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Parameter estimation of exponentiated exponential distribution under selective ranked set sampling

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ABSTRACT

Partial ranked set sampling (PRSS) is a cost-effective sampling method. It is a combination of simple random sample (SRS) and ranked set sampling (RSS) designs. The PRSS method allows flexibility for the experimenter in selecting the sample when it is either difficult to rank the units within each set with full confidence or when experimental units are not available. In this article, we introduce and define the likelihood function of any probability distribution under the PRSS scheme. The performance of the maximum likelihood estimators is examined when the available data are assumed to have an exponentiated exponential (EE) distribution via some selective RSS schemes as well as SRS. The suggested ranked schemes include the PRSS, RSS, neoteric RSS (NRSS), and extreme RSS (ERSS). An intensive simulation study was conducted to compare and explore the behaviour of the proposed estimators. The study demonstrated that the maximum likelihood estimators via PRSS, NRSS, ERSS, and RSS schemes are more efficient than the corresponding estimators under SRS. A real data set is presented for illustrative purposes.

Key words: exponentiated exponential distribution, partial ranked set sampling, neoteric ranked set sampling, maximum likelihood method.

Mathematical Subject Classification: 62F10

1. Introduction

In many studies where sampling is used, such as environmental management, ecology, sociology, and agriculture, exact measurement of a selected unit is either difficult or costly and time-consuming. However, the ranking of a small set of selected units can be carried out easily either by visual inspection with respect to the study

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variable or on the basis of an auxiliary variable. The RSS scheme was first proposed by McIntyre (1952) to obtain a sample from a population in his study for estimating the yield of pastures. The RSS scheme outweighs the conventionally used SRS scheme in terms of the superior efficiency of the RSS estimators of population mean and variance (see Wolfe (2010)). Several studies have shown that the calculated estimators based on RSS are more efficient than their counterparts in SRS. For example, Bhoj and Ahsanullah (1996) used the RSS scheme to estimate the generalized geometric distribution parameters. Al-Odat and Al-Saleh (2001) considered estimation of the population mean using a variation of the RSS procedure. Mahdizadeh and Arghami (2010) discussed entropy estimation in RSS design and compared the results with those in SRS design. Hassan (2013) obtained a Bayesian estimator for the shape and scale parameters of the EE distribution using RSS. Abu-Dayyeh et al. (2013) used RSS to estimate the shape and scale parameters of the Pareto distribution. Samuh and Qtait (2015) used median RSS (MRSS) to estimate the shape and scale parameters of the EE distribution. Tahmasebi et al. (2017) provided Bayesian estimation for Rayleigh distribution based on SRS, RSS, and maximum RSS procedures with unequal samples in two cases: one cycle and *r*-cycles. Bantan et al. (2020) derived Zubair Lomax distribution parameter estimators under the RSS scheme. Al-Omari et al. (2020) considered stress-strength reliability estimator of the exponentiated Pareto model using MRSS and RSS designs. Almarashi et al. (2021) studied stress-strength reliability estimator for the Topp–Leone distribution using advanced sampling methods. Hassan et al. (2022) considered estimating system reliability using NRSS and MRSS data for generalized exponential distribution.

Some variations of the RSS scheme were proposed by several authors. The PRSS requires fewer sampling units and less ranking than the RSS and proves to be more efficient than the SRS (see Haq et al. (2013)). In the PRSS scheme, the experimenter selects (A) sample units using SRS and (B) sample units using RSS, producing a final sample of size M=A+B units. Thus, it requires fewer sampling units and fewer rankings than the RSS. The ERSS design has been suggested by Samawi et al. (1996) for estimating the population mean. Studies based on the ERSS scheme have been studied by several authors (see, for example, Hassan (2012), Hassan et al. (2014), (2015)). The NRSS scheme was suggested by Zamanzade and Al-Omari (2016) and it differs from the original RSS scheme by the composition of a single set of n^2 units instead of n sets of size n. This strategy has been shown to be effective, producing more efficient estimators for the population mean and variance than the SRS and RSS schemes. Several studies have been conducted based on the NRSS scheme by several authors (see, for example, Koyuncu and Karagöz (2018) and Sabry and Shaaban (2020)).

The EE distribution was introduced by Gupta and Kundu (1999) as a generalization of an exponential distribution. It is of great interest and is popularly used in analyzing lifetime or survival data. The cumulative distribution function (cdf) and the probability density function (pdf) of the EE distribution are given, respectively, by:

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; \qquad x,\alpha,\lambda > 0, \tag{1}$$

and

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}; \qquad x,\alpha,\lambda > 0, \tag{2}$$

where α and λ are shape and scale parameters, respectively. Many authors have studied the properties and applications of the EE distribution, including Raqab and Ahsanullah (2001), Gupta and Kundu (2007), Nadarajah (2011), Ristić and Balakrishnan (2012), Abu-Youssef et al. (2015), de Andrade et al. (2016) and Chesneau et al. (2022).

In this study, we introduce, for the first time, the likelihood function for any random variable X based on the PRSS scheme, which has not been considered in the literature yet. Further, the population parameter estimators of the EE distribution are considered based on the maximum likelihood (ML) method. Simulation studies are carried out to compare the behaviour of the proposed estimators based on PRSS, RSS, NRSS, ERSS, and SRS designs. Finally, we present an application to real data. The rest of the article is organized as follows. Section 2 describes the RSS, ERSS, NRSS, and PRSS schemes. Section 3 provides the ML estimator of the EE model based on the suggested schemes. Section 4 gives a numerical study as well as application to real data. Finally, concluding remarks are handled in Section 5.

2. Some Ranked Set Sampling Schemes

This section provides the notion and a short description of the proposed RSS, ERSS, NRSS, and the PRSS schemes.

2.1. Ranked Set Sampling

The basic idea behind selecting a sample under RSS can be described as follows:

Step 1: Allocate n^2 randomly selected units from the target population into n sets, each of size n.

Step 2: Without knowing any values for the variable of interest, rank the units within each set in terms of the variable of interest using your professional judgment.

Step 3: Choose a sample for actual quantification by including the smallest ranked unit in the first set and the second smallest ranked unit in the second set. The process is continued in this way until the largest ranked unit is selected from the last set.

Step 4: Repeat Steps 1–3 for r cycles to obtain a sample of size $m = nr$ for measurement.

2.2. Partial Ranked Set Sampling

The PRSS scheme is used when the experimenter is unable to inspect the required number of units or when the inspection cost per unit is high. At the same time, the PRSS scheme requires fewer identified units as compared with a RSS, also it provides more precise estimates than the commonly used SRS scheme. Thus, the PRSS scheme helps in reducing the total cost and expenditure that are involved in sampling. In order to select a PRSS of size m, the following steps are carried out:

Step 1: Define a coefficient k such that $k = an$, where $0 \le a < 0.5$.

Step 2: Select $2k$ SRS each of size one from the parent population. In order to select the remaining $n-2k$ units, select $n–2k$ sets each of size n from the parent population. Rank the units within each set and select the i^{th} ranked unit of the i^{th} sample, for $i = k+1,\ldots,n-k$. This completes one cycle of a PRSS of size n.

Step 3: To obtain PRSS of size $m = nr$, we repeat steps 1 and 2 r times. The total number of units that are involved in selecting a PRSS of size $n^2 - 2k(n-1)$. Note that for $k = 0$, PRSS is equivalent to RSS.

