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Zero-modified Poisson-Modification of Quasi Lindley distribution and its application

Ramajeyam Tharshan[1], Pushpakanthie Wijekoon[2]

ABSTRACT

The Poisson-Modification of Quasi Lindley (PMQL) distribution is a newly introduced mixed Poisson distribution for over-dispersed count data. The aim of this article is to introduce the Zero-modified PMQL (ZMPMQL) distribution as an alternative to the PMQL distribution in order to accommodate zero inflation/deflation. The method of obtaining the ZMPMQL distribution jointly with some of its important properties, namely the probability mass and distribution functions, mean, variance, index of dispersion, and quantile function are presented. Furthermore, some of its special cases are discussed. The maximum likelihood (ML) estimation method is used for the unknown parameter estimation. A simulation study is conducted in order to evaluate the asymptotic theory of the ML estimation method and to show the superiority of the ML method over the method of moments estimation. The applicability of the introduced distribution is illustrated by using a real-world data set.

Key words: over-dispersion, mixed Poisson distribution, PMQL distribution, zero modification, maximum likelihood estimation

1. Introduction

The Poisson distribution is the most commonly used distribution for modelling count data. One of the important properties of the Poisson distribution is that the mean and variance of the random variable are equal. This property is commonly referred as to equidispersion. However, in some real-world applications, especially actuarial, biomedical, engineering, ecological sciences, and others, observed data do not obey the equidispersion property. Here, the variance of the observed data exceeds the mean. This phenomenon is called over-dispersion (Greenwood and Yule, 1920). In such a situation, the mixed Poisson distributions are often adopted for modelling the count data as an alternative to the Poisson distribution. The literature provides various mixed Poisson distributions as negative binomial/Poisson-gamma (Greenwood and Yule, 1920), Poisson-Gamma product ratio (Irwin, 1975), Poisson-Generalized gamma (Albrecht, 1984), Poisson-Lindley (Sankaran, 1970), Poisson-Sujatha (Shanker, 2016c), Poisson-Quasi Lindley (Grine et al., 2017) distributions, among others.

Even though such mixed Poisson distributions can accommodate the longer right-tails and observed over-dispersion by heterogeneous populations, they do not perform well for observed over-dispersion by zero-inflation/deflation. To tackle this problem, the researchers

[1]Postgraduate Institute of Science, University of Peradeniya, Peradeniya, Sri Lanka & Department of Mathematics and Statistics, University of Jaffna, Jaffna, Sri Lanka. E-mail: tharshan10684@gmail.com. ORCID: https:/orcid.org/0000-0002-6112-2517.

[2]Department of Statistics and Computer Science, University of Peradeniya, Peradeniya, Sri Lanka. E-mail: pushpaw@pdn.ac.lk. ORCID: http://orcid.org/0000-0003-4242-1017.

have proposed several zero-modified mixed Poisson distributions. Here, we point out some notable examples for the zero-modified mixed Poisson distributions. Greenwood and Yule (1920) described the zero-inflated negative binomial (ZINB) distribution; Ghitany et al. (2008) proposed the Zero-truncated Poisson-Lindley distribution; Silva et al. (2018) introduced the Zero-modified Poisson-Sujatha distribution; Xavier et al. (2018) proposed the Zero-modified Poisson-Lindley (ZMPL) distribution.

Recently, Tharshan and Wijekoon (2021) introduced a lifetime distribution, namely the Modification of Quasi Lindley (MQL) distribution. Its probability density function (pdf) is given as

$$f_Y(y;\theta,\alpha,\delta)=\frac{\theta e^{-\theta y}}{(\alpha^3+1)\Gamma(\delta)}\left(\Gamma(\delta)\alpha^3+(\theta y)^{\delta-1}\right)\,;y>0,\theta>0,\alpha^3>-1,\delta>0, \quad (1)$$

where α and δ are shape parameters and θ is a scale parameter, and y is the respective random variable. Equation (1) presents the mixture of two non-identical distributions, exponential (θ), and gamma (δ,θ) with the mixing proportion, $p=\frac{\alpha^3}{\alpha^3+1}$. Then, the same authors (Tharshan and Wijekoon, 2022) obtained the Poisson-Modification of Quasi Lindley (PMQL) distribution by amalgamating the Poisson distribution and the MQL distribution. Its explicit form of the probability mass function (pmf) and some other important statistics are given in Section 2.

This paper aims to modify the PMQL distribution at zero probability to adopt the situation with an excessive number of zeros or a smaller number of zeros. The new distribution will be called the Zero-modified PMQL (ZMPMQL) distribution. The ZMPMQL distribution's unknown parameters will be estimated by the maximum likelihood estimation method. Further, the asymptotic property of the estimation method will be evaluated by a Monte Carlo simulation study.

This paper is structured as follows. Section 2 briefly presents the PMQL distribution and some of its statistical properties. In Section 3, we introduce the ZMPMQL distribution with some of its important structural properties. Its quantile function is discussed in Section 4. Section 5 covers the simulation of the random variables and the maximum likelihood estimator (MLE) for the ZMPMQL distribution. Section 6 studies the asymptotic property of the MLE and the applicability of the ZMPMQL distribution by designing a Monte Carlo simulation study and using a real-world data set, respectively.

2. PMQL distribution

Suppose the random variable $X|\Lambda$ is said to have the Poisson distribution with parameter λ. Then, its pmf can be written as

$$f_{X|\Lambda}(x|\lambda)=\frac{e^{-\lambda}\lambda^x}{x!};\ x=0,1,2,...,\ \lambda>0. \quad (2)$$

As defined by Tharshan and Wijekoon (2022), the PMQL distribution is the resultant distribution of X by assuming the Poisson parameter λ to be followed the MQL distribution.

