STATISTICS IN TRANSITION new series, December 2022
Vol. 23, No. 4, pp. 149–160, DOI 10.2478/stattrans-2022-0047
Received – 23.12.2021; accepted – 22.08.2022

Comparison of confidence intervals for variance components in an unbalanced one-way random effects model

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ABSTRACT

The purpose of this paper is to study and compare the methods for constructing confidence intervals for variance components in an unbalanced one-way random effects model. The methods are based on a classical exact, generalised pivotal quantity, a fiducial inference and a fiducial generalised pivotal quantity. The comparison of criteria involves the empirical coverage probability that maintains at the nominal confidence level of 0.95 and the shortest average length of the confidence interval. The simulation results show that the method based on the generalised pivotal quantity and the fiducial inference perform very well in terms of both the empirical coverage probability and the average length of the confidence interval. The classical exact method performs well in some situations, while the fiducial generalised pivotal quantity performs well in a very unbalanced design. Therefore, the method based on the generalised pivotal quantity is recommended for all situations.

Key words: variance components, unbalanced one-way random effects model, pivotal quantity, fiducial inference, coverage probability.

1. Introduction

The one-way random effects model is studied in many applications, such as medical treatment, animal breeding studies, agricultural genetics and industrial process management, etc. The variance components of this model are used to consider the different sources of variation. For example, radiotherapy doses for cancer treatment are determined by process variation due to difference in area of organs of individual patients and diagnosis of individual physician (Demetrashvili et al., 2016). Thus, the inferences for variance components in the model is of interest. Consider the one-way random effects model

$$y_{ij} = \mu + a_i + e_{ij}, \;\; i = 1, \ldots, g, \; j = 1, \ldots, n_i, \tag{1}$$

where y_{ij} is the random observation, μ is the overall mean. The random group effects a_i and the random errors e_{ij} are mutually independent random variables, and distributed

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as $N(0,\sigma_a^2)$ and $N(0,\sigma_e^2)$, respectively. In addition, let $n=\sum_{i=1}^{g} n_i$ denote the number of the total observations. When the number of observations n_i of each group is equal, model (1) is called balanced model. Otherwise, it is called unbalanced model. The source of variation is known as the variance components, namely, σ_a^2 and σ_e^2. In general, σ_a^2 is called between-group variance component, and σ_e^2 is called within-group variance component. The proportion of the between-group variance component and the total variation can be written in the form $\rho=\sigma_a^2/(\sigma_a^2+\sigma_e^2)$, which measures the importance of one effect related to the other effect.

One very important property of an estimator is minimal sufficient statistics. A closed-form function of the minimal sufficient statistics is available in balanced random model. However, these functions are unavailable in unbalanced random model as described by Searle et al. (2006). Furthermore, solving the closed-form functions of the minimal sufficient statistics in the unbalanced case is computationally complicated for estimation of the variance components. There are several works in the literature that studied inferences for variance components in unbalanced model, such as Wald (1940), Thomas and Hultquist (1978), Park and Burdick (2003), and Arendacká (2005) which are based on a pivotal quantity approach. Ting et al. (1990) and Hartung and Knapp (2000) studied that by the classical exact method. Li and Li (2007) and Lidong et al. (2008) used the idea of a fiducial generalized confidence interval for variance components. Liu et al. (2016) proposed the concept of the fiducial generalized pivotal quantity for constructing the confidence interval for variance components in unbalanced model.

The aim of this paper is to compare five methods which are applicable to confidence intervals for between-group variance component in unbalanced one-way random effects model. These five methods are as follows: the Ting and others (TG) method (Ting et al., 1990), the Hartung-Knapp (HK) method (Hartung and Knapp, 2000), the Park-Burdick (PB) method (Park and Burdick, 2003), the Li-Li (LL) method (Li and Li, 2007), and the Liu-Xu-Hannig (LXH) method (Liu et al., 2016).

The paper is organized as follows. Section 2 describes the model and notation. Section 3 presents the methods for constructing a confidence interval for σ_a^2. Section 4 shows the results of a simulation study and compare the performance of the methods. Section 5 provides previously published data example. In the final Section 6, a conclusion is given.