2.3. Neoteric Ranked Set Sampling

The NRSS design is applied in situations where the ranking of sample observations is much easier than obtaining their precise values (Zamanzade and Al-Omari (2016)). The NRSS method can be described as follows:

Step 1: Allocate n^2 randomly selected units from the target population and rank the sample units based on the pre-established ordering criterion.

Step 2: If n is odd, then select the $[\left((n+1)/2\right)+(i-1)n]^{th}$ ranked unit for $i = 1,\ldots,n$. But if n is even, select the $[J+(i-1)n]^{th}$ ranked unit, where $J = \left(n/2\right)$ if i is an even and $J = \left((n+2)/2\right)$ if i is an odd for $i = 1,\ldots,n$.

Step 3: Again, steps 1–2 can be repeated r times to obtain a final sample of size $m = nr$.

2.4. Extreme Ranked Set Sampling

The ERSS scheme is performed by quantifying the smallest and largest order statistics (Samawi et al. (1996)). The ERSS procedure is as follows:

Step 1: Allocate the n^2 selected units randomly from the target population into n sets, each of size n.

Step 2: Without yet knowing any values for the variable of interest, rank the units within each set with respect to a variable of interest.

Step 3: If the set size is odd, select the smallest unit from the first $(n-1)/2$ samples, from the other $(n-1)/2$ the largest unit and for the last sample select the median of the sample for actual measurement. If the set size is even, select the smallest unit from the first $n/2$ samples and from the other $n/2$ samples the largest unit for actual measurement.

Step 4: The steps 1 to 3 can be repeated *r* times to obtain a sample of size $m = nr$.

3. Parameter Estimation

In this section, the ML estimators of the EE distribution parameters are obtained based on SRS, RSS, ERSS, NRSS, and PRSS designs.

3.1. ML Estimator based on SRS

Let $X_1, X_2, \ldots, X_m$ be independent and identically distributed random variables from the EE distribution with pdf (2). The log-likelihood function of α and λ is specified by:

$$lnL_1 = m\,ln\,\alpha + m\,ln\,\lambda + (\alpha-1)\sum_{i=1}^{m} ln(1-e^{-\lambda x_i}) - \lambda\sum_{i=1}^{m} x_i.$$

The first partial derivatives of L_1 for each parameter are given by:

$$\frac{\partial ln\,L_1}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} ln\,(1-e^{-\lambda x_i}), \tag{3}$$

$$\frac{\partial lnL_1}{\partial \lambda} = \frac{m}{\lambda} + (\alpha-1)\sum_{i=1}^{m} \frac{x_i}{e^{\lambda x_i}-1} - \sum_{i=1}^{m} x_i. \tag{4}$$

Setting Equations (3) and (4) with zero and solving them numerically, we get the ML estimators of α and λ.

3.2. ML Estimator based on RSS

Here, we derive the ML estimators of the EE distribution parameters based on the RSS scheme. Assume that $X = \{X_{i(i)s}; i = 1,2,\ldots,n, s = 1,2,\ldots,r\}$ is a RSS observed from the EE distribution with sample size *nr*, *n* being the set size and *r* being the number of cycles. The likelihood function based on the RSS scheme is given by:

$$L_2 = \prod_{s=1}^{r}\prod_{i=1}^{n} C_1 f(x_{i(i)s})\left[F(x_{i(i)s}\right]^{i-1}\left[1-F(x_{i(i)s}\right]^{n-i}, \tag{5}$$

where $C_1 = n!/[(i-1)!(n-i)!]$. The log-likelihood function of (5), based on RSS, is yielded by substituting pdf (2) and cdf (1) in (5) as follows:

$$\ln L_2 \propto r\,n \ln\alpha + r\,n\ln\lambda + \sum_{s=1}^{r}\sum_{i=1}^{n}(\alpha i-1)\ln(T_{i(i)s}) - \lambda\sum_{s=1}^{r}\sum_{i=1}^{n}x_{i(i)s} + \sum_{s=1}^{r}\sum_{i=1}^{n}(n-i)\ln(1-(T_{i(i)s})^{\alpha}),$$

where $T_{i(i)s} = (1-e^{-\lambda x_{i(i)s}})$. The first derivatives of L_2 with respect to α and λ are given by:

$$\frac{\partial lnL_2}{\partial\alpha} = \frac{r\,n}{\alpha} + \sum_{s=1}^{r}\sum_{i=1}^{n} i\; lnT_{i(i)s} - \sum_{s=1}^{r}\sum_{i=1}^{n}\frac{(n-i)(T_{i(i)s})^{\alpha}\, lnT_{i(i)s}}{1-(T_{i(i)s})^{\alpha}}, \tag{6}$$

$$\frac{\partial\, ln\, L_2}{\partial\lambda} = \frac{r\,n}{\lambda} + \sum_{s=1}^{r}\sum_{i=1}^{n}\frac{(\alpha\, i-1)\,x_{i(i)s}}{e^{\lambda x_{i(i)s}}-1} - \sum_{s=1}^{r}\sum_{i=1}^{n}x_{i(i)s} - \sum_{s=1}^{r}\sum_{i=1}^{n}\frac{\alpha\,(n-i)\,(T_{i(i)s})^{\alpha-1}\,x_{i(i)s}e^{-\lambda x_{i(i)s}}}{1-(T_{i(i)s})^{\alpha}}. \tag{7}$$

Differentiate (6) and (7) and equate by zero, the estimators of α and λ, say $\hat{\alpha}$ and $\hat{\lambda}$, are obtained through an appropriate numerical technique.

In the following, the pdf of a random variable X based on PRSS, as well as its likelihood function, are introduced in the case of any continuous probability distribution. Then, we obtain the pdf of the EE distribution, under PRSS, as well as we provide its likelihood function. Furthermore, based on the log-likelihood function, we obtain the ML estimator of the EE distribution via the PRSS scheme.

3.2.1. Likelihood Function via PRSS

Here, we will define the likelihood function for the PRSS scheme depending on Lemma 1 using the order statistics theory.

Lemma 1:

Let $X=(X_1,X_2,...,X_k)$, and $X^*=(X_{n-k+1},X_{n-k+2},...,X_n)$, be k independent simple random samples each of size k. Also, let $X^{**}=(X_{(k+1)n},X_{(k+2)n},...,X_{(n-k)n})$, be the order statistics of size n–$2k$. We define the joint pdf of a random variable $X_{i(i)}$, under the PRSS scheme, as follows:

$$f_{X_{i(i)}}(x) = \begin{cases} f_{1X_i}(x) & ,i=1,...,k, \\ f_{2X^{**}_{(i)}}(x) & ,i=k+1,...,n-k, \\ f_{3X^*_i}(x) & ,i=n-k+1,...,n, \end{cases} \tag{8}$$

where $f_{1X_i}(x)$ is the pdf of SRS, $X=(X_1,X_2,...,X_k)$, and $f_{3X^*_i}(x)$ is the pdf of SRS $X^*=(X_{n-k+1},X_{n-k+2},...,X_n)$, while $f_{2X^{**}_{(i)}}(x)$ is the pdf of i^{th} order statistics of sample $X^{**} = (X_{(k+1)n}, X_{(k+2)n}, \ldots, X_{(n-k)n})$, where $(i = k+1, \ldots, n-k)$. Hence, we define, for the first time, the pdf of $X_{i(i)}$ under the PRSS scheme as follows:

$$f_{X_{i(i)}}(x) = C^* f_1(x) f_2(x) \left[F_2(x)\right]^{i-k-1} \left[1 - F_2(x)\right]^{n-i-k} f_3(x), \; -\infty < x < \infty, \tag{9}$$

where $C^* = \dfrac{(n-2k)!}{(i-k-1)!(n-i-k)!}$

Proposition 1:

Let $X_{i(i)s} = \{X_s \cup X_s^{**} \cup X_s^{*}\} = \{X_{is}, i = 1, \ldots, k\} \cup \{X_{i(i)s}, i = k+1, \ldots, n-k\} \cup \{X_{is}, i = n-k+1, \ldots, n\}$, $s = 1, \ldots, r$ be a PRSS observed from continuous distribution, with a sample size $m = nr$, where n is the set size and r is the number of cycles. Based on pdf (9), the likelihood function of random variable $X_{i(i)s}$ based on the PRSS design is as follows:

$$L_3 = \prod_{s=1}^{r} \left[\prod_{i=1}^{k} f_{1X_i}(x) \prod_{i=k+1}^{n-k} f_{2X^{**}_{(i)s}}(x) \prod_{i=n-k+1}^{n} f_{3X^{*}_i}(x) \right], \tag{10}$$

where $f_{1X_i}(x)$ is the pdf of SRS, $X = (X_1, X_2, \ldots, X_k)$, and $f_{3X^*_i}(x)$ is the pdf of SRS $X^* = (X_{n-k+1}, X_{n-k+2}, \ldots, X_n)$, while $f_{2X^{**}_{(i)s}}(x)$ is the pdf of i^{th} order statistics of sample $X^{**} = (X_{(k+1)n}, X_{(k+2)n}, \ldots, X_{(n-k)n})$, where $(i = k+1, \ldots, n-k)$.

3.2.2. ML Estimator of EE Distribution

Here, the ML estimators of α and λ for the EE distribution are derived based on the PRSS scheme. Assume that $X_{i(i)s} = \{X_{is}, i = 1, \ldots, k, s = 1, \ldots, r\} \cup \{X_{i(i)s}, i = k+1, \ldots, n-k, s = 1, \ldots, r\} \cup \{X_{is}, i = n-k+1, \ldots, n, s = 1, \ldots, r\}$ is a PRSS observed from the EE distribution with sample size *nr*, *n* being the set size and *r* being the number of cycles. The likelihood function, via RSS scheme, is obtained by inserting (1) and (2) in (10) as follows:

$$L_3 = \prod_{s=1}^{r} \left[\begin{array}{l} \displaystyle\prod_{i=1}^{k} \alpha\lambda e^{-\lambda x_{is}} (1 - e^{-\lambda x_{is}})^{\alpha-1} \prod_{i=k+1}^{n-k} C^* \alpha\lambda e^{-\lambda x_{i(i)s}} (1 - e^{-\lambda x_{i(i)s}})^{\alpha(i-k)-1} \left[1 - (1 - e^{-\lambda x_{i(i)s}})^{\alpha}\right]^{n-i-k} \\ \displaystyle\times \prod_{i=n-k+1}^{n} \alpha\lambda e^{-\lambda x_{is}} (1 - e^{-\lambda x_{is}})^{\alpha-1} \end{array} \right].$$

Hence, the logarithm of L_3, under the PRSS design, is as follows:

$$\begin{aligned} ln\, L_3 &= r(n-2k)\, ln\, C^* + r\, n(ln\, \alpha + ln\, \lambda) + (\alpha - 1)\sum_{s=1}^{r}\left(\sum_{i=1}^{k} ln(z_{is}) + \sum_{i=n-k+1}^{n} ln(z_{is}) \right) - \lambda \sum_{s=1}^{r} \sum_{i=n-k+1}^{n} x_{is} \\ &- \lambda \sum_{s=1}^{r}\sum_{i=1}^{k} x_{is} + \sum_{s=1}^{r}\sum_{i=k+1}^{n-k} [\alpha(i-k) - 1] ln(T_{i(i)s}) - \lambda \sum_{s=1}^{r}\sum_{i=k+1}^{n-k} x_{i(i)s} + \sum_{s=1}^{r}\sum_{i=k+1}^{n-k} (n-i-k) ln(1 - (T_{i(i)s})^{\alpha}), \end{aligned}$$

where $Z_{is}=1-e^{-\lambda x_{is}}$ and $T_{i(i)s}=(1-e^{-\lambda x_{i(i)s}})$. The first partial derivatives of L_3, for each parameter are:

$$\frac{\partial ln\, L_3}{\partial\alpha}=\frac{rn}{\alpha}+\sum_{s=1}^{r}\left(\sum_{i=1}^{k}ln(Z_{is})+\sum_{i=n-k+1}^{n}ln(Z_{is})\right)+\sum_{s=1}^{r}\left(\sum_{i=k+1}^{n-k}(i-k)ln(T_{i(i)s})-\sum_{i=k+1}^{n-k}\frac{(n-i-k)ln(T_{i(i)s})}{(T_{i(i)s})^{-\alpha}-1}\right),$$

$$\frac{\partial lnL_3}{\partial\lambda}=\frac{rn}{\lambda}+\sum_{s=1}^{r}\left(\sum_{i=1}^{k}\frac{(\alpha-1)x_{is}}{e^{\lambda x_{is}}-1}+\sum_{i=n-k+1}^{n}\frac{(\alpha-1)x_{is}}{e^{\lambda x_{is}}-1}\right)+\sum_{s=1}^{r}\sum_{i=k+1}^{n-k}\frac{(\alpha(i-k)-1)x_{i(i)s}}{e^{\lambda x_{i(i)s}}-1}$$

$$-\sum_{s=1}^{r}\left(\sum_{i=1}^{k}x_{is}+\sum_{i=k+1}^{n-k}x_{i(i)s}+\sum_{i=n-k+1}^{n}x_{is}\right)-\sum_{s=1}^{r}\sum_{i=k+1}^{n-k}\frac{\alpha(n-i-k)(T_{i(i)s})^{\alpha-1}x_{i(i)s}e^{-\lambda x_{i(i)s}}}{1-(T_{i(i)s})^{\alpha}}.$$

Clearly, it is not easy to obtain a closed form solution for $\partial ln\, L_3/\partial\alpha, \partial ln\, L_3/\partial\lambda$ after setting them to zero. Therefore, an iterative technique must be applied to solve these equations numerically.

3.3. ML Estimator based on NRSS

Using the NRSS technique, we obtain the ML estimators of the EE distribution parameters. Let $\{X_{b(i)s}, i=1,2,...,n; s=1,2,...,r\}$ and $w=n^2$ be a NRSS where n is the set size, r is the number of cycles, and $b(i)$ is chosen as:

$$b(i)=\begin{cases}\frac{n+1}{2}+(i-1)n, & if\ n\ odd\\ \frac{n}{2}+(i-1)n, & if\ n\, even, i\, even\\ \frac{n+2}{2}+(i-1)n, & if\ n\, even, i\, odd\end{cases}$$

According to Sabry and Shaaban (2020), the likelihood function, under the NRSS scheme, is given by:

$$L_4=\prod_{s=1}^{r}C_3\left[\prod_{i=1}^{n}f(x_{b(i)s})\prod_{i=1}^{n+1}[F(x_{b(i)s})-F(x_{b(i-1)s})]^{b(i)-b(i-1)-1}\right], \tag{11}$$

where $C_3=\frac{w!}{\prod_{i=1}^{n+1}(b(i)-b(i-1)-1)!}$, $b(0)=0$, $b(n+1)=w+1$ $x_{(b(0))}=-\infty, w=n^2$ and $x_{(b(i+1))}=\infty$.