Its pmf is given as

$$f_X(x) = \frac{\theta\left(\Gamma(\delta)\Gamma(x+1)\alpha^3(1+\theta)^{\delta-1} + \theta^{\delta-1}\Gamma(x+\delta)\right)}{x!(\alpha^3+1)(1+\theta)^{x+\delta}\Gamma(\delta)}; \tag{3}$$

$$x = 0,1,2,...,\theta > 0,\delta > 0,\alpha^3 > -1.$$

It can be shown that equation (3) represents a two-component mixture of geometric $(\frac{\theta}{1+\theta})$ and negative binomial $(\delta, \frac{1}{1+\theta})$ with the mixing proportion $p = \frac{\alpha^3}{\alpha^3+1}$.

Its corresponding cumulative distribution function (cdf) is given as

$$F_X(x) = \sum_{t=0}^{x} f(x) =$$

$$\frac{\delta(1+\theta)^{\delta-1}\Gamma(\delta)\alpha^3\Gamma(x+1)((1+\theta)^{x+1}-1) + \theta^{\delta}\Gamma(x+\delta+1)_2F_1(1,x+\delta+1;\delta+1;\frac{\theta}{1+\theta})}{(\alpha^3+1)\Gamma(\delta)x!\delta(1+\theta)^{x+\delta+1}} \tag{4}$$

$$;x = 0,1,2,...,\theta > 0,\delta > 0,\alpha^3 > -1,$$

where ${}_2F_1(c,d;r;w)$ is the Gaussian hypergeometric function defined as

$${}_2F_1(c,d;r;w) = \sum_{i=0}^{\infty} \frac{(c)_i(d)_i w^i}{(r)_i i!},$$

which is a special case of the generalized hypergeometric function given by the expression

$${}_aF_b(p_1,p_2,...p_a;q_1,q_2,...q_b;w) = \sum_{i=0}^{\infty} \frac{(p_1)_i...(p_a)_i w^i}{(q_1)_i...(q_b)_i i!},$$

and $(p)_i = \frac{\Gamma(p+i)}{\Gamma(p)} = p(p+1)...(p+i+1)$ is the Pochhammer symbol.

Tharshan and Wijekoon (2022) showed that its r^{th} factorial moment is given as

$$\mu'_{(r)} = \frac{\Gamma(\delta)\Gamma(r+1)\alpha^3 + \Gamma(\delta+r)}{(\alpha^3+1)\Gamma(\delta)\theta^r}. \tag{5}$$

By using the following relationship

$$\mu'_r = E(X^r) = \sum_{i=0}^{r} S(r,i)\mu'_{(i)}\ ;\ \ r = 1,2,...,$$

where $S(r,i)$ is the Stirling numbers of the second kind, which is defined as

$$S(r,i) = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}j^r\ ,\ \ 0 < i < r,$$

they have obtained the raw moments of X. Then, they have shown that its mean and variance are

$$\mu_{(PMQL)} = \frac{\alpha^3+\delta}{(\alpha^3+1)\theta},\quad \text{and } \sigma^2_{(PMQL)} = \mu_{(PMQL)} + \mu^2_{(PMQL)}\left(\frac{\alpha^3(\alpha^3+2+\delta(\delta-1))+\delta}{(\alpha^3+\delta)^2}\right),$$

respectively. Its index of dispersion (ID) was derived as

$$ID_{(PMQL)} = \frac{\sigma^2_{(PMQL)}}{\mu_{(PMQL)}} = 1 + \frac{\alpha^3(\alpha^3+2+\delta(\delta-1))+\delta}{(\alpha^3+1)(\alpha^3+\delta)\theta}. \tag{6}$$

It is clear that the $ID_{(PMQL)} > 1$. Then, equation (6) implies that the PMQL distribution is an over-dispersed distribution. Further, the authors derived 2^{nd}, 3^{rd}, and 4^{th} raw moments of the PMQL distribution as

$$\mu_2' = \frac{\theta(\alpha^3+\delta)+2\alpha^3+\delta(\delta+1)}{(\alpha^3+1)\theta^2},$$

$$\mu_3' = \frac{\theta^2(\alpha^3+\delta)+3\theta(2\alpha^3+\delta(\delta+1))+6\alpha^3+\delta(\delta+1)(\delta+2)}{(\alpha^3+1)\theta^3},$$

$$\mu_4' = \frac{1}{(\alpha^3+1)\theta^4}\Big(\theta^3(\alpha^3+\delta)+7\theta^2(2\alpha^3+\delta(\delta+1))+6\theta(6\alpha^3+\delta(\delta+1)(\delta+2)) \\ +24\alpha^3+\delta(\delta+1)(\delta+2)(\delta+3)\Big).$$

3. Zero-modified PMQL distribution

The pmf of a zero-modified count distribution is given as

$$f_X(x) = \begin{cases} \phi+(1-\phi)g(0) & \text{for } x=0 \\ (1-\phi)g(x) & \text{for } x=1,2,..., \end{cases}$$

where $g(.)$ is the pmf of the parent count distribution and the parameter ϕ is the zero-modified parameter. Then, the random variable X is said to have the ZMPMQL $(\phi,\theta,\alpha,\delta)$ if its pmf is given as

$$f_X(x) = \begin{cases} \phi+(1-\phi)\dfrac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{(\alpha^3+1)(1+\theta)^{\delta}} & \text{for } x=0 \\ (1-\phi)\dfrac{\theta\Big(\Gamma(\delta)\Gamma(x+1)\alpha^3(1+\theta)^{\delta-1}+\theta^{\delta-1}\Gamma(x+\delta)\Big)}{x!(\alpha^3+1)(1+\theta)^{x+\delta}\Gamma(\delta)} & \text{for } x=1,2,..., \end{cases} \tag{7}$$

where $\delta>0, \theta>0, \alpha^3>-1$, and $\frac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{\theta^{\delta}-(1+\theta)^{\delta-1}(1+\theta+\alpha^3)} \le \phi \le 1$.