2. Model and notation

A matrix formulation of the model (1) is given by

$$\mathbf{Y}=\mathbf{1}_n\mu+\mathbf{ZA}+\mathbf{E}, \tag{2}$$

where $\mathbf{Y}=(\mathbf{Y}_1',\ldots,\mathbf{Y}_g')'$ with $\mathbf{Y}_i=(Y_{i1},\ldots,Y_{in_i})'$ for $i=1,\ldots,g$, $\mathbf{1}_n=(\mathbf{1}_{n_1}',\ldots,\mathbf{1}_{n_g}')'$ with $\mathbf{1}_{n_i}$ is a $n_i\times 1$ vector of ones, and $n=\sum_{i=1}^{g} n_i$. The matrix $\mathbf{Z}=\mathrm{diag}(\mathbf{1}_{n_1},\ldots,\mathbf{1}_{n_g})$ is known as incidence matrix of size $n\times g$. The random group effects vector $\mathbf{A}=(A_1,\ldots,A_g)'$ is distributed as $N(\mathbf{0}_g,\sigma_a^2\mathbf{I}_g)$ and the random errors vector $\mathbf{E}=(\mathbf{E}_1',\ldots,\mathbf{E}_g')'$ with $\mathbf{E}_i=(E_{i1},\ldots,E_{in_i})'$ is distributed as $N(\mathbf{0}_n,\sigma_e^2\mathbf{I}_n)$, where $\mathbf{0}_c$ is a $c\times 1$ vector of zeros, and $\mathbf{I}_c$ is a $c\times c$ identity matrix. The random vectors $\mathbf{A}$ and $\mathbf{E}$ are mutually independent. Denote $r_1 = \text{rank}(\mathbf{X}) - \text{rank}(\mathbf{1}_n)$ and $r_2 = n - \text{rank}(\mathbf{X})$, where $\mathbf{X} = (\mathbf{1}_n, \mathbf{ZZ}')$ is the horizontal concatenation of matrices $\mathbf{1}_n$ and $\mathbf{ZZ}'$. Under model (2), the distribution function of $\mathbf{Y}$ is $\mathbf{Y} \sim N(\mu\mathbf{1}_n, \sigma_a^2\mathbf{ZZ}' + \sigma_e^2\mathbf{I}_n)$, then $\mathbf{H}'\mathbf{Y} \sim N(\mathbf{0}, \sigma_a^2\mathbf{W} + \sigma_e^2\mathbf{I})$, where $\mathbf{H}$ is matrix whose columns span the space orthogonal to the space spanned by the column vector of ones (Burch, 2011), $\mathbf{W}$ is the part of the variance-covariance matrix associated with σ_a^2, $\mathbf{0}$ is vector of zeros, and $\mathbf{I}$ is identity matrix. The quadratic form is denoted by $T = \mathbf{Y}'\mathbf{BY}$, where $\mathbf{B}$ is an appropriately chosen symmetric matrix of constants called the matrix of the quadratic form (Milliken and Johnson, 2009).

Graybill (1976) described the properties of quadratic forms for estimation of the variance components. The independently quadratic forms, denoted by $T_1, \ldots, T_d$, T_{d+1}, are minimal sufficient statistics for (σ_a^2, σ_e^2) under multivariate normal distribution of $\mathbf{Y}$. Burch (2011) showed that the sum of squares due to between groups SS_a and the sum of squares due to within groups SS_e can be expressed as quadratic forms
$\mathbf{Y}'(\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' - \mathbf{1}_n(\mathbf{1}_n'\mathbf{1}_n)^-\mathbf{1}_n')\mathbf{Y} = T_1 + \cdots + T_d$ and $\mathbf{Y}'(\mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}')\mathbf{Y} = T_{d+1}$, respectively. The mean square for between groups and the mean square for within groups are denoted by $MS_a = SS_a/r_1$ and $MS_e = SS_e/r_2$, respectively. Furthermore, MS_a and MS_e are independent, and SS_e/σ_e^2 has a chi-squared distribution with r_2 degrees of freedom.

3. Approximate confidence intervals for σ_a^2

Several existing methods for constructing the confidence interval for σ_a^2 are reviewed in this section.