The logarithm of (11), based on the NRSS scheme, is obtained as follows:

$$ln\, L_4\propto rn\left(ln\,\alpha+ln\,\lambda\right)+\sum_{s=1}^{r}\sum_{i=1}^{n}(\alpha-1)ln(N_{b(i)s})-\sum_{s=1}^{r}\sum_{i=1}^{n}\lambda x_{b(i)s}$$

$$+\sum_{s=1}^{r}\sum_{i=1}^{n+1}[b(i)-b(i-1)-1]ln[(N_{b(i)s})^{\alpha}-(N_{b(i-1)s})^{\alpha}],$$

where $N_{b(i)s} = 1 - e^{-\lambda x_{b(i)s}}$ and $N_{b(i-1)s} = 1 - e^{-\lambda x_{b(i-1)s}}$.

The first partial derivatives of L_4 with respect to each parameter are given by:

$$\frac{\partial lnL_4}{\partial \alpha} = \frac{r\,n}{\alpha} + \sum_{s=1}^{r}\sum_{i=1}^{n} ln(N_{b(i)s}) + \sum_{s=1}^{r}\sum_{i=1}^{n+1} \frac{(b(i)-b(i-1)-1)\left[(N_{b(i)s})^{\alpha} ln(N_{b(i)s}) - (N_{b(i-1)s})^{\alpha} ln(N_{b(i-1)s})\right]}{[(N_{b(i)s})^{\alpha} - (N_{b(i-1)s})^{\alpha}]}, \tag{12}$$

$$\frac{\partial ln\,L_4}{\partial \lambda} = \frac{r\,n}{\lambda} + \sum_{s=1}^{r}\sum_{i=1}^{n} \frac{(\alpha-1)\,x_{b(i)s}}{e^{\lambda x_{b(i)s}} - 1} - \sum_{s=1}^{r}\sum_{i=1}^{n} x_{b(i)s} + \sum_{s=1}^{r}\sum_{i=1}^{n+1} \frac{(b(i)-b(i-1)-1)[\alpha(N_{b(i)s})^{\alpha-1} e^{-\lambda x_{b(i)s}} x_{b(i)s} - \alpha(N_{b(i-1)s})^{\alpha-1} e^{-\lambda x_{b(i-1)s}} x_{b(i-1)s}]}{[(N_{b(i)s})^{\alpha} - (N_{b(i-1)s})^{\alpha}]}. \tag{13}$$

There is no closed form solution to (12) and (13), so a numerical technique will be used to obtain the ML estimators for α and λ, represented by $\hat{\alpha}$, and $\hat{\lambda}$.

3.4. ML Estimator based on ERSS

In this section, the ML estimation approach will be used to estimate the EE distribution parameters on the basis of the ERSS scheme.

3.4.1 ML Estimator for Odd *Set Size*

Suppose that

$X = \{X_{i(1)s}, i = 1,2,...g-1, s = 1,2,...,r\} \cup \{X_{i(n)s}, i = g, g+1,...,n-1, s = 1,2,...,r\} \cup \{X_{n(g)s},$ $, g = n+1/2, s = 1,...,r\}$ is an odd ERSS (ERSSO) design observed from the EE distribution, with sample size $m = nr$, where n is the set size, r is the number of cycles.

Then the likelihood function, under the ERSSO scheme, is given as follows:

$$L_5 \propto \prod_{s=1}^{r}\prod_{i=1}^{g-1} f_1(x_{i(1)s}) \prod_{s=1}^{r}\prod_{i=g}^{n-1} f_n(x_{i(n)s}) \prod_{s=1}^{r} f_g(x_{n(g)s}),$$

where $f_1(x_{i(1)s})$ and $f_n(x_{i(n)s})$ are the pdfs of the smallest and largest order statistics, respectively, and $f_g(x_{n(g)s})$ is the pdf of the median. Hence, the logarithm of L_5, based on ERSSO, is obtained as follows:

$$ln\,L_5 \propto r\,n\left[ln\alpha + ln\lambda\right] - \lambda \sum_{s=1}^{r}\sum_{i=1}^{g-1} x_{i(1)s} + (n-1)\sum_{s=1}^{r}\sum_{i=1}^{g-1} ln[1-(W_{i(1)s})^{\alpha}] - \lambda \sum_{s=1}^{r}\sum_{i=g}^{n-1} x_{i(n)s} + (\alpha-1)\sum_{s=1}^{r}\sum_{i=1}^{g-1} ln(W_{i(1)s}) + (\alpha n-1)\sum_{s=1}^{r}\sum_{i=g}^{n-1} ln(V_{i(n)s}) - \lambda \sum_{s=1}^{r} x_{n(g)s} + (\alpha g-1)\sum_{s=1}^{r} ln(E_{n(g)s}) + (g-1)\sum_{s=1}^{r} ln(1-(E_{n(g)s})^{\alpha}),$$

where $W_{i(1)s}=1-e^{-\lambda x_{i(1)s}}$, $V_{i(n)s}=1-e^{-\lambda x_{i(n)s}}$ and $E_{n(g)s}=1-e^{-\lambda x_{n(g)s}}$. The first partial derivatives of L_5 owing to α and λ are given, respectively, by:

$$\begin{aligned}\frac{\partial lnL_5}{\partial\alpha}&=\frac{r\,n}{\alpha}-\sum_{s=1}^{r}\sum_{i=1}^{g-1}\frac{(n-1)(W_{i(1)s})^{\alpha}ln(W_{i(1)s})}{1-(W_{i(1)s})^{\alpha}}+\sum_{s=1}^{r}\sum_{i=1}^{g-1}ln(W_{i(1)s})\\&+\sum_{s=1}^{r}\sum_{i=g}^{n-1}n\,ln(V_{i(n)s})+\sum_{s=1}^{r}g\,ln(E_{n(g)s})-\sum_{s=1}^{r}\frac{(g-1)(E_{n(g)s})^{\alpha}ln(E_{n(g)s})}{1-(E_{n(g)s})^{\alpha}},\end{aligned}\tag{14}$$

$$\begin{aligned}\frac{\partial lnL_5}{\partial\lambda}&=\frac{r\,n}{\lambda}-\sum_{s=1}^{r}\left[\sum_{i=1}^{g-1}x_{i(1)s}-\sum_{i=g}^{n-1}x_{i(n)s}-x_{n(g)s}\right]+\sum_{s=1}^{r}\sum_{i=1}^{g-1}\frac{(\alpha-1)x_{i(1)s}}{e^{\lambda x_{i(1)s}}-1}\\&-\sum_{s=1}^{r}\sum_{i=1}^{g-1}\frac{\alpha(n-1)(W_{i(1)s})^{\alpha-1}e^{-\lambda x_{i(1)s}}x_{i(1)s}}{1-(W_{i(1)s})^{\alpha}}+\sum_{s=1}^{r}\sum_{i=g}^{n-1}\frac{(\alpha n-1)x_{i(n)s}}{e^{\lambda x_{i(n)s}}-1}\\&+\sum_{s=1}^{r}\frac{(\alpha g-1)x_{n(g)s}}{e^{\lambda x_{n(g)s}}-1}-\sum_{s=1}^{r}\frac{\alpha(g-1)(E_{n(g)s})^{\alpha-1}e^{-\lambda x_{n(g)s}}x_{n(g)s}}{1-(E_{n(g)s})^{\alpha}}.\end{aligned}\tag{15}$$

Using an iterative technique for (14) and (15) after setting them with zero to produce the ML estimators of α and λ.