The corresponding cdf is given as

$$F_{(ZMPMQL)}(x) = \phi+(1-\phi)F_X(x), \tag{8}$$

where $F_X(x)$ is the cdf of the PMQL distribution, which is defined in equation (4).

Note that equation (7) is not a finite mixture model since ϕ can take negative values. Further, various ϕ values adopt various zero-modifications of the PMQL distribution.

Remarks:

(i) When $\phi = \frac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{\theta^{\delta}-(1+\theta)^{\delta-1}(1+\theta+\alpha^3)}$, the ZMPMQL distribution reduces to the zero-truncat -ed PMQL distribution. Here, ϕ no longer appears. The zero-truncated models are commonly used to study the length of hospital stay.

(ii) For $\frac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{\theta^{\delta}-(1+\theta)^{\delta-1}(1+\theta+\alpha^3)} < \phi < 0$, the ZMPMQL distribution reduces to the zero-deflated PMQL distribution, and zero-deflated models are very rare in practice.

(iii) When $\phi = 0$, the ZMPMQL distribution is the PMQL distribution.

(iv) For $0 < \phi < 1$, the ZMPMQL distribution reduces to the zero-inflated PMQL distribution. This can accommodate more zeros than the actual PMQL distribution.

(v) When $\phi = 1$, the ZMPMQL distribution is degenerated at zero, i.e. all probabilities of the distribution are concentrated at zero.

The pmf of the ZMPMQL distribution is shown in Figure 1. We can observe that the parameter ϕ controls the observed counts of zeros.

The mean, variance, and index of dispersion of the ZMPMQL distribution are given,

$$\mu_{(ZMPMQL)} = (1-\phi)\mu_{(PMQL)}, \quad \sigma^2_{(ZMPMQL)} = (1-\phi)\left(\sigma^2_{(PMQL)} + \phi\mu^2_{(PMQL)}\right)$$

and

$$ID_{(ZMPMQL)} = \frac{\sigma^2_{(PMQL)}}{\mu_{(PMQL)}} + \phi\mu_{(PMQL)} = ID_{(PMQL)} + \phi\mu_{(PMQL)},$$

respectively, where $\mu_{(PMQL)}$, $\sigma^2_{(PMQL)}$, and $ID_{(PMQL)}$ are mean, variance, and index of dispersion of the PMQL distribution, respectively. Further, the 2^{nd}, 3^{rd}, and 4^{th} raw moments of the ZMPMQL distribution are $(1-\phi)\mu_2'$, $(1-\phi)\mu_3'$, and $(1-\phi)\mu_4'$, respectively, where μ_2', μ_3', and μ_4' are 2^{nd}, 3^{rd}, and 4^{th} raw moments of the PMQL distribution, respectively discussed in Section 2.