3.1. The TG method

Ting et al. (1990) suggested the method for constructing the confidence interval for the variance components in random effect model applying results provided by Howe (1974) and using cross-product terms in Ting et al. (1989). Let $\mathbf{W}_{\text{TG}} = \mathbf{H}_{\text{TG}}\mathbf{ZZ}'\mathbf{H}_{\text{TG}}$, where $\mathbf{H}_{\text{TG}}$ is a $n \times n$ matrix such that $\mathbf{H}_{\text{TG}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' - \mathbf{1}_n(\mathbf{1}_n'\mathbf{1}_n)^-\mathbf{1}_n'$. Let $\lambda_1 > \cdots > \lambda_d > 0$ be the distinct positive eigenvalues of $\mathbf{W}_{\text{TG}}$ having multiplicities $s_1, \ldots, s_d$. Define $SS_a = \mathbf{Y}'\mathbf{H}_{\text{TG}}'\mathbf{W}_{\text{TG}}^-\mathbf{H}_{\text{TG}}\mathbf{Y}$.

The approximate $100(1-\alpha)\%$ confidence interval for σ_a^2 is derived by

$$[MS_a - \frac{1}{b}MS_e - (G_1^2 MS_a^2 + \frac{1}{b^2}C_2^2 MS_e^2 + \frac{1}{b}G_{12}MS_a MS_e)^{1/2},$$
$$MS_a - \frac{1}{b}MS_e + (C_1^2 MS_a^2 + \frac{1}{b^2}G_2^2 MS_e^2 + \frac{1}{b}C_{12}MS_a MS_e)^{1/2}],$$

where $b = r_1(\sum_{\ell=1}^{d} s_\ell/\lambda_\ell)^{-1}$, $G_1 = 1 - 1/F_{1-\alpha,(r_1,\infty)}$, $C_2 = 1/F_{\alpha,(r_2,\infty)} - 1$,
$G_{12} = [(F_{1-\alpha,(r_1,r_2)} - 1)^2 - G_1^2 F_{1-\alpha,(r_1,r_2)}^2 - C_2^2]/F_{1-\alpha,(r_1,r_2)}$, $C_1 = 1/F_{\alpha,(r_1,\infty)} - 1$,
$G_2 = 1 - 1/F_{1-\alpha,(r_2,\infty)}$, $C_{12} = [(1 - F_{\alpha,(r_1,r_2)})^2 - C_1^2 F_{\alpha,(r_1,r_2)}^2 - G_2^2]/F_{\alpha,(r_1,r_2)}$.

Note that $F_{\alpha,(r_1,r_2)}$ and $F_{1-\alpha,(r_1,r_2)}$ are the α and $1-\alpha$ quantiles of the F-distribution with degrees of freedom r_1 and r_2, respectively. Furthermore, $F_{\alpha,(r_1,\infty)} = \chi^2_{\alpha,r_1}/r_1$, $F_{1-\alpha,(r_1,\infty)} = \chi^2_{1-\alpha,r_1}/r_1$, $F_{\alpha,(r_2,\infty)} = \chi^2_{\alpha,r_2}/r_2$, and $F_{1-\alpha,(r_2,\infty)} = \chi^2_{1-\alpha,r_2}/r_2$ (Milliken and Johnson, 2009).

3.2. The HK method

Hartung and Knapp (2000) developed the method for constructing the confidence interval for the between-group variance component using the concept of Wald (1940). The sufficient statistics of the HK method are defined as $T_{\mathrm{HK}_1},\ldots,T_{\mathrm{HK}_g}$, where $T_{\mathrm{HK}_i} = (\mathbf{1}'_{n_i}\mathbf{1}_{n_i})^{-}\mathbf{1}'_{n_i}\mathbf{Y}_i$, $i = 1,\ldots,g$.

The approximate $100(1-\alpha)\%$ confidence interval for σ_a^2 is derived by

$$[MS_eR_1, MS_eR_2].$$

Note that R_1 and R_2 are the root of the equations as follows:

$$f(R_1) = \frac{\sum_{i=1}^{g} w_i(T_{\mathrm{HK}_i} - \sum_{i=1}^{g} w_iT_{\mathrm{HK}_i}/\sum_{i=1}^{g} w_i)^2}{r_1MS_e} \sim F_{1-\alpha/2,(r_1,r_2)} \text{ and}$$
$$f(R_2) = \frac{\sum_{i=1}^{g} v_i(T_{\mathrm{HK}_i} - \sum_{i=1}^{g} v_iT_{\mathrm{HK}_i}/\sum_{i=1}^{g} v_i)^2}{r_1MS_e} \sim F_{\alpha/2,(r_1,r_2)},$$

where $w_i = n_i/(1+n_iR_1)$ and $v_i = n_i/(1+n_iR_2)$.