3.4.2. ML Estimator for Even Set Size

Suppose that $X=\{X_{i(1)s}, i=1,2,...g_1,\ s=1,2,...,r\}\cup\{X_{i(n)s},\ i=g_1+1,\ g_1+2,...,n,$ $s=1,2,...,r\}$ is an even ERSS (ERSSE) scheme observed from an EE distribution, with a sample of size $m=nr$, where n is the set size, r is the number of cycles and $g_1=n/2$. The likelihood function of the EE distribution from the ERSSE scheme is given by:

$$L_6\propto\prod_{s=1}^{r}\prod_{i=1}^{g_1}f_1(x_{i(1)s})\prod_{s=1}^{r}\prod_{i=g_1+1}^{n}f_n(x_{i(n)s}).$$

The logarithm of L_6 for the EE distribution, using the ERSSE scheme, is given by.

$$\begin{aligned}lnL_6&\propto r\,n\left(ln\,\alpha+ln\,\lambda\right)+(n-1)\sum_{s=1}^{r}\sum_{i=1}^{g_1}ln[1-(W_{i(1)s})^{\alpha}]+(\alpha-1)\sum_{s=1}^{r}\sum_{i=1}^{g_1}ln(W_{i(1)s})\\&-\lambda\sum_{s=1}^{r}\sum_{i=1}^{g_1}x_{i(1)s}+(\alpha n-1)\sum_{s=1}^{r}\sum_{i=g_1+1}^{n}ln(V_{i(n)s})-\lambda\sum_{s=1}^{r}\sum_{i=g_1+1}^{n}x_{i(n)s}.\end{aligned}$$

The first partial derivatives of L_6 owing to α and λ are given, respectively, by:

$$\frac{\partial lnL_6}{\partial\alpha}=\frac{r\,n}{\alpha}-\sum_{s=1}^{r}\sum_{i=1}^{g_1}\frac{(n-1)(W_{i(1)s})^{\alpha}ln(W_{i(1)s})}{1-(W_{i(1)s})^{\alpha}}+\sum_{s=1}^{r}\sum_{i=1}^{g_1}ln(1-e^{-\lambda x_{i(1)s}})+\sum_{s=1}^{r}\sum_{i=g_1+1}^{n}n\,ln(V_{i(n)s}),\tag{16}$$

and,

$$\frac{\partial lnL_6}{\partial \lambda}=\frac{r\,n}{\lambda}-\sum_{s=1}^{r}\sum_{i=1}^{g_1}\frac{\alpha\,(n-1)(W_{i\,(1)s})^{\alpha-1}e^{-\lambda x_{i\,(1)s}}\,x_{i\,(1)s}}{1-(W_{i\,(1)s})^{\alpha}}+\sum_{s=1}^{r}\sum_{i=1}^{g_1}\frac{(\alpha-1)x_{i\,(1)s}}{e^{\lambda x_{i\,(1)s}}-1}$$
$$-\sum_{s=1}^{r}\left[\sum_{i=1}^{g_1}x_{i\,(1)s}+\sum_{i=g_1+1}^{n}x_{i\,(n)s}\right]+\sum_{s=1}^{r}\sum_{i=g_1+1}^{n}\frac{(\alpha n-1)x_{i\,(n)s}}{e^{\lambda x_{i\,(n)s}}-1}. \tag{17}$$

After setting (16) and (17) to zero, there is no closed form solution, hence the ML estimators α and λ are derived using a numerical technique.

4. Numerical Study and Application

In this section, a numerical study is provided to evaluate the behaviour of ML estimates (MLEs) of the EE distribution based on the SRS, RSS, PRSS, NRSS, and ERSS schemes. Also, an application to one real data set is provided.

4.1. Numerical Study

A numerical evaluation is carried out to examine the performance of the MLEs. The MLEs are evaluated based on absolute biases (ABs), mean squared errors (MSEs), and relative efficiencies (REs). The simulation procedure is achieved via the MATHEMATICA software. The simulation algorithm is performed as follows:

Step 1: An SRS scheme $X_1, X_2, \ldots, X_n$ of sample sizes; m = 20, 40, 60 and 100 are considered; and these random samples are generated from the EE distribution by using the inversion method.

Step 2: An RSS scheme is considered as: $X_{1(1)s}, X_{2(2)s}, \ldots, X_{n(n)s}$; $s=1,\ldots,r$ having sample sizes; m = 20, 40, 60 and 100 with the number of cycles *r* = 5, 10, and 20 and set sizes n = 4, 5 and 6.

Step 3: A PRSS scheme is considered as:

$X_{1s}, X_{2s}, \ldots, X_{ks}$, $X_{k+1(k+1)s}, X_{k+2(k+2)s}, \ldots,$ $X_{n-k(n-k)s}$, $X_{n-k+1s}, X_{n-k+2s}, \ldots, X_{ns}$; *s*=1,…, *r* of sample sizes; m = 20, 40, 60 and 100, where (*n*, *r*)= (4,5), (4,10), (6,10) and (5,20).

Step 4: An NRSS scheme is considered as $X_{b(1)s}, X_{b(2)s}, \ldots, X_{b(n)s}$; *s* =1,…, *r* of sample sizes; m = 20, 40, 60 and 100, where (*n*, *r*)= (4,5), (4,10), (6,10) and (5,20).

Step 5: An ERSSO scheme is considered as

$X_{1(1)s}, \ldots, X_{g-1(1)s}, X_{g(n)s}, \ldots, X_{n-1(n)s}, X_{n(g)s}; g=\frac{n+1}{2}$, $s=1,\ldots,r$ of sample sizes; m = 20, 40, 60 and 100, where (*n*, *r*) = (5,4), (5,8), (5,12) and (5,20).

Step 6: An ERSSE scheme is considered as

$X_{1(1)s},\ldots,X_{g_1(1)s},X_{g_1+1(n)s},\ldots,X_{n(n)s};g_1=\frac{n}{2},s=1,\ldots,r$ of sample sizes; m = 20, 40, 60 and 100, where (n, r) = (4,5), (4,10), (6,10) and (4,25).

Step 7: Parameters' values are selected as (α =0.5, λ =0.4), (α =1, λ =0.4), (α =2, λ =2) and (α =3, λ =2). The MSEs and ABs of $\hat{\alpha}$ and $\hat{\lambda}$ are evaluated for different sample sizes.

Step 8: The efficiencies of different estimates under selective schemes with respect to SRS are defined by $RE_{\zeta}(\hat{\theta})=\frac{MSE_{SRS}(\hat{\theta})}{MSE_{\zeta}(\hat{\theta})}$, where $\hat{\theta}=(\hat{\alpha},\hat{\lambda}),\zeta$ = RSS, PRSS, NRSS, ERSSE, and ERSSO.

Step 9: The process is repeated 1000 times. The MLEs of $\hat{\alpha}$ and $\hat{\lambda}$ are inspected via ABs, MSEs, and their efficiencies.

Step 10: Empirical results are listed in Tables 1–3. Tables 1 and 2 list the observed results of ABs and MSEs of both estimates based on selective schemes. Also, Table 3 gives the efficiency of different schemes with respect to SRS.