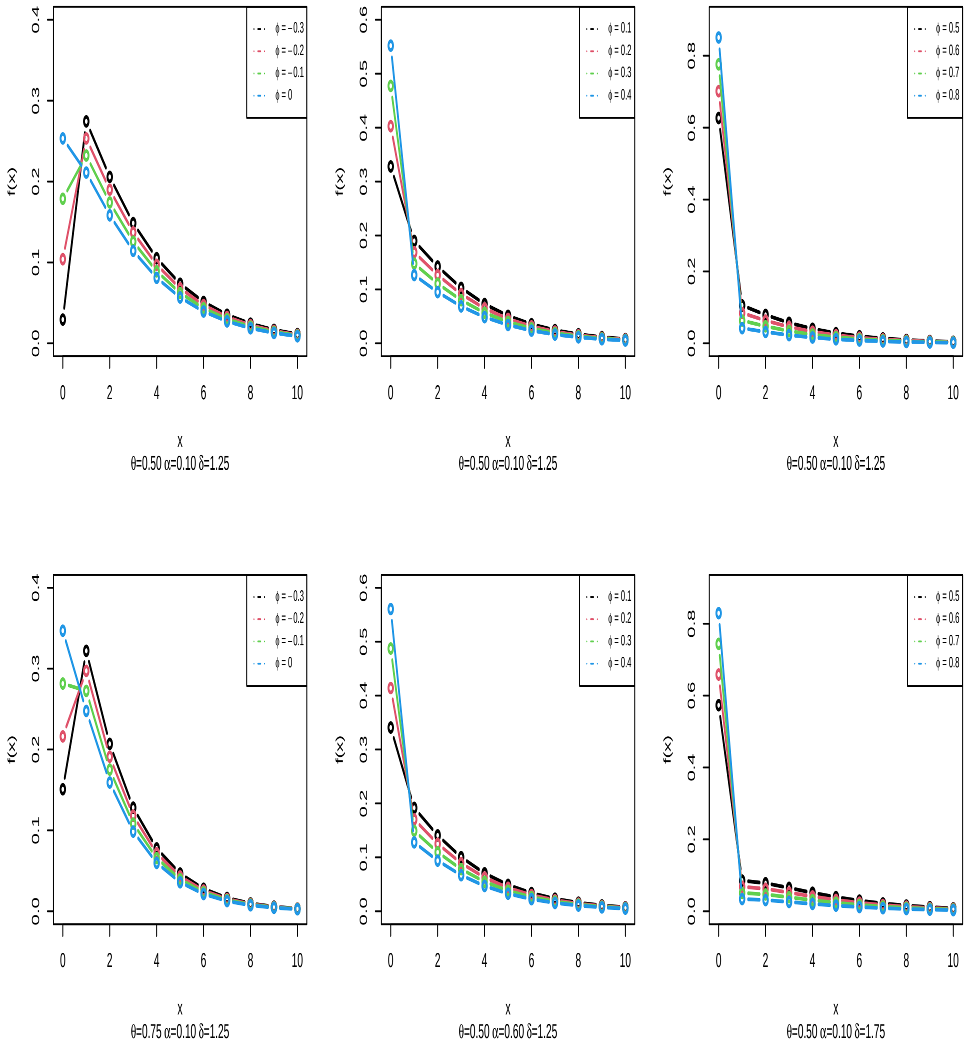

Figure 1: The pmf of the ZMPMQL distribution at different parameter values of ϕ, θ, α, and δ

4. Quantile function

The u^{th} quantile of the ZMPMQL distribution can be derived by solving $F(x_u) = u$ for $0 < u < 1$. It is defined as

$$\phi\beta_1(x_u) + (1-\phi)\left(\beta_2(x_u) + \theta^{\delta}\Gamma(x_u+\delta+1)_2F_1((1, x_u+\delta+1; \delta+1; \frac{\theta}{1+\theta}\right) - u\beta_1(x_u) = 0, \tag{9}$$

where

$$\beta_1(x_u) = (\alpha^3+1)\Gamma(\delta)x_u!\delta(1+\theta)^{x_u+\delta+1},$$

and

$$\beta_2(x_u) = \delta(1+\theta)^{\delta-1}\Gamma(\delta)\alpha^3\Gamma(x_u+1)((1+\theta)^{x_u+1}-1).$$

Since equation (9) is not a linear function with respect to x_u, the estimates of quantiles can be evaluated by using the Newton Raphson method. Further, the first three quantiles can be found by substituting $u = 0.25,\ 0.50$, and 0.75 in equation (9) and solving the respective non-linear equations.

5. Simulation and parameter estimation

5.1. Simulation of random variables

Here, we provide an algorithm to simulate the random variables $x_1, x_2, ..., x_n$ from the ZMPMQL $(\phi, \theta, \alpha, \delta)$ with size n based on the inverse transform method.

Algorithm

i Simulate random variables, $U_i \sim$uniform$(0,1);\ i = 1,2,...,n.$

ii Solve the non-linear equation for $[x_{u_i}]$;
$\phi\beta_1(x_{u_i}) + (1-\phi)\left(\beta_2(x_{u_i}) + \theta^\delta\Gamma(x_{u_i}+\delta+1)_2F_1((1,x_{u_i}+\delta+1;\delta+1;\frac{\theta}{1+\theta}\right) - u_i\beta_1(x_{u_i})$ $= 0$, where $\beta_1(x_{u_i})$ and $\beta_2(x_{u_i})$ are defined as in Section 4. Further, $[.]$ denotes the integer part.

5.2. Parameter estimation of the ZMPMQL distribution

This subsection presents the unknown parameter estimation of the ZMPMQL distribution based on the method of moments and the maximum likelihood estimation method.

5.2.1 Method of moments estimator (MME)

Let $x_1, x_2, ..., x_n$ be a random sample of size n from the ZMPMQL distribution. Then, the method of moments estimators of ϕ, θ, α, and δ, abbreviated as $\hat{\phi}_{MME}, \hat{\theta}_{MME}, \hat{\alpha}_{MME}$, and $\hat{\delta}_{MME}$ are found by equating the first four raw moments, $\mu_r^{'}$ $(r = 1,2,3,4)$ to the sample moments, say $\frac{\sum_{i=1}^{n} x_i}{n}, r = 1,2,3,4$, and solving the system of non-linear equations. The system of non-linear equations are as follows:

$$n(1-\phi)(\alpha^3+\delta) - (\alpha^3+1)\theta\sum_{i=1}^{n} x_i = 0,$$

$$n(1-\phi)\bigg(\theta(\alpha^3+\delta)+2\alpha^3+\delta(\delta+1)\bigg)-(\alpha^3+1)\theta^2\sum_{i=1}^{n}x_i^2=0,$$

$$n(1-\phi)\bigg(\theta^2(\alpha^3+\delta)+3\theta(2\alpha^3+\delta(\delta+1))+6\alpha^3+\delta(\delta+1)(\delta+2)\bigg)$$

$$-(\alpha^3+1)\theta^3\sum_{i=1}^{n}x_i^3=0,$$

$$n(1-\phi)\bigg(\theta^3(\alpha^3+\delta)+7\theta^2(2\alpha^3+\delta(\delta+1))+6\theta(6\alpha^3+\delta(\delta+1)(\delta+2))$$

$$+24\alpha^3+\delta(\delta+1)(\delta+2)(\delta+3)\bigg)-(\alpha^3+1)\theta^4\sum_{i=1}^{n}x_i^4=0.$$

