L| \leq 1$ and so $\{Y_t\}$ is stationary. The expansion of $(L^n - L^{1+n}) - d^*(L^n - L^{2+n})$ resulted in (10) when $n \to \infty$, that was $(L^n - L^{1+n}) - d^*(L^n - L^{2+n}) \sim \left((1-L)(1-d^*(1+L)\right)$.

b. $\{Y_t\}$ is invertible and has infinite AR function writen as:

$$\Phi(L)Y_t = \textstyle\sum_{n=0}^{\infty} \Phi_n L^n Y_t = \sum_{n=0}^{\infty} \Phi_n Y_{t-n} = \varepsilon_t, \tag{13}$$

where Φ_n is defined similar to ψ_n in (9) with the invertibility

$$\textstyle\sum_{n=0}^{\infty} |\Phi_n| < \infty \tag{14}$$

Proof.

The proof was similar to (a).

c. The spectral density function of $\{Y_t\}$ is

$$f(\lambda) = \sum_{k=0}^{\infty} e^{i\lambda k}\,\gamma(k) \tag{15}$$

where $\gamma(k)$ was the autocovariance function of $\{Y_t\}$. $f_{(\lambda)} \sim |\lambda|^{-d} C_f$ described the pole at the zero frequency of the spectral density as $C_f > 0$ and $1 < d < 2$.

d. The autocovariance function of $\{Y_t\}$ is

$$\gamma(k) = E(Y_t Y_{t-k}) = \frac{d^*\gamma_{k-2} - \gamma_{k-1}}{d^*}, \tag{16}$$

where $\gamma_{(k)} \sim K^{d-1}$, $1 < d < 2$ described the very slow decay in the autocorrelation function of Y_t as $k \to \infty$.

Proof.

Using

$$\begin{aligned}\gamma_k = E(Y_t Y_{t-k}) &= Y_{t-k}\big(Y_t - Y_{t-1} - d^*(Y_t - Y_{t-2})\big)\\ &= \gamma_k - \gamma_{k-1} - d^*(\gamma_k - \gamma_{k-2})\\ &= \gamma_k(1 - d^*) - \gamma_{k-1} + d^*\gamma_{k-2},\end{aligned}$$

and re-arranging the equation as $-\gamma_k(1 - d^*) = d^*\gamma_{k-2} - \gamma_{k-1}$, we get $\gamma_k d^* = d^*\gamma_{k-2} - \gamma_{k-1}$ and therefore,

$$\gamma_k = \frac{d^*\gamma_{k-2} - \gamma_{k-1}}{d^*}.$$

e. The autocorrelation function of $\{Y_t\}$ is

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{d^*\gamma_{k-2} - \gamma_{k-1}}{d^*\gamma_2 - \gamma_1} \tag{17}$$

where d^* is as defined in (4).

Proof.

Using

$$\begin{aligned}\gamma_0 = E(Y_t Y_t) &= Y_t\big(Y_t - Y_{t-1} - d^*(Y_t - Y_{t-2})\big)\\ &= \gamma_0 - \gamma_1 - d^*(\gamma_0 - \gamma_2)\\ &= \gamma_0(1 - d^*) - \gamma_1 + d^*\gamma_2.\end{aligned}$$

Re-arranging the equation will get $\gamma_0 - \gamma_0(1 - d^*) = d^*\gamma_2 - \gamma_1$.
Therefore,

$$\gamma_0 = \frac{d^*\gamma_2 - \gamma_1}{d^*}. \tag{18}$$

Therefore, (17) is resulted from substituting (16) and (18).

2.3. The Nonstationary ARFURIMA(*p*,*d*,*q*) model

Consider the stationary ARFURIMA(*p*,*d*,*q*) model, written as:

$$\varphi(L)\left((1-L)\left(1-d^{*}(1+L)\right)\right)(Y_t-\mu)=\theta(L)\varepsilon_t, \tag{19}$$

where $\varphi(L)=1-\varphi_1 L-\varphi_2 L^2-\cdots-\varphi_p L^p$ and $\theta(L)=1-\theta_1 L-\theta_2 L^2-\cdots-\theta_q L^q$. For (19) to be stationary and invertible, each zero of $\varphi(L)$ and $\theta(L)$ must be outside the unit circle respectively. Noticed that when $d^{*}=0$, the ARFURIMA is reduced to the ARIMA model.