Based on Tables 1–3 and Figures 1–11, we conclude the following:

1- For all sampling schemes, as m increases, the MSE and AB of $\hat{\alpha}$ and $\hat{\lambda}$ decreases (see Tables 1, 2).

2- The MLEs of $\hat{\alpha}$ and $\hat{\lambda}$ under the NRSS scheme provide more efficient estimates than the corresponding estimates in other schemes.

3- The MLEs of $\hat{\alpha}$ and $\hat{\lambda}$ under all modifications of the RSS schemes are more efficient than the corresponding estimates under the SRS scheme (see Figure 1 and Figure 2).

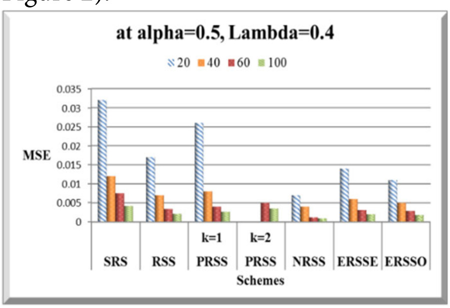

Figure 1. MSE of $\hat{\alpha}$ for all schemes at $\alpha=0.5$ and $\lambda=0.4$

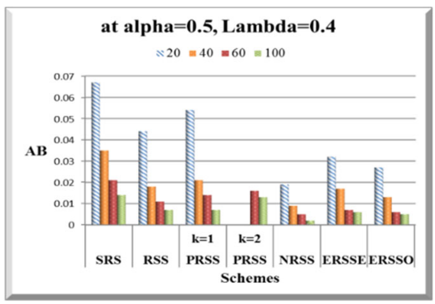

Figure 2. AB of $\hat{\alpha}$ for all schemes at $\alpha=0.5$ and $\lambda=0.4$

4- The MLEs of $\hat{\alpha}$ and $\hat{\lambda}$ under NRSS are more efficient than the others based on the RSS, PRSS (at $k=1$ and $k=2$) and ERSS schemes (see Figure 3 and Table 3).

5- The MLEs of $\hat{\alpha}$ and $\hat{\lambda}$ under the PRSS scheme at k =1, 2 are more efficient than the corresponding estimates under the SRS for all different values of m (see Figure 4).

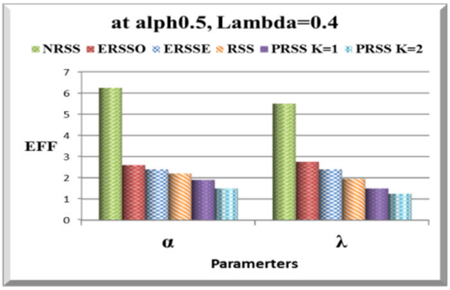

Figure 3. Efficiency of MLEs for all schemes at m = 60 at $\alpha = 0.5$, $\lambda = 0.4$

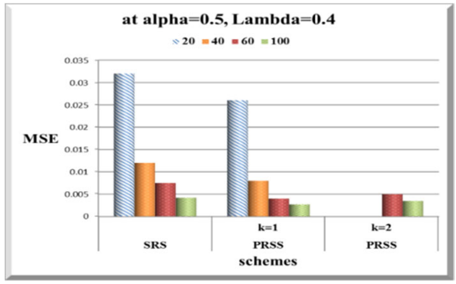

Figure 4. MSE of $\hat{\alpha}$ under SRS and PRSS schemes at $\alpha = 0.5$, $\lambda = 0.4$

6- The MSE of $\hat{\alpha}$ under PRSS increases as the value of k increases from k =1 to k =2, because the number of observations under SRS increases when selecting the PRSS. In this regard, we notice that as the value of k increases, the MSE of MLEs approaches the MSE of those under SRS (see Figures 4 and 5).

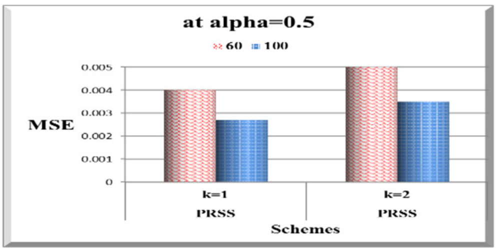

Figure 5. MSE of $\hat{\alpha}$ under PRSS for m = 60 and 100

7- As the value of α increases, the MSE of $\hat{\alpha}$ increases, while the MSE of $\hat{\lambda}$ decreases under different sampling schemes (see Figures 6, 7 and Tables 1, 2).

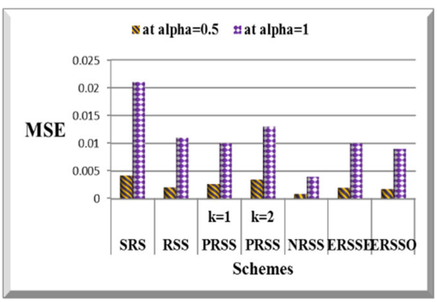

Figure 6. MSE of $\hat{\alpha}$ for all schemes at $m = 100$

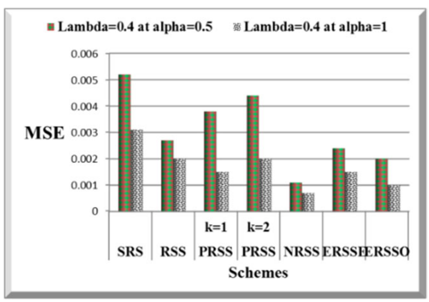

Figure 7. MSE of $\hat{\lambda}$ for all schemes at $m = 100$

8- As the value of α increases, from 0.5 to 1, the MSE and the AB of $\hat{\alpha}$ increase, while the MSE and the AB of $\hat{\lambda}$ decrease at $m=100$ (see Figures 8, 9 and Tables 1, 2).

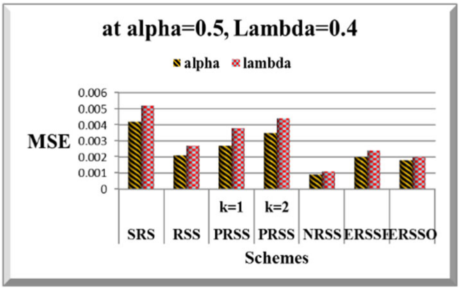

Figure 8. MSE of $\hat{\alpha}$ and $\hat{\lambda}$ for all schemes at $m=100$

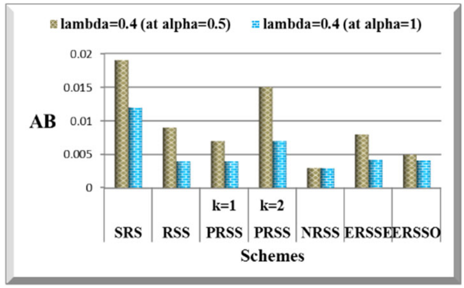

Figure 9. AB of $\hat{\lambda}$ when $\alpha=0.5$ and 1 for all schemes at $m=100$

9- The MLE of $\hat{\alpha}$ under the ERSSO scheme is more efficient than the others under the ERSSE for all *m* (see Figure 10 and Tables 1, 2).

10- As the sample size *m* increases, the efficiency of estimates also increases (see Figure 11 and Table 3).