3.3. The PB method

Park and Burdick (2003) proposed the generalized pivotal quantity for constructing the confidence interval for the between-group variance component using results provided by Olsen et al. (1976). Let $\mathbf{W}_{\mathrm{PB}} = \mathbf{H}_{\mathrm{PB}}\mathbf{ZZ}'\mathbf{H}_{\mathrm{PB}}$, where $\mathbf{H}_{\mathrm{PB}}$ is a $n \times n$ matrix such that $\mathbf{H}_{\mathrm{PB}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' - \mathbf{1}_n(\mathbf{1}'_n\mathbf{1}_n)^{-}\mathbf{1}'_n$. Let $\lambda_1 > \cdots > \lambda_d > 0$ be the distinct positive eigenvalues of $\mathbf{W}_{\mathrm{PB}}$ having multiplicities $s_1,\ldots,s_d$. Let $\mathbf{P}_{\mathrm{PB}} = [\mathbf{P}_{\mathrm{PB}_1},\ldots,\mathbf{P}_{\mathrm{PB}_d}]$ be $n \times n$ orthogonal matrix such that $\mathbf{P}'_{\mathrm{PB}}\mathbf{W}_{\mathrm{PB}}\mathbf{P}_{\mathrm{PB}} = \mathrm{diag}(\lambda_1\mathbf{1}'_{s_1},\ldots,\lambda_d\mathbf{1}'_{s_d})$, where $\mathbf{P}_{\mathrm{PB}_\ell}$, $\ell = 1,\ldots,d$ corresponding to λ_ℓ is of dimension $n \times s_\ell$.

The minimal sufficient statistics of the PB method are defined as $T_{\mathrm{PB}_1},\ldots,T_{\mathrm{PB}_d}$, where $T_{\mathrm{PB}_\ell} = \mathbf{Y}'\mathbf{H}'_{\mathrm{PB}}\mathbf{P}_{\mathrm{PB}_\ell}(\mathbf{P}'_{\mathrm{PB}_\ell}\mathbf{P}_{\mathrm{PB}_\ell})^{-}\mathbf{P}'_{\mathrm{PB}_\ell}\mathbf{H}_{\mathrm{PB}}\mathbf{Y}$, $\ell = 1,\ldots,d$. Lamotte (1976) showed that $SS_a = \sum_{\ell=1}^{d} T_{\mathrm{PB}_\ell}$, where $T_{\mathrm{PB}_\ell}/(\lambda_\ell\sigma_a^2 + \sigma_e^2)$, $\ell = 1,\ldots,d$ has the chi-squared distribution with s_ℓ degrees of freedom. The function of the generalized pivotal quantity is defined by R as the solution for σ_a^2 in the non-linear equation given by

$$U = \sum_{\ell=1}^{d} \frac{T_{\mathrm{PB}_\ell}}{\lambda_\ell R + r_2MS_e/K}, \tag{3}$$

where $U \sim \chi^2_{r_1}$ and $K \sim \chi^2_{r_2}$.

The approximate $100(1-\alpha)\%$ confidence interval for σ_a^2 is derived by

$$[\max(0, R_{\alpha/2}), \max(0, R_{1-\alpha/2})],$$

where $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of R in equation (3), respectively. Note that the solutions of $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are based on pivotal quantities.

3.4. The LL method

Li and Li (2007) presented the concept of the fiducial inference for constructing the confidence interval for the between-group variance component in random effect model applying results provided by Li and Li (2005). Let $\mathbf{W}_{\text{LL}} = \mathbf{H}_{\text{LL}}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{H}'_{\text{LL}}$, where $\mathbf{H}_{\text{LL}}$ is a $(g-1)\times g$ matrix such that $\mathbf{H}_{\text{LL}}\mathbf{H}'_{\text{LL}} = \mathbf{I}_{g-1}$ and $\mathbf{H}'_{\text{LL}}\mathbf{H}_{\text{LL}} = \mathbf{I}_g$. Let $\lambda_1 > \cdots > \lambda_d \geq 0$ be the distinct eigenvalues of $\mathbf{W}_{\text{LL}}$ having multiplicities $s_1,\ldots,s_d$. Let $\mathbf{P}_{\text{LL}} = [\mathbf{P}_{\text{LL}_1},\ldots,\mathbf{P}_{\text{LL}_d}]$ be $(g-1)\times(g-1)$ orthogonal matrix such that $\mathbf{P}'_{\text{LL}}\mathbf{W}_{\text{LL}}\mathbf{P}_{\text{LL}} = \text{diag}(\lambda_1\mathbf{1}'_{s_1},\ldots,\lambda_d\mathbf{1}'_{s_d})$, where $\mathbf{P}_{\text{LL}_\ell}$, $\ell = 1,\ldots,d$ corresponding to λ_ℓ is of dimension $(g-1)\times s_\ell$.