5.2.2 Maximum likelihood estimator (MLE)

Given a random sample $x_1,x_2,...,x_n$ with size n from the ZMPMQL$(\phi,\theta,\alpha,\delta)$, the likelihood function of the i^{th} sample value x_i is given as

$$L(\phi,\theta,\alpha,\delta|x_i)=\left(\phi+(1-\phi)\frac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{(\alpha^3+1)(1+\theta)^{\delta}}\right)^{I_{(X=0)}(x_i)}\times$$

$$\left((1-\phi)\frac{\theta(\Gamma(\delta)\Gamma(x_i+1)\alpha^3(1+\theta)^{\delta-1}+\theta^{\delta-1}\Gamma(x_i+\delta)}{x_i!(\alpha^3+1)(1+\theta)^{x_i+\delta}\Gamma(\delta)}\right)^{(1-I_{(X=0)}(x_i))},$$

where $I_S(.)$ is the indicator function of subset S. Then, the log-likelihood function is given as

$$\ell(\phi,\theta,\alpha,\delta|x)=$$

$$n_0\log\left(\phi+(1-\phi)\frac{\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})}{(\alpha^3+1)(1+\theta)^{\delta}}\right)+(n-n_0)\log\left(\frac{(1-\phi)\theta}{(\alpha^3+1)\Gamma(\delta)}\right)+$$

$$\sum_{i=1}^{n}(1-I_{(X=0)}(x_i))\Bigg(\log\bigg(\Gamma(\delta)\Gamma(x_i+1)\alpha^3(1+\theta)^{\delta-1}+\theta^{\delta-1}\Gamma(x_i+\delta)\bigg)-$$

$$\log\bigg(x_i!(1+\theta)^{x_i+\delta}\bigg)\Bigg),$$

where $n_0=\sum_{i=1}^{n}I_{(X=0)}(x_i)$, which is the zero counts of the sample.

The score functions are:

$$\frac{\partial\ell(\phi,\theta,\alpha,\delta|x)}{\partial\phi}=\frac{T_1}{T_2}-\frac{n-n_0}{1-\phi},$$

$$\frac{\partial \ell(\phi,\theta,\alpha,\delta|x)}{\partial \theta} =$$

$$\frac{n_0(1-\phi)\Big(T_3-T_4\Big)}{(1+\theta)^{\delta}T_2} + \frac{n-n_0}{\theta} + \sum_{i=1}^{n}(1-I_{(X=0)}(x_i))\frac{T_5}{T_6} - \sum_{i=1}^{n}(1-I_{(X=0)}(x_i))\frac{(x_i+\delta)}{1+\theta},$$

$$\frac{\partial \ell(\phi,\theta,\alpha,\delta|x)}{\partial \alpha} =$$

$$\frac{T_7}{(\alpha^3+1)T_2} - \frac{3\alpha^2(n-n_0)}{\alpha^3+1} + \sum_{i=1}^{n}(1-I_{(X=0)}(x_i))\frac{3\alpha^2\Gamma(\delta)\Gamma(x_i+1)(1+\theta)^{\delta-1}}{T_6},$$

and

$$\frac{\partial \ell(\phi,\theta,\alpha,\delta|x)}{\partial \delta} =$$

$$\frac{n_0(1-\phi)\theta\Big(T_8-T_9\Big)}{(1+\theta)^{\delta}T_2} - (n-n_0)(\psi(\delta)+\log(1+\theta)) + \sum_{i=1}^{n}(1-I_{(X=0)}(x_i))\frac{T_{10}+T_{11}}{T_6},$$

where

$$T_1 = n_0((\alpha^3+1)(1+\theta)^{\delta} - \theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})),$$

$$T_2 = \phi(\alpha^3+1)(1+\theta)^{\delta} + (1-\phi)\theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1}),$$

$$T_3 = (1+\theta)^{\delta}(\alpha^3(\theta(\delta-1)(1+\theta)^{\delta-2}+(1+\theta)^{\delta-1})+\delta\theta^{\delta-1}),$$

$$T_4 = \theta((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})\delta(1+\theta)^{\delta-1},$$

$$T_5 = \Gamma(\delta)\Gamma(x_i+1)\alpha^3(\delta-1)(1+\theta)^{\delta-2} + (\delta-1)\theta^{\delta-2}\Gamma(x_i+\delta),$$

$$T_6 = \Gamma(\delta)\Gamma(x_i+1)\alpha^3(1+\theta)^{\delta-1} + \theta^{\delta-1}\Gamma(x_i+\delta),$$

$$T_7 = n_0(1-\phi)\theta((\alpha^3+1)(3\alpha^2(1+\theta)^{\delta-1}) - 3\alpha^2((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})),$$

$$T_8 = (1+\theta)^{\delta}(\alpha^3(1+\theta)^{\delta-1}\log(1+\theta)+\theta^{\delta-1}\log(\theta)),$$

$$T_9 = ((1+\theta)^{\delta-1}\alpha^3+\theta^{\delta-1})(1+\theta)^{\delta}\log(1+\theta),$$

$$T_{10} = \Gamma(x_i+1)\alpha^3(\Gamma(\delta)(1+\theta)^{\delta-1}\log(1+\theta)+(1+\theta)^{\delta-1}\Gamma(\delta)\psi(\delta)),$$

and

$$T_{11} = \Gamma(x_i+\delta)\theta^{\delta-1}\log(\theta) + \theta^{\delta-1}\Gamma(x_i+\delta)\psi(x_i+\delta).$$

By setting the score functions equal to zero and solving the system of non-linear equations, the MLEs of ϕ, θ, α, and δ abbreviated as $\hat{\phi}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\alpha}_{MLE}$, and $\hat{\delta}_{MLE}$ can be derived. The system of non-linear equations with respect to the parameters can be solved by the Newton Raphson method. Here, the solutions of the parameter estimates will be obtained by using the *optim* function in the R package *stats*.