Theorem 2

The ARFURIMA(*p*,*d*,*q*) model as mentioned by (19) is

a. stationary if

$$Y_t=\left((1-L)\left(1-d^{*}(1+L)\right)\right)^{-1}\varphi(L)^{-1}\theta(L)\varepsilon_t. \tag{20}$$

b. It is invertible when

$$\varphi(L)\theta(L)^{-1}\left((1-L)\left(1-d^{*}(1+L)\right)\right)Y_t=\varepsilon_t. \tag{21}$$

c. Its spectral density function is given by

$$f(\omega)=\frac{\sigma_e^2}{2\pi}\frac{\left|\theta\left(e^{-i\omega}\right)\right|^2}{\left|\varphi\left(e^{-i\omega}\right)\right|^2}\left((1-L)\left(1-d^{*}(1+L)\right)\right)\left(e^{i2\pi\omega}\right). \tag{22}$$

d. Then, the non-stationary ARFURIMA model can be represented as

$$\varphi(L)\left((1-L)\left(1-d^{*}(1+L)\right)\right)Y_t=\upsilon+\theta(L)\varepsilon_t \tag{23}$$

where υ is a constant.

2.4. Maximum Likelihood Estimation Method for ILM Model and Its Hybrid

Consider series $Y=(y_1,\dots,y_t)'$, where $y_1,\dots,y_t$was the FURI process. In order to obtain the estimates of the ARFURIMA, the series $Y_t$was filtered by the non-power operator, ε_t, where

$$\varepsilon_t=\left\{\left\{(1-L)\left(1-d^{*}(1+L)\right)\right\}Y_t\right\}. \tag{24}$$

Following Kang and Yoon (2013), ε_t in equation (24) was assumed to be normal. The parameters of the ARFURIMA (*p*,*d*,*q*) model were estimated by using the Maximum Likelihood Estimation (MLE) and nonlinear optimization procedures. The maximized of the logarithm of the normal likelihood function was given in equation (25).

$$\ln\{L(\mu,d,\varphi,\theta,\sigma^2)\}=-\frac{n}{2}\ln(2\pi)-\frac{1}{2}\ln|\Sigma|-\frac{1}{2}Y'\Sigma^{-1}Y, \tag{25}$$

where n is the number of observations, Σ represents the n x n covariance matrix of Y dependent on $\mu, d, \varphi, \theta, \sigma^2$ and $|\Sigma|$is the determinant of Σ.

2.5. The SEMIFARMA*(p,d,q)* model

With reference to the Semiparametric Fractional Autoregressive Moving Average (SEMIFARMA) model by Beran and Feng (2002), we used the definition of the SEMIFARMA*(p,d,q)* model as:

$$\varphi(L)(1-L)^d\{(1-L)^m Y_t - \mu\} = \theta(L)\varepsilon_t \tag{26}$$

where μ is the mean of Y_t, $\varphi(L) = 1 - \varphi_1 L - \varphi_2 L^2 - \cdots \varphi_p L^p$, $\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \cdots \theta_q L^q$, m and d defined as $\delta = m + d$ such that $d \in (-0.5, 0.5)$ and $m \in \{0,1\}$.

3. Simulation properties of ARFURIMA (p,d,q) model

This section discusses the simulation to assess the ILM, large sample, conceptual and unbiased properties of the ARFURIMA models. The simulated models are summarized in Table 1.

Table 1. Different models with their different selection of d,$\varphi_1\theta_1$ and n

Model	d	φ_1	θ_1	Sample size, n
ARFURIMA(1,d,0)	1.1	0.5 to 0.9	-	6000
ARIMA(1,1,0)	1		-	6000
ARFURIMA(1,d,0)	1.1,1.5,1.9	-0.9,0.7,0,0.7,0.9	-	6000
ARFURIMA(1,d,0)			-	375
ARFURIMA(1,d,1)		0.4	0.6	375, 750, 1500, 3000, 6000

Referring to Figure 1, all the ACF indicate a very strong hyperbolic decay, implying evidence of ILM. Therefore, on average, all the simulated series are not stationary. Also, the degree of dependency between observations may produce fractional differencing value in interval of $1 < d < 2$.