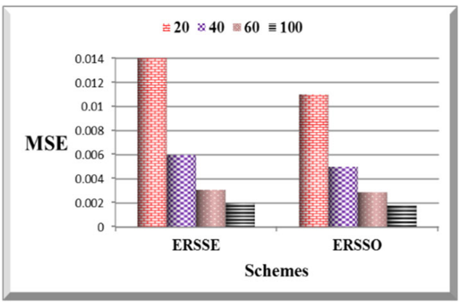

Figure 10. MSE of $\hat{\alpha}$ under ERSSE and ERSSO for all *m*

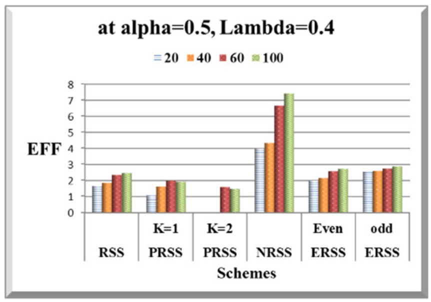

Figure 11. Efficiency of the MLEs for all schemes at all sample sizes

Table 1. The MSEs and ABs of the EE distribution based on different RSS schemes

m: n	m: r	scheme		$\alpha=0.5,\ \lambda=0.4$ MSE $\hat{\alpha}$	MSE $\hat{\lambda}$	AB $\hat{\alpha}$	AB $\hat{\lambda}$	$\alpha=1,\ \lambda=0.4$ MSE $\hat{\alpha}$	MSE $\hat{\lambda}$	AB $\hat{\alpha}$	AB $\hat{\lambda}$
20		SRS		0.028	0.039	0.057	0.074	0.201	0.021	0.169	0.051
4	5	RSS		0.017	0.022	0.044	0.047	0.102	0.012	0.097	0.032
4	5	PRSS	*k*=1	0.026	0.031	0.054	0.063	0.159	0.019	0.135	0.045
4	5	NRSS		0.007	0.010	0.019	0.025	0.043	0.006	0.052	0.016
4	5	ERSSE		0.014	0.017	0.032	0.038	0.081	0.010	0.092	0.029
5	4	ERSSO		0.011	0.014	0.027	0.037	0.077	0.009	0.088	0.028
40		SRS		0.013	0.017	0.032	0.041	0.078	0.009	0.088	0.030
4	10	RSS		0.007	0.009	0.018	0.022	0.038	0.005	0.047	0.015
4	10	PRSS	*k*=1	0.008	0.012	0.021	0.026	0.059	0.008	0.072	0.023
4	10	NRSS		0.003	0.004	0.007	0.012	0.016	0.002	0.026	0.009
4	10	ERSSE		0.006	0.007	0.017	0.019	0.03	0.004	0.028	0.011
5	8	ERSSO		0.005	0.006	0.013	0.013	0.028	0.003	0.021	0.010
60		SRS		0.0075	0.0088	0.021	0.024	0.037	0.006	0.047	0.016
6	10	RSS		0.0034	0.0045	0.011	0.012	0.017	0.003	0.021	0.007
6	10	PRSS	*k*=1	0.004	0.006	0.014	0.018	0.022	0.003	0.026	0.009
			k=2	0.005	0.007	0.016	0.020	0.033	0.004	0.041	0.013
6	10	NRSS		0.0012	0.0016	0.005	0.007	0.007	0.0011	0.013	0.005
6	10	ERSSE		0.0031	0.0036	0.007	0.008	0.014	0.0026	0.018	0.006
5	12	ERSSO		0.0029	0.0032	0.006	0.006	0.013	0.0018	0.015	0.0045
100		SRS		0.0042	0.0052	0.014	0.019	0.024	0.004	0.035	0.012
5	20	RSS		0.0021	0.0027	0.007	0.009	0.011	0.002	0.015	0.004
5	20	PRSS	*k*=1	0.0027	0.0038	0.007	0.007	0.010	0.0015	0.014	0.004
			k=2	0.0035	0.0044	0.013	0.015	0.013	0.0020	0.021	0.007
5	20	NRSS		0.0007	0.001	0.002	0.0002	0.004	0.0007	0.008	0.003
4	25	ERSSE		0.0019	0.0023	0.007	0.007	0.009	0.0015	0.014	0.0042
5	20	ERSSO		0.0018	0.002	0.005	0.005	0.008	0.0010	0.013	0.0041

Table 2. The MSEs and ABs of the EE distribution based on different RSS schemes

m		scheme		$\alpha=2,\ \lambda=2$				$\alpha=3,\ \lambda=2$			
n	r			MSE		AB		MSE		AB	
				$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
20		SRS		0.979	0.383	0.382	0.212	3.420	0.287	0.788	0.178
4	5	RSS		0.575	0.204	0.252	0.132	2.169	0.196	0.509	0.135
4	5	PRSS	k=1	0.958	0.282	0.369	0.160	2.565	0.260	0.768	0.168
4	5	NRSS		0.262	0.105	0.142	0.072	0.808	0.093	0.279	0.070
4	5	ERSSE		0.493	0.174	0.209	0.111	1.379	0.149	0.334	0.090
5	4	ERSSO		0.367	0.147	0.169	0.085	1.293	0.134	0.330	0.077
40		SRS		0.468	0.187	0.232	0.114	1.192	0.140	0.379	0.120
4	10	RSS		0.228	0.097	0.147	0.083	0.679	0.079	0.191	0.049
4	10	PRSS	k=1	0.340	0.127	0.180	0.084	0.885	0.112	0.327	0.079
4	10	NRSS		0.117	0.051	0.070	0.034	0.258	0.039	0.116	0.029
4	10	ERSSE		0.169	0.084	0.103	0.064	0.450	0.066	0.168	0.045
5	8	ERSSO		0.165	0.072	0.097	0.047	0.402	0.055	0.151	0.041
60		SRS		0.256	0.111	0.160	0.087	0.663	0.083	0.212	0.060
6	10	RSS		0.103	0.048	0.070	0.041	0.298	0.041	0.108	0.030
6	10	PRSS	k=1	0.156	0.069	0.079	0.040	0.459	0.058	0.161	0.039
			k=2	0.207	0.088	0.126	0.067	0.554	0.070	0.201	0.051
6	10	NRSS		0.033	0.017	0.019	0.013	0.091	0.019	0.051	0.017
6	10	ERSSE		0.098	0.048	0.072	0.042	0.240	0.038	0.102	0.025
5	12	ERSSO		0.089	0.041	0.067	0.033	0.206	0.032	0.099	0.031
100		SRS		0.222	0.078	0.092	0.050	0.314	0.050	0.139	0.040
5	20	RSS		0.072	0.033	0.058	0.029	0.172	0.025	0.061	0.019
5	20	PRSS	k=1	0.096	0.044	0.078	0.039	0.253	0.036	0.119	0.033
			k=2	0.120	0.053	0.089	0.048	0.303	0.042	0.133	0.034
5	20	NRSS		0.024	0.011	0.014	0.012	0.063	0.011	0.019	0.006
4	25	ERSSE		0.057	0.029	0.038	0.020	0.149	0.024	0.059	0.018
5	20	ERSSO		0.052	0.025	0.023	0.010	0.147	0.022	0.034	0.009