The sufficient statistics of the LL method are defined as $\mathbf{T}_{\text{LL}} = \mathbf{P}_{\text{LL}}\mathbf{H}_{\text{LL}}(\mathbf{Z}'\mathbf{Z})^{-}\mathbf{Z}'\mathbf{Y}$. The function of the fiducial inference is given by

$$R = \frac{\mathbf{T}'_{\text{LL}}\mathbf{T}_{\text{LL}} - \mathbf{Q}'\mathbf{C}\mathbf{Q}SS_e/K}{\mathbf{Q}'\mathbf{Q}}, \tag{4}$$

where $\mathbf{C} = \mathbf{P}_{\text{LL}}\mathbf{W}_{\text{LL}}\mathbf{P}'_{\text{LL}}$, $\mathbf{Q} \sim N(\mathbf{0},\mathbf{I}_{r_1})$, and $K \sim \chi^2_{r_2}$.

The approximate $100(1-\alpha)\%$ confidence interval for σ_a^2 is derived by

$$[\max(0,R_{\alpha/2}),\max(0,R_{1-\alpha/2})],$$

where $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of R in equation (4), respectively. Note that the solutions of $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are based on pivotal quantities.

3.5. The LXH method

Liu et al. (2016) proposed the least squares idea of the fiducial generalized pi-votal quantity for constructing the confidence interval for the variance components in random effect model. Let $\mathbf{W}_{\text{LXH}} = \mathbf{H}'_{\text{LXH}}\mathbf{Z}\mathbf{Z}'\mathbf{H}_{\text{LXH}}$, where $\mathbf{H}_{\text{LXH}}$ is a $n\times(n-1)$ matrix such that $\mathbf{H}_{\text{LXH}}\mathbf{H}'_{\text{LXH}} = \mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n$ and $\mathbf{H}'_{\text{LXH}}\mathbf{H}_{\text{LXH}} = \mathbf{I}_{n-1}$. Let $\lambda_1 > \cdots > \lambda_d \geq 0$ be the distinct eigenvalues of $\mathbf{W}_{\text{LXH}}$ having multiplicities $s_1,\ldots,s_d$. Let $\mathbf{P}_{\text{LXH}} = [\mathbf{P}_{\text{LXH}_1},\ldots,\mathbf{P}_{\text{LXH}_d}]$ be $(n-1)\times(n-1)$ orthogonal matrix such that $\mathbf{P}'_{\text{LXH}}\mathbf{W}_{\text{LXH}}\mathbf{P}_{\text{LXH}} = \text{diag}(\lambda_1\mathbf{1}'_{s_1},\ldots,\lambda_d\mathbf{1}'_{s_d})$, where $\mathbf{P}_{\text{LXH}_\ell}$, $\ell = 1,\ldots,d$ corresponding to λ_ℓ is of dimension $(n-1)\times s_\ell$.

The minimal sufficient statistics of the LXH method are defined as $T_{\text{LXH}_1},\ldots,T_{\text{LXH}_d}$, where $T_{\text{LXH}_\ell} = \mathbf{Y}'\mathbf{H}_{\text{LXH}}\mathbf{P}_{\text{LXH}_\ell}\mathbf{P}'_{\text{LXH}_\ell}\mathbf{H}'_{\text{LXH}}\mathbf{Y}$, $\ell = 1,\ldots,d$. The variables U_ℓ, $\ell = 1,\ldots,d$ are mutually independent and $U_\ell = T_{\text{LXH}_\ell}/(\lambda_\ell\sigma_a^2 + \sigma_e^2)$, $\ell = 1,\ldots,d$ has the chi-squared distribution with s_ℓ degrees of freedom. The function of the least squares fiducial inference is given by

$$R = \frac{\sum_{\ell=1}^d U_\ell^2 \sum_{\ell=1}^d \lambda_\ell T_{\text{LXH}_\ell} U_\ell - \sum_{\ell=1}^d \lambda_\ell U_\ell^2 \sum_{\ell=1}^d T_{\text{LXH}_\ell} U_\ell}{\sum_{\ell=1}^d U_\ell^2 \sum_{\ell=1}^d \lambda_\ell^2 U_\ell^2 - (\sum_{\ell=1}^d \lambda_\ell U_\ell^2)^2}. \tag{5}$$

The approximate $100(1-\alpha)\%$ confidence interval for σ_a^2 is derived by

$$[\max(0,R_{\alpha/2}),\max(0,R_{1-\alpha/2})],$$

where $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are the $\alpha/2$ and $1-\alpha/2$ quantiles of the distribution of R in equation (5), respectively. Note that the solutions of $R_{\alpha/2}$ and $R_{1-\alpha/2}$ are based on pivotal quantities.