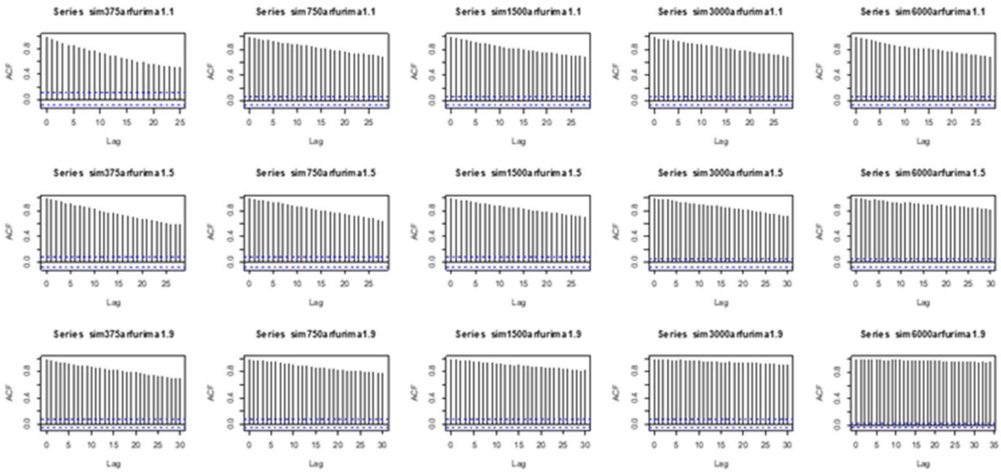

Figure 1. ACF for simulated series using ARFURIMA *(1,d,1)* based on $\varphi = 0.4$, $\theta = 0.6$, $d = \{1.1, 1.5, 1.9\}$ and sample size $n = \{375, 750, 1500, 3000, 6000\}$.

Table 2. Autocorrelation of ARFURIMA (1,1.1,0) and ARIMA (1,1,0) process for different values of φ with n=6000

	ρ_k **of ARFURIMA(1, 1.1, 0)**					ρ_k **of ARIMA(1, 1, 0)**				
k	$\varphi = 0.5$	$\varphi = 0.6$	$\varphi = 0.7$	$\varphi = 0.8$	$\varphi = 0.9$	$\varphi = 0.5$	$\varphi = 0.6$	$\varphi = 0.7$	$\varphi = 0.8$	$\varphi = 0.9$
1	1	1	1	1	1	0.999	0.999	1	1	1
2	0.999	0.999	1	1	1	0.998	0.998	0.999	1	1
3	0.999	0.999	0.997	0.999	1	0.997	0.997	0.998	0.999	0.999
4	0.999	0.999	0.996	0.999	0.999	0.995	0.995	0.997	0.999	0.999
5	0.999	0.999	0.994	0.999	0.999	0.994	0.994	0.996	0.999	0.998

The autocorrelation values of ARFURIMA (1,d,0) and ARIMA (1,1,0), as shown in Table 2, indicated a perfect relationship and strong dependency between observations. On the average, the dependence degrees captured by ARFURIMA was higher compared to the ARIMA models. Therefore, the simulations have provided adequate explanations about the quality of the proposed ARFURIMA model in simulating ILM and FURI series and thus proved the ILM properties of the model.

Meanwhile, for $k \leq 3$, the autocorrelation values of ARFURIMA (1,1.1,0) when $\varphi_1 = 0.9$, as shown, indicate that both the large theoretical fractional difference and φ parameter value have influenced the degree of dependence among the simulated series. Also, for $k \leq 3$, the autocorrelation values of ARFURIMA (1,d,0) for $d = 1.9$ and $\varphi_1 = 0.9$, as shown in Table 3, was perfect indicating that the large theoretical fractional difference and φ parameter value has influenced the degree of dependence among the simulated series. Meanwhile, by comparing Table 3 and 4, the occurrence of perfect autocorrelations among simulated series with $n = 6000$ was found higher compared to $n = 375$. This implied the existence of large sample size properties of the ARFURIMA model.

Table 3. Autocorrelation of the ARFURIMA (1,d,0) process for various values of d and φ , with n=6000

k	d	$\varphi = -0.9$	$\varphi = -0.7$	$\varphi = 0$	$\varphi = 0.7$	$\varphi = 0.9$
1	1.1	0.999	0.999	0.999	0.999	0.999
2		0.999	0.999	0.999	0.999	0.999
3		0.999	0.999	0.997	0.999	0.999
4		0.998	0.999	0.996	0.999	0.999
5		0.997	0.999	0.994	0.999	0.999
1	1.5	0.999	0.999	0.999	1	1
2		0.999	0.999	0.999	0.999	1
3		0.999	0.999	0.999	0.999	1
4		0.999	0.999	0.999	0.999	0.999
5		0.998	0.999	0.999	0.999	0.999
1	1.9	0.999	0.999	0.999	1	1
2		0.999	0.999	0.999	1	1
3		0.999	0.999	0.999	0.999	1
4		0.998	0.999	0.999	0.999	0.999
5		0.998	0.998	0.999	0.999	0.999

The results of the simulation for ARFURIMA(p,d,q) with various settings mentioned in Table 1 showed that means and variances of all the estimated ARFURIMA models confirmed and supported the assumption that the residuals are normally distributed since all the means are zero with variances in the interval of $0.5 < \sigma_e^2 < 1.2$ specifically for $n \geq 1000$. Again, this proves the large sample and also the conceptual properties of the proposed ARFURIMA model. The authors can be contacted for a complete result of these simulations.

4. The application

This section presents the application of the proposed ARFURIMA model by using data of energy series, bitcoin exchange rates and some financial data.