The asymptotic confidence intervals for the parameters ϕ, θ, α, and δ are derived by the asymptotic theory. The estimates are asymptotic four-variate normal with mean $(\phi,\ \theta,\ \alpha,\ \delta)$ and the observed information matrix is

$$I(\phi,\theta,\alpha,\delta)=\begin{pmatrix} -\frac{\partial^2\ell}{\partial\phi^2} & -\frac{\partial^2\ell}{\partial\phi\partial\theta} & -\frac{\partial^2\ell}{\partial\phi\partial\alpha} & -\frac{\partial^2\ell}{\partial\phi\partial\delta} \\ -\frac{\partial^2\ell}{\partial\theta\partial\phi} & -\frac{\partial^2\ell}{\partial\theta^2} & -\frac{\partial^2\ell}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell}{\partial\theta\partial\delta} \\ -\frac{\partial^2\ell}{\partial\alpha\partial\phi} & -\frac{\partial^2\ell}{\partial\alpha\partial\theta} & -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\delta} \\ -\frac{\partial^2\ell}{\partial\delta\partial\phi} & -\frac{\partial^2\ell}{\partial\delta\partial\theta} & -\frac{\partial^2\ell}{\partial\delta\partial\alpha} & -\frac{\partial^2\ell}{\partial\delta^2} \end{pmatrix}$$

at $\phi=\hat{\phi}_{MLE},\theta=\hat{\theta}_{MLE},\alpha=\hat{\alpha}_{MLE}$, and $\delta=\hat{\delta}_{MLE}$.

That is $(\hat{\phi}_{MLE},\hat{\theta}_{MLE},\hat{\alpha}_{MLE},\hat{\delta}_{MLE})\sim N_4((\phi,\theta,\alpha,\delta),I^{-1}(\phi,\theta,\alpha,\delta))$. Since the mathematical expressions of the second order partial derivatives of the log-likelihood function are very long, we do not present the elements of the observed information matrix, $I(\phi,\theta,\alpha,\delta)$.

Therefore, $(1-a)100\%$ confidence intervals for the parameters ϕ,θ,α, and δ are given by

$$\hat{\phi}_{MLE}\pm z_{a/2}\sqrt{Var(\hat{\phi}_{MLE})},\qquad \hat{\phi}_{MLE}\pm z_{a/2}\sqrt{Var(\hat{\phi}_{MLE})},$$

$$\hat{\alpha}_{MLE}\pm z_{a/2}\sqrt{Var(\hat{\alpha}_{MLE})},\qquad \hat{\delta}_{MLE}\pm z_{a/2}\sqrt{Var(\hat{\delta}_{MLE})},$$

where the $Var(\hat{\phi}_{MLE}),Var(\hat{\theta}_{MLE}),Var(\hat{\alpha}_{MLE})$, and $Var(\hat{\delta}_{MLE})$ are the variance of $\hat{\phi}_{MLE}$, $\hat{\theta}_{MLE},\hat{\alpha}_{MLE}$, and $\hat{\delta}_{MLE}$, respectively. They can be derived by the diagonal elements of $I^{-1}(\phi,\theta,\alpha,\delta)$ and $z_{a/2}$ is the critical value at a level of significance.

6. Monte Carlo simulation study and real-world application

6.1. Monte Carlo simulation study

Here, we evaluate the performance of the MLEs ($\hat{\phi}_{MLE},\hat{\theta}_{MLE},\hat{\alpha}_{MLE}$, and $\hat{\delta}_{MLE}$) and MMEs ($\hat{\phi}_{MME},\hat{\theta}_{MME},\hat{\alpha}_{MME}$, and $\hat{\delta}_{MME}$) with respect to the sample size n by designing a simulation study. We consider sample sizes of 60, 100, 200, and 300, and the simulation is repeated 1000 times. The simulation study is designed as follows:

(i) Simulate 1000 samples of size n.

(ii) Compute the MLEs and MMEs for the 1000 samples, say $(\hat{\phi}_i,\hat{\theta}_i,\hat{\alpha}_i,\hat{\delta}_i)$, $i=1,2,...,$ 1000.

(iii) Compute the average MLEs, MMEs, biases, and mean square errors (MSEs) by using the following equations:

$$\hat{S}(n)=\tfrac{1}{1000}\textstyle\sum_{i=1}^{1000}\hat{S}_i,\qquad bias_S(n)=\tfrac{1}{1000}\textstyle\sum_{i=1}^{1000}(\hat{S}_i-S),$$

$$MSE_S(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{S}_i - S)^2, \text{ for } S = \phi, \theta, \alpha, \delta, \text{ and } n = 60, 100, 200, 300.$$

Tables A1 and A2 present the performance of the MLEs and MMEs of ϕ, θ, α, and δ for different values of ϕ which are -0.4,-0.2,0.2, and 0.4. Here, the population values of θ, α, and δ are 0.10,0.50, and 2.50, respectively. Note that here the average MSEs are presented in parentheses in both tables. We can observe that in both estimation methods, the biases and MSEs decrease as n increases for all parameters in all given situations. This implies that the maximum likelihood estimation method and the method of moments estimation verify the asymptotic property for all given parameter estimates. Further, when comparing the performance of the MLEs and MMEs based on the estimators' biases and MSEs, it is clear that the maximum likelihood estimation method is better than the method of moment estimation.