4. Simulation study

In this section, a comparison of the methods for constructing the confidence interval for σ_a^2 with the methods described in Section 3 is studied by the Monte Carlo simulation. Without loss of generality, it is assumed that $\mu = 0$ in model (2). The values chosen for (σ_a^2, σ_e^2) are $(0.001, 0.999)$, $(0.1, 0.9)$, $(0.2, 0.8)$, $(0.3, 0.7)$, $(0.4, 0.6)$, $(0.5, 0.5)$, $(0.6, 0.4)$, $(0.7, 0.3)$, $(0.8, 0.2)$, $(0.9, 0.1)$, and $(0.999, 0.001)$. The ratio of va-riance components, $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ varies from small to large. The nominal confidence level of 0.95 is considered. The simulation study is based on $5,000$ iterations for each setting of the values (σ_a^2, σ_e^2) and the sample size pattern $(n_i,\ i = 1, \ldots, g)$.

The criteria for analysing the performance of the methods are the empirical coverage probability that maintains at the nominal confidence level, and the shortest average length of the confidence interval. The empirical coverage prob-ability is firstly considered, and the average length of the confidence interval is later compared. The degree of imbalance is $\Phi = (g/\sum_{i=1}^{g} n_i)(g/\sum_{i=1}^{g} 1/n_i)$, which is used to measure imbalance in one-way model (Ahrens and Pincus, 1981). Note that $0 < \Phi \leq 1$ is equal to 1 if and only if the model is bal-anced, and Φ is close to 0 when the model is very unbalanced. The coverage probability of confidence interval for σ_a^2 depends on the degree of imbalance and the design $(n_1, \ldots, n_g)$. The simulation patterns are shown in Table 1.

Table 1. Unbalanced patterns used in simulations

Pattern	Φ	g	n_i									
1	0.044	3	1	1	100							
2	0.570	3	3	7	20							
3	0.818	3	5	10	15							
4	0.068	6	1	1	1	1	1	100				
5	0.700	6	5	10	15	20	25	30				
6	0.957	6	6	6	8	8	10	10				
7	0.525	10	1	1	4	4	6	6	8	8	10	10
8	0.835	10	3	3	4	5	6	6	8	8	10	10

The simulation results are represented in the boxplots of Figures 1 and 2. The empirical coverage probabilities of the confidence interval for σ_a^2 with the number of groups $g = 3$, 6, and 10, where $\rho < 0.5$ and $\rho \geq 0.5$, are shown in Figure 1. The relative difference of the average length of the confidence interval for σ_a^2 with the number of groups $g = 3$, 6, and 10, where $\rho < 0.5$ and $\rho \geq 0.5$ is shown in Figure 2. The relative length is defined as $(L_{\mathrm{M}} - L_{\mathrm{PB}})/L_{\mathrm{PB}}$, where L_{M} denotes the average interval length of competing methods and