4.1. Data

The description of nine series of data consisted of energy prices series, bitcoin exchanged rates, a financial index and few currencies exchange rates, which are displayed in Table 4. As shown in Figure 1-3, the time series plots of the studied series exhibited nonlinearity deterministic trends. All the ACF showed a very slow decay in the long term with positive autocorrelations, which provided evidence of the LM process. In view of this, there exists LM in the studied series, and it can be described as an ILM. On average, all the nine series are not stationary.

Table 4. Daily Time Series Used for Analysis

S/No.	Type of Data	Abbreviation	Sample Size	Date
1	Brazil Diesel Distributors BRLLTR Prices	BDDP	3915	26/01/04 - 25/01/19
2	Dubai Crude Oil Prices	DBCP	3896	26/01/04 - 25/01/19
3	WTI Crude Oil Prices	WTCP	3896	26/01/04 - 25/01/19
4	Bitcoin to 1000 Euro Exchange Rate	BEUR	1056	15/12/14 - 31/12/18
5	Bitcoin to 1000 Pound Exchange Rate	BPOU	1056	15/12/14 - 31/12/18
6	Bitcoin to 1000 US Dollar Exchange Rate	BDOL	1056	15/12/14 - 31/12/18
7	ATHEX Composite Index	ATIN	7891	03/10/98 - 31/12/18
8	Kuwait Dinar to US Dollar Exchange Rate	KUSD	5196	01/02/99 - 31/12/18
9	Uruguay Peso to UK Pound Exchange Rate	UUKP	5196	01/02/99 - 31/12/18

Source: datastream of Thomson Reuters and Morgan Stanley Capital International (MSCI).

Table 5 presented the descriptive statistics, serial correlation, and normality test for the nine series. It showed that the mean for the Brazil Diesel, Dubai and WTI price each is 2.05, 71.97 and 71.06 respectively. Meanwhile, the standard deviation, which measured the variability or volatility of bitcoin exchange rate for each Euro, British pound sterling, United State (U.S) dollar and Japan Yen is 1.68, 2.32 and 1.50 respectively. The skewness and kurtosis for all the series indicated non-normality. Similarly, the Ljung-Box Q-statistic at lag 50 and Jarque-Bera statistic showed that for all the studied series, the null hypotheses of no serial correlation and normality were rejected at the 0.05 significance level respectively.

In testing and estimating the LM, the bandwidth was chosen between 0 and 1 (GPH, 1983). Hurvich *et al.* (1998) suggests that the best bandwidth (bw) is 0.8, however, Baillie and Morana (2012) uses 0.6 and as for this study, we considered the bw as 0.5, the average of all possible fractional values in interval $0 < d < 1$. For the purpose of comparison, we also produced the estimations based on the bw suggested in Baillie and Morana (2012) and Hurvich et al. (1998) and results were presented in Table 6. The GPH and LWE confirmed the incidence of ILM at level among the studied series. The null hypothesis of no ILM was rejected due to the p-values are less than the significant level of 0.05. Besides, notice that the two estimators, GPH and LWE with bandwidth of 0.5 and 0.8 respectively, produced inconsistent results at level series, with $0 < d < 1$ and $1 < d < 2$. Also, on average, the GPH produced higher fractional differencing values that can adequately eliminate the unwanted noise signals across the nine series.

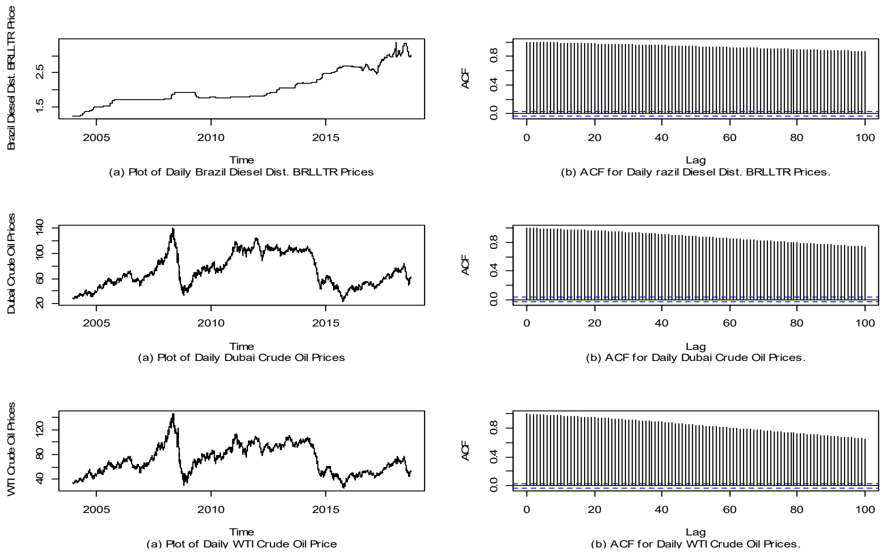

Figure 1. Time Series Plot and ACF for Brazil Diesel, Dubai and WTI Crude Oil Prices (in Dollar per barrel)