6.2. Real-world application

This subsection is devoted to show the applicability of the ZMPMQL distribution over the negative binomial (NB), Zero-modified NB (ZMNB), Poisson-Lindley (PL), Zero-modifi -ed PL (ZMPL), and PMQL distributions. The best-fitted distribution is selected based on the negative log-likelihood $(-2logL)$, Akaike information criterion (AIC), and chi-square goodness of fit test statistic (χ^2). Further, the maximum likelihood estimation method is used to estimate the unknown parameters of the distributions.

The example data set presents the number of units of consumers' goods purchased by households over 26 weeks (Lindsey, 1995). The proportion of the zeros in the data set is 80.60%, which indicates that there exists inflation of zeros. The sample dispersion index of 4.761 shows that extreme over-dispersion is present. Further, the skewness and excess kurtosis of the data are 3.895 and 16.306, respectively. These results imply that the distribution of the data set is highly positively skewed having a very long right tail. Table 1 summarizes the comparability of the ZMPMQL distribution with the NB, ZINB, PL, ZMPL, and PMQL distributions. Based on the results, the ZMPMQL distribution having AIC=3419.95, $\chi^2 = 12.22$, p-value=0.06 gives a better fit than the other distributions. Further, when we compare the PMQL distribution and the ZMPMQL distribution, the likelihood ratio (LR) test statistic for the hypothesis testing $H_0 : \phi = 0$ vs $H_a : \phi \neq 0$ for this data set is 50.50, and it is greater than $\chi^2_{1,0.05} = 3.84$. Then, the parameter estimate $\hat{\phi}$ is significantly different from zero.

Table 1. Units of consumers good

Counts	Observed	Expected					
		NB	ZINB	PL	ZMPL	PMQL	ZMPMQL
0	1612	1239.62	1617.92	1272.56	1611.94	1585.72	1611.78
1	164	240.36	146.26	470.84	122.00	207.56	165.09
2	71	130.05	80.89	168.06	88.57	45.30	74.66
3	47	85.41	50.02	58.50	61.44	32.99	39.88
4	28	61.04	32.56	19.99	41.33	33.90	26.36
5	17	45.74	21.83	6.73	27.20	30.93	20.24
6	12	35.33	14.93	2.24	17.59	24.36	16.34
7	12	27.88	10.34	0.74	11.23	16.90	13.03
8	5	22.35	7.24	0.24	7.10	10.53	10.03
9	7	18.12	5.11	0.07	4.45	5.97	7.41
10	25	94.10	12.90	0.03	7.15	5.84	15.18
Total	2000	2000	2000	2000	2000	2000	2000
MLE		$\hat{\beta}=0.21$ (0.21) $\hat{\alpha}=0.12$ (0.01)	$\hat{\phi}=0.33$ (0.30) $\hat{\beta}=0.23$ (0.04) $\hat{\alpha}=0.20$ (0.13)	$\hat{\theta}=2.33$ (0.07)	$\hat{\phi}=0.73$ (0.01) $\hat{\theta}=0.75$ (0.04)	$\hat{\theta}=7.00$ (0.67) $\hat{\alpha}=2.12$ (0.07) $\hat{\delta}=32.88$ (2.81)	$\hat{\phi}=-0.61$ (0.03) $\hat{\theta}=1.45$ (0.25) $\hat{\alpha}=1.78$ (0.11) $\hat{\delta}=8.51$ (1.32)
χ^2		311.64	18.86	17992.93	77.89	110.99	12.22
p-value		0.00	0.01	0.00	0.00	0.00	0.06
$-2logL$		3800.90	3427.16	4216.21	3455.33	3462.45	**3411.95**
AIC		3804.90	3433.16	4218.21	3459.33	3468.45	**3419.95**

7. Conclusion

In this article, the zero-modified Poisson-Modification of Quasi Lindley distribution was introduced to model the over-dispersed count data having zero inflation/deflation. We derived some of its structural properties. Further, in order to estimate its unknown parameters, we derived its log-likelihood function and score functions. We showed that the maximum likelihood estimation method is a suitable method to estimate its unknown parameters via a Monte Carlo simulation study. The usefulness of the introduced distribution was illustrated by fitting it to a real-world data set. The results revealed its superiority over some other existing mixed Poisson and zero-modified mixed Poisson distributions.

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Appendix

Table A1. Performance of MLEs for ZMPMQL$(\phi, \theta = 0.10, \alpha = 0.50, \delta = 2.25)$