Table 3. Efficiency of the estimators based on RSS, PRSS (at k =1, 2), ERSSE, ERSSO, and NRSS

n	scheme		$\alpha=0.5, \lambda=0.4$		$\alpha=1, \lambda=0.4$		$\alpha=2, \lambda=2$		$\alpha=3, \lambda=2$	
			$EFF(\hat{\alpha})$	$EFF(\hat{\lambda})$	$EFF(\hat{\alpha})$	$EFF(\hat{\lambda})$	$EFF(\hat{\alpha})$	$EFF(\hat{\lambda})$	$EFF(\hat{\alpha})$	$EFF(\hat{\lambda})$
20	RSS		1.65	1.77	1.97	1.75	1.7	1.87	1.57	1.46
	PRSS	k=1	1.07	1.25	1.26	1.11	1.02	1.35	1.33	1.10
	NRSS		4	3.9	4.67	3.5	3.73	3.65	4.23	3.08
	ERSSE		2	2.29	2.48	2.1	1.98	2.20	2.48	1.92
	ERSSO		2.55	2.78	2.61	2.33	2.66	2.60	2.64	2.14
40	RSS		1.85	1.88	2.05	1.8	2.05	1.93	1.75	1.77
	PRSS	k=1	1.63	1.42	1.32	1.13	1.37	1.47	1.34	1.25
	NRSS		4.33	4.25	4.87	4.5	4	3.67	4.62	3.58
	ERSSE		2.16	2.43	2.6	2.25	2.76	2.22	2.64	2.12
	ERSSO		2.6	2.8	2.78	3	2.83	2.59	2.96	2.54
60	RSS		2.35	2	2	2	2.48	2.31	2.22	2.02
	PRSS	k=1	2	1.5	1.7	2	1.64	1.61	1.44	1.43
		k=2	1.6	1.28	1.12	1.5	1.23	1.26	1.19	1.18
	NRSS		6.66	5.62	5.3	4.45	7.75	6.52	7.28	4.63
	ERSSE		2.58	2.5	2.64	2.3	2.61	2.31	2.76	2.18
	ERSSO		2.75	2.81	2.85	3.3	2.87	2.71	3.21	2.59
100	RSS		2.47	2.22	2.18	2	3.08	2.36	2.83	2.32
	PRSS	k=1	1.92	1.57	2.4	2.66	12.31	1.77	1.92	1.61
		k=2	1.48	1.36	1.85	2	1.85	1.47	1.60	1.38
	NRSS		7.42	6	6	5.7	9.25	7.09	7.73	5.27
	ERSSE		2.73	2.6	2.67	2.66	3.89	2.68	3.26	2.41
	ERSSO		2.88	3	3	4	2.26	3.12	3.31	2.63

4.2. Application to Real Data

Here, a real data set is considered, and all the details for illustrative purposes are described. The data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). To check the validity of the fitted model, the Kolmogorov-Smirnov (KS) goodness of fit test and its

P-value are obtained. It is observed that the KS distance is 0.0931 with a corresponding P-value of 0.561. Additionally, some criteria measurements including values of -2lnL = 188.472, Akaike information criterion (AIC) = 192.472, correct AIC (AICc) =192.646, Bayesian information criterion (BIC) = 192.187 and Hannan-Quinn information criterion (HQIC) =194.285 were used to acquire more information. These results show that the EE model fits the data reasonably well.

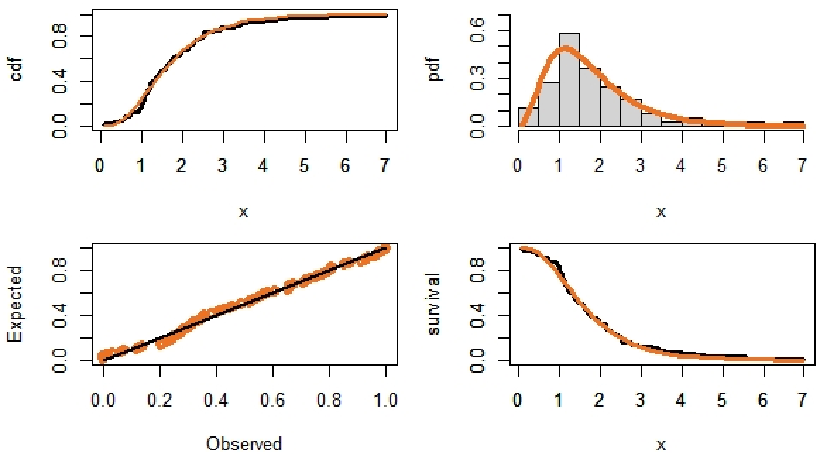

Figure 12. Plots of pdf, cdf, PP plots, and empirical survival function of the EE model

Table 4 gives the observed ranked values according to different sampling method techniques.

Table 4. The observation of different ranked sampling from real data set

Observation	**Schemes**						
	NRSS	**RSS**	**PRSS, *k*=1**	**PRSS, *k*=2**	**SRS**	**ERSSE**	**ERSSO**
1	0.10	0.10	0.10	0.10	0.10	0.56	0.10
2	0.74	0.77	0.44	0.33	0.33	0.92	0.72
3	1.00	1.05	0.39	0.59	0.44	1.07	0.77
4	1.15	1.12	1.07	1.00	0.56	1.09	0.93
5	1.24	1.22	1.15	1.05	0.59	1.22	1.05
6	1.46	1.46	1.20	1.07	0.72	1.36	1.07
7	1.53	1.53	1.21	1.07	0.74	1.63	1.08
8	1.71	1.72	1.22	1.08	0.77	1.76	1.15
9	1.97	2.13	1.46	1.09	0.92	2.15	1.20
10	2.53	2.45	1.71	1.22	0.93	2.40	1.22
11	3.42	3.27	2.02	1.30	0.96	2.93	1.36
12	5.55	5.55	2.15	1.34	1.00	4.02	1.44

Based on the theoretical study, we obtain the MLEs of α and λ under the PRSS, RSS, NRSS, ERSS, and SRS sampling from the considered data set. Table 5 gives the parameter estimators and their corresponding standard error (SE) of the EE model via the PRSS, RSS, NRSS, ERSS, and SRS schemes.

Table 5. Estimated parameters and SE of the EE distribution based on selective RSS schemes

Scheme		Estimators		SE		$RE_{\zeta}(\hat{\theta})$	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
NRSS		1.759	0.747	0.730	0.243	2.99	3.10
RSS		1.789	0.755	0.744	0.244	2.94	3.08
ERSS (even)		2.948	1.890	1.269	0.517	1.80	1.20
ERSS (odd)		2.217	1.323	0.926	0.394	2.40	1.80
PRSS	***k*=1**	3.809	1.963	1.440	0.781	1.40	1.19
	***k*=2**	3.998	1.986	1.525	0.888	1.30	1.17
SRS		5.260	2.330	2.882	0.974	1	1

Table 5 shows that the SE of $\hat{\alpha}$ and $\hat{\lambda}$ based on NRSS, RSS, ERSSE, ERSSO, and PRSS (at $k = 1$ and $k = 2$) are smaller than the corresponding estimates based on SRS for the considered data.

5. Conclusion

This paper introduces and defines the density and likelihood function for a random variable under the PRSS scheme. The maximum likelihood estimators of exponentiated exponential distribution are discussed under selective RSS schemes and the SRS scheme. The proposed sampling schemes are SRS, RSS, PRSS, NRSS, and ERSS. An intensive numerical study was conducted to compare the performances of different estimators using some accuracy measures. Generally, based on a numerical study, we conclude that all ranked schemes (RSS, PRSS, NRSS, and ERSS) are more efficient than the SRS scheme as evidenced by the results in Table 3. Also, PRSS is not the best method compared to the other ranked schemes, but it is important in some cases, in selecting the sample, when it is either difficult to rank the units within each set with full confidence or due to non-availability of experimental units.

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