Consequently, we suggest that using GPH estimator with a bandwidth equal to 0.5 will produce an adequate fractional differencing value. The adequate fractional differencing value would eliminate the deterministic trend and help in producing a series with less variability. Table 7 presents standard errors of the means of the series. The series were differenced using the $\{(1-L)(1-d^{*}(1+L))\}Y_t$, $(1-L)^{d}Y_t$ and $(1-L)^{\propto+\delta}Y_t$ operators or fractional filters of Rahman and Jibrin (2019), Granger and Joyuex (1981) and Dolado and Marmol (1997) respectively.

Note that the filters of Dolado and Marmol (1997) and Beran and Feng (2002), shown in the last two columns respectively, produced almost the same standard errors and can be considered to be similar to the current operator used in this study.

A comparison of the standard errors of the mean produced by these three differenced series have shown evidence of a better performance of the fractional unit root difference filter, in which it gave the most minimum standard errors of mean compared to the other two filters. Although the KUSD series indicated that the three filters produced the same standard error, there is a reason to believe that the fractional unit root differenced filter procedure used in this study for fractionally differencing FURI time series was the most appropriate among the three filters because it has reduced the volatility, dependency and linearity structures in all the considered series.

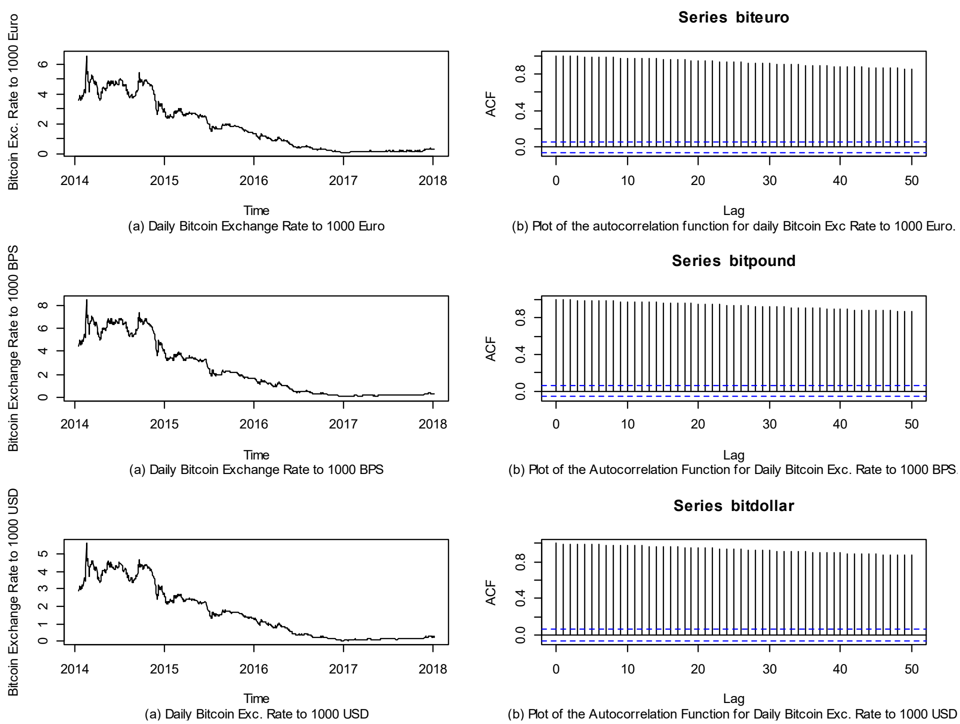

Figure 2. Time Series Plot and ACF for Daily Bitcoin Exchange Rate to 1000 Euro, USD and USP

4.2. Models identification

The AIC values for ARFURIMA and SEMIFARMA models were presented in Table 8 and 9 respectively. The best model according to the least AIC value (the values were bold) was identified for each series among the candidate models of ARFURIMA and SEMIFARMA.

4.3. Estimation, diagnostic tests and forecast

In this section, the estimated parameters of the mean model ARFURIMA and SEMIFARMA for each studied series are presented.

4.3.1. Estimation of the ARFURIMA and SEMIFARMA Model

The results of the estimated parameters of both models ARFURIMA(1,d,1) and SEMIFARMA(1,d,1) for each series and their log-likelihood values are reported in Table 10. All the parameters of the ARFURIMA models were found significant due to their minimum standard errors. Furthermore, the ARFURIMA had larger log-likelihood values compared to the SEMIFARMA model, implying that the ARFURIMA have fitted the data well. Also, the proposed non power operator in ARFURIMA had successfully eliminated large inherent noise signals in all the considered series.