	$n = 60$		$n = 100$		$n = 200$		$n = 300$	
	MLE	Bias (MSE)	MLE	Bias (MSE)	MLE	Bias (MSE)	MLE	Bias (MSE)
$\phi = -0.40$								
ϕ	-0.0313	0.3686	-0.0770	0.3229	-0.0950	0.3049	-0.1328	0.2671
		(0.1402)		(0.1092)		(0.1043)		(0.0809)
θ	0.0373	-0.0626	0.0629	-0.0370	0.0865	-0.0134	0.0916	-0.0083
		(0.0040)		(0.0030)		(0.0026)		(0.0019)
α	0.5429	0.0429	0.5359	0.0359	0.4820	-0.0179	0.4903	-0.0096
		(0.0933)		(0.0710)		(0.0379)		(0.0333)
δ	0.9333	-1.5666	1.1108	-1.3891	1.1776	-1.3223	1.2210	-1.2789
		(2.6203)		(2.3570)		(2.0278)		(1.8462)
$\phi = -0.20$								
ϕ	-0.0566	0.1433	-0.0818	0.1181	-0.0975	0.1024	-0.1129	0.0870
		(0.0242)		(0.0291)		(0.0118)		(0.0097)
θ	0.0451	-0.0548	0.0848	-0.0151	0.0912	-0.0087	0.0961	-0.0038
		(0.0035)		(0.0025)		(0.0018)		(0.0009)
α	0.5629	0.0629	0.5900	0.0900	0.4547	-0.0452	0.4701	-0.0298
		(0.1958)		(0.1790)		(0.0805)		(0.0412)
δ	1.0944	-1.4055	1.2009	-1.2990	1.3474	-1.1525	1.4268	-1.0731
		(2.2541)		(1.9939)		(1.6011)		(1.2742)
$\phi = 0.20$								
ϕ	0.2582	0.0582	0.2427	0.0427	0.2257	0.0257	0.2119	0.0119
		(0.0133)		(0.0053)		(0.0023)		(0.0004)
θ	0.1473	0.0473	0.1379	0.0379	0.1193	0.0193	0.1063	0.0063
		(0.0195)		(0.0061)		(0.0022)		(0.0002)
α	0.4138	-0.0861	0.4308	-0.0691	0.4495	-0.0504	0.4891	-0.0108
		(0.0953)		(0.0701)		(0.0565)		(0.0314)
δ	2.9704	0.4704	2.7098	0.2098	2.6435	0.1435	2.5623	0.0623
		(1.2737)		(0.7320)		(0.3341)		(0.1134)
$\phi = 0.40$								
ϕ	0.4335	0.0335	0.4213	0.0213	0.4174	0.0174	0.4099	0.0099
		(0.0105)		(0.0027)		(0.0016)		(0.0008)
θ	0.1447	0.0447	0.1280	0.0280	0.1177	0.0177	0.1051	0.0051
		(0.0121)		(0.0022)		(0.0013)		(0.0002)
α	0.4109	-0.0890	0.4245	-0.0754	0.4309	-0.0690	0.4467	-0.0532
		(0.0988)		(0.0722)		(0.0650)		(0.0424)
δ	3.0761	0.5761	2.7313	0.2313	2.6573	0.1573	2.5932	0.0932
		(1.4439)		(0.7488)		(0.4785)		(0.2135)

Table A2. Performance of MMEs for ZMPMQL$(\phi, \theta = 0.10, \alpha = 0.50, \delta = 2.25)$

	$n = 60$		$n = 100$		$n = 200$		$n = 300$	
	MME	Bias (MSE)	MME	Bias (MSE)	MME	Bias (MSE)	MME	Bias (MSE)
$\phi = -0.40$								
ϕ	-0.0223	0.3776	-0.0547	0.3452	-0.0769	0.3230	-0.1062	0.2937
		(0.1427)		(0.1199)		(0.1186)		(0.1034)
θ	0.1832	0.0832	0.1649	0.0649	0.1376	0.0376	0.1335	0.0335
		(0.0103)		(0.0076)		(0.0044)		(0.0028)
α	0.7600	0.2600	0.6028	0.1028	0.5921	0.0921	0.5777	0.0777
		(0.1022)		(0.0843)		(0.0642)		(0.0440)
δ	4.5414	2.2914	4.1311	1.8811	3.6185	1.3685	3.8331	1.3331
		(6.7310)		(4.2928)		(2.3961)		(2.2493)
$\phi = -0.20$								
ϕ	-0.3827	-0.1827	-0.3612	-0.1612	-0.3493	-0.1493	-0.3243	-0.1243
		(0.0441)		(0.0301)		(0.0231)		(0.0173)
θ	0.1967	0.0967	0.1650	0.0650	0.1492	0.0492	0.1407	0.0407
		(0.0131)		(0.0082)		(0.0034)		(0.0024)
α	0.9864	0.4864	0.9691	0.4691	0.7889	0.2889	0.7324	0.2324
		(0.2694)		(0.3253)		(0.1238)		(0.0729)
δ	5.7357	3.4857	5.1516	2.9016	3.9256	1.6756	3.6355	1.3855
		(15.7459)		(12.4246)		(3.7138)		(2.6906)
$\phi = 0.20$								
ϕ	-0.0349	-0.2349	0.0166	-0.1833	0.0700	-0.1299	0.0993	-0.1006
		(0.0706)		(0.0530)		(0.0216)		(0.0139)
θ	0.2253	0.1253	0.1759	0.0759	0.1562	0.0562	0.1459	0.0459
		(0.0246)		(0.0101)		(0.0047)		(0.0030)
α	1.0917	0.5917	1.0101	0.5101	1.0036	0.5036	0.8582	0.3582
		(0.3969)		(0.3522)		(0.2782)		(0.1620)
δ	6.6309	4.3809	5.5530	3.3030	4.3311	2.0811	4.1850	1.9350
		(27.6692)		(15.6071)		(5.8163)		(4.5527)
$\phi = 0.40$								
ϕ	0.2199	-0.1800	0.2350	-0.1649	0.3151	-0.0848	0.3576	-0.0423
		(0.0465)		(0.0406)		(0.0114)		(0.0053)
θ	0.2581	0.1581	0.1962	0.0962	0.1670	0.0670	0.1553	0.0553
		(0.0398)		(0.0141)		(0.0067)		(0.0048)
α	1.0907	0.5907	1.0502	0.5502	0.9728	0.4728	0.8745	0.3745
		(0.4056)		(0.3492)		(0.2560)		(0.1674)
δ	7.5974	5.3474	6.2256	3.9756	4.6281	2.3781	4.2158	1.9658
		(42.3990)		(20.4941)		(7.9752)		(5.2737)