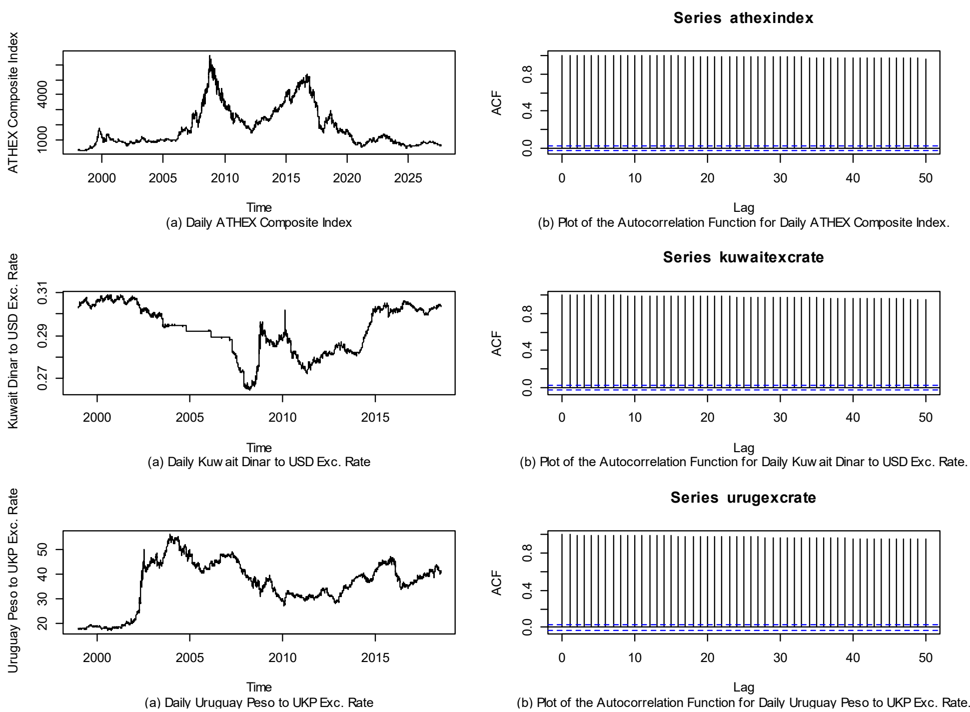

Figure 3. Time Series Plot and ACF for Daily ATHEX Index, Kuwait, and Uruguay Exchange Rate to USD and UKP

Table 5. Descriptive Statistics

Variables	Minimum	Maximum	Mean	SD	Skewness	Kurtosis	Q-Test (50)	JB Test
BDDP	1.22	3.40	2.05	0.49	0.76	-0.30	185894.69***	387.62
DBCP	22.79	140.56	71.97	26.17	0.31	-1.08	177008.13***	250.34
WTCP	26.19	145.31	71.06	22.88	0.38	-0.66	170140.41***	162.49
BEUR	0.06	6.58	1.78	1.68	0.67	-0.93	46627.64***	117.96
BPOU	0.07	8.51	2.34	2.32	0.75	-0.89	47436.33***	132.84
BDOL	0.05	5.59	1.60	1.50	0.65	-0.98	47423.24***	115.83
ATIN	269.45	6633.90	1825.10	1334	1.15	0.40	382594.01***	1793.57
KUSD	0.26	0.31	0.29	0.01	-0.48	-0.70	250067.39***	310.36
UUKP	17.36	56.03	36.10	9.71	-0.43	-0.59	248247.15***	237.57

Notes: SD=Standard Deviation, the Jarque–Bera test corresponds to the test statistic for the null hypothesis of normality in the distribution of sample data. The Ljung–Box statistic, Q(n), check for serial correlation of the series up to the nth order.

Table 6. Tests and Estimation of ILM and LM

	Average bw, bw=0.5	BM (2012), bw=0.6	H (1998), bw=0.8	Average of bw, bw=0.5	BM (2012), bw=0.6	H (1998), bw=0.8
Energy Data		LWE			GPH	
BDDP	1.1202(0.000)	1.1202(0.000)	0.9945(0.000)	1.1182(0.000)	1.1182(0.000)	1.0077(0.000)
DBCP	1.2667(0.000)	1.0885(0.000)	1.0170(0.000)	1.2613(0.000)	1.0983(0.000)	1.0469(0.000)
WTCP	1.2348(0.000)	1.0635(0.000)	1.0095(0.000)	1.2294(0.000)	1.0582(0.000)	1.0220(0.000)
BEUR	1.1463(0.000)	0.9809(0.000)	1.0011(0.000)	1.2872(0.000)	1.0002(0.000)	1.0257(0.000)
BPOU	1.1756(0.000)	0.9940(0.000)	1.0061(0.000)	1.3156(0.000)	1.0229(0.000)	1.0187(0.000)
BDOL	1.1832(0.000)	1.0019(0.000)	1.0074(0.000)	1.3150(0.000)	1.0275(0.000)	1.0215(0.000)
ATIN	1.1695(0.000)	1.0750(0.000)	1.0775(0.000)	1.1738(0.000)	1.0592(0.000)	1.0229(0.000)
KUSD	1.2580(0.000)	1.1486(0.000)	0.9875(0.000)	1.2523(0.000)	1.1376(0.000)	0.9858(0.000)
UUKP	1.1184(0.000)	1.0682(0.000)	0.9816(0.000)	1.1020(0.000)	1.1112(0.000)	0.9751(0.000)

Note: p-values are in parenthesis (.), bw denotes the bandwidth for the LWE and GPH tests. BM is Bailie and Morana, meanwhile H is Hurvich)

Table 7. Standard Errors of the Mean for Fractional Unit Root and Fractional Differenced of the Studied Series

Variables	$\{(1-L)(1-d^*(1+L))\}Y_t$	$(1-L)^d Y_t$	$(1-L)^{\alpha+\delta} Y_t$
BDDP	0.00015	0.00027	0.00028
DBCP	0.01926	0.02633	0.02842
WTCP	0.02036	0.02657	0.02830
BEUR	0.00226	0.00343	0.00366
BPOU	0.00291	0.00442	0.00476
BDOL	0.00191	0.00284	0.00306
ATIN	0.36372	0.47519	0.48515
KUSD	0.00001	0.00001	0.00001
UUKP	0.00405	0.00573	0.00582

Table 8. AIC Values for ARFURIMA(p,d,q) Models

Variables	ARFURIMA(1,d,0)	ARFURIMA(1,d,1)	ARFURIMA(2,d,0)
BDDP	-25616.84	**-25618.43**	-25616.48
DBCP	11716.44	**11584.88**	11607.83
WTCP	12444.65	12379.17	**12378.06**
BEUR	-2618.18	**-2639.72**	-2627.75
BPOU	-2111.12	**-2150.71**	-2130.78
BDOL	-2998.07	**-3038.86**	-3019.29
ATIN	77220.03	**77208.76**	77212.13
KUSD	-66159.66	-66206.75	**-66369.59**
UUKP	1915.799	**1905.627**	1916.047

Table 9. AIC Values for SEMIFARMA(p,d,q) Models

Variables	SEMIFARMA(1,d,0)	SEMIFARMA(1,d,1)	SEMIFARMA(2,d,0)
BDDP	-24550.76	**-24573.65**	-24564.02
DBCP	14170.81	**14007.84**	14071.67
WTCP	14589.55	**14473.07**	14507.58
BEUR	-1870.80	**-1918.11**	-1880.35
BPOU	-1283.53	**-1345.28**	-1297.33
BDOL	-2173.91	**-2237.29**	-2189.36
ATIN	80076.43	**79966.76**	80018.9
KUSD	-65401.79	**-65787.28**	-65691.27
UUKP	3066.39	**3017.78**	3050.02

Table 10. Estimation of ARFURIMA(p,q) and SEMIFARMA(p,q) with their Log-likelihood Values

Variables	Candidate Models	φ_1	φ_2	θ_1	Log-Likelihood
BDDP	ARFURIMA(1,d,1)	0.09(0.0003)	-------	0.27(0.0005)	12813.22
	SEMIFARMA(1,d,1)	0.41(0.0821)	-------	-0.54(0.0757)	12290.81
DBCP	ARFURIMA(1,d,1)	-0.09(0.0003)	-------	0.53(0.0007)	-5788.442
	SEMIFARMA(1,d,1)	0.24(0.0412)	-------	-0.62(0.0345)	-7031.430
WTCP	ARFURIMA(2,d,0)	-0.37(0.0006)	-0.14(0.0004)	-------	-6185.031
	SEMIFARMA(1,d,1)	0.40(0.0532)	-------	-0.68(0.0443)	−7263.33
BEUR	ARFURIMA(1,d,1)	0.13(0.0004)	-------	0.48(0.0007)	1323.859
	SEMIFARMA(1,d,1)	0.65(0.0568)	-------	-0.87(0.0391)	963.440
BPOU	ARFURIMA(1,d,1)	0.13(0.0003)	-------	0.55(0.0007)	1079.354
	SEMIFARMA(1,d,1)	0.63(0.0531)	-------	-0.88(0.0352)	676.159

Table 10. Estimation of ARFURIMA(p,q) and SEMIFARMA(p,q) with their Log-likelihood Values (cont.)

Variables	Candidate Models	φ_1	φ_2	θ_1	Log-Likelihood
BDOL	ARFURIMA(1,d,1)	0.12(0.0004)	-------	0.54(0.0007)	1523.427
	SEMIFARMA(1,d,1)	0.61(0.0562)	-------	-0.87(0.0383)	1121.097
ATIN	ARFURIMA(1,d,1)	0.59(0.0008)	-------	0.69(0.0008)	-38600.380
	SEMIFARMA(1,d,1)	0.84(0.0202)	-------	-0.90(0.0157)	−39979.420
KUSD	ARFURIMA(2,d,0)	-0.12(0.0003)	0.2(0.0004)	-------	33188.790
	SEMIFARMA(1,d,1)	0.12(0.0265)	-------	-0.62(0.0213)	32897.460
UUKP	ARFURIMA(1,d,1)	0.43(0.0007)	-------	0.59(0.0007)	-948.814
	SEMIFARMA(1,d,1)	0.68(0.0521)	-------	-0.77(0.0454)	−1504.901

Note: standard errors are in (·) except in the second column

4.3.2. The Diagnostic Test

Tests based on the residual's normality test of Jarque-Bera, the Ljung-Box and ARCH-LM tests were applied, and the results showed evidence of non-normality, serial correlation, and heteroscedasticity in both the ARFURIMA and SEMIFARMA models due to large statistic and p-values less than 0.05. However, a comparison of the statistics from the three tests showed that the ARFURIMA model performed better due to larger test statistic for each test. A table of this analysis can be provided by the author on request.

4.3.3. Forecasting accuracy

The Mean Absolute Error (MAE), Mean Percentage Error (MPE) and Mean Absolute Percentage Error (MAPE) were used to evaluate the forecast performance. The results are presented in Table 11 and showed that the ARFURIMA model produced a better forecast with minimum MAE, MPE and MAPE.

Table 11. Forecasts Accuracy Values of ARFURIMA and SEMIFARMA Model

Variables	Candidate Models	MAE	MPE	MAPE
BDDP	ARFURIMA(1,d,1)	0.001542	0.021271	0.063209
	SEMIFARMA(1,d,1)	0.001953	123.7131	149.1917
DBCP	ARFURIMA(1,d,1)	1.045593	-0.028338	1.598621
	SEMIFARMA(1,d,1)	1.046859	59.90192	193.3132
WTCP	ARFURIMA(2,d,0)	1.079942	-0.043681	1.625366
	SEMIFARMA(1,d,1)	1.084338	117.8703	221.0361
BEUR	ARFURIMA(1,d,1)	0.040129	1.028172	3.097527
	SEMIFARMA(1,d,1)	0.049917	108.1391	397.9863

Table 11. Forecasts Accuracy Values of ARFURIMA and SEMIFARMA Model (cont.)

Variables	Candidate Models	MAE	MPE	MAPE
BPOU	ARFURIMA(1,d,1)	0.032087	1.044535	3.116316
	SEMIFARMA(1,d,1)	0.054838	49.77582	222.7510
BDOL	ARFURIMA(1,d,1)	0.006603	-0.923786	3.006184
	SEMIFARMA(1,d,1)	0.036540	89.33419	188.2536
ATIN	ARFURIMA(1,d,1)	11.91682	-0.039669	1.227970
	SEMIFARMA(1,d,1)	19.86253	72.46209	153.1615
KUSD	ARFURIMA(2,d,0)	0.000223	-0.000138	0.076933
	SEMIFARMA(1,d,1)	0.137226	800.8632	1001.293
UUKP	ARFURIMA(1,d,1)	0.007133	0.002391	0.568356
	SEMIFARMA(1,d,1)	0.296838	122.9570	154.4095

Similarly, Diebold and Mariano (1995) accuracy tests indicated that the ARFURIMA was better in forecasting all the series at 0.05 level of significance. A table of this analysis can be provided by the author on request.

5. Conclusions

In this work, we defined the family of the ARFURIMA (p,d,q) model and the stationarity, invertibility and basic properties of the models were derived and presented. The presented simulations studies confirmed superiority of the ARFURIMA over the ARIMA in simulating nonstationary and the FURI series and thus proved the ILM properties of the ARFURIMA model and its large sample properties too. Besides, some applications of the model were presented and further confirmed a better fit of the ARFURIMA compared to the SEMIFARMA model.

In conclusion, this study provided another perspective in analysing large time series data for modelling and forecasting, and the findings suggested that the ARFURIMA model should be considered if the data show a type of the ILM process with a degree of fractional difference in the interval of $1 < d < 2$.

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