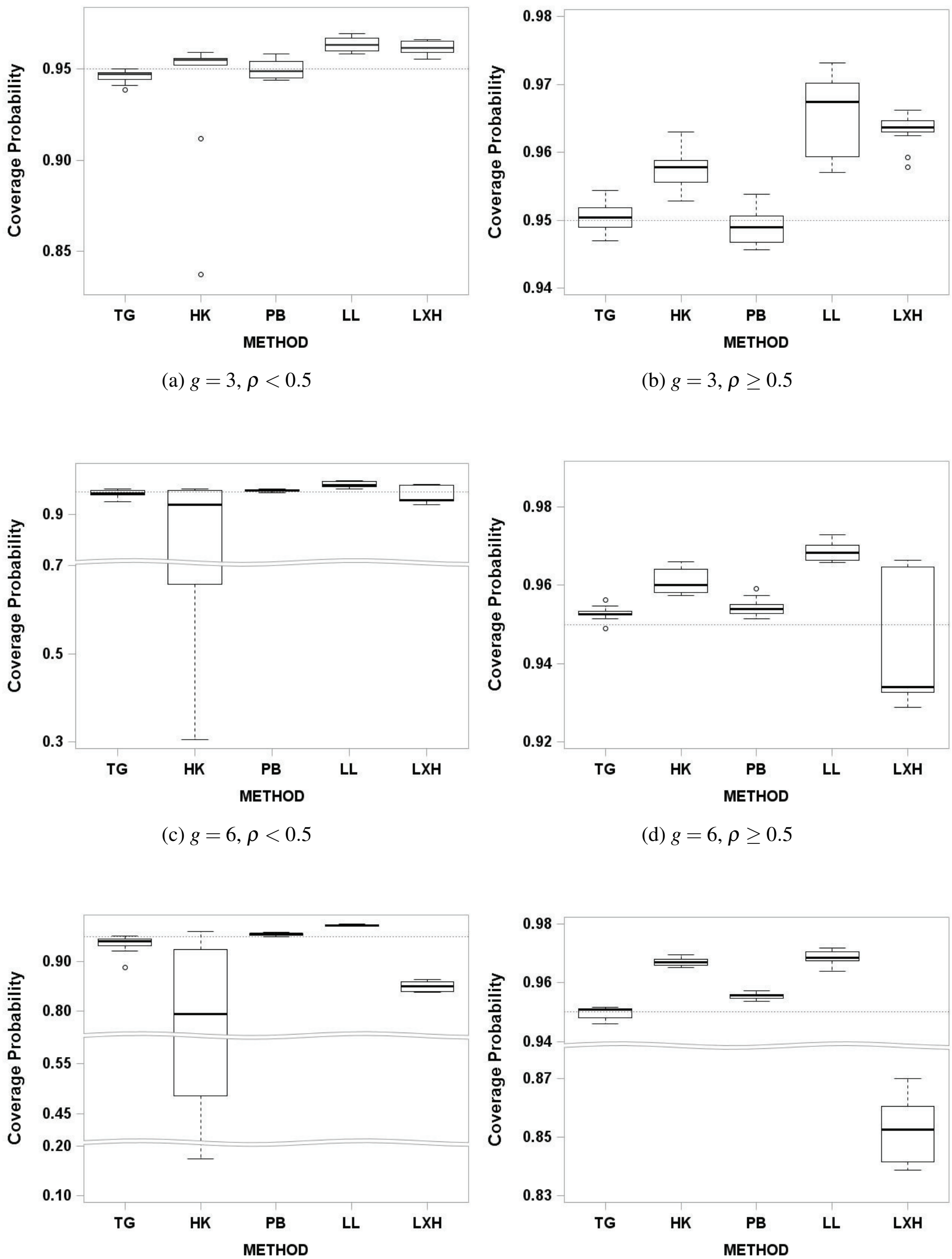
L_{PB} denotes the average interval length of the PB method. Clearly, the positive value of the relative length implies that L_{PB} is shorter than L_{M}. On the contrary, the negative value of the relative length implies that L_{M} is shorter than L_{PB}. Moreover, the relative length equal to 0 implies that L_{M} and L_{PB} are equal.

Regarding the empirical coverage probabilities, from Figure 1, the PB procedure maintains the nominal confidence level for all situations. The LL procedure provides a larger than the nominal confidence level for all situations. The TG procedure maintains the nominal confidence level for all situations except for $g = 10$. However, the TG procedure provides a smaller than the nominal confidence level when $\rho < 0.5$ for $g = 10$. The HK procedure mostly maintains the nominal confidence level when $\rho < 0.5$ and it provides a larger than the nominal confidence level when $\rho \geq 0.5$ for $g = 3$. The HK procedure provides a smaller than the nominal confidence level when $\rho < 0.5$ and it provides a larger than the nominal confidence level when $\rho \geq 0.5$ for $g = 6$ and 10. The LXH procedure provides a larger than the nominal confidence level for all ρ for $g = 3$. The LXH procedure provides a smaller than the nominal confidence level for all ρ for $g = 6$ and 10 except in a very unbalanced design (pattern 4), that is, the LXH procedure maintains the nominal confidence level in a very unbalanced design.

Comparing the average length of the confidence interval, Figure 2 clearly indicates that the average lengths of the TG, LL, and PB intervals behave very similar. The average length of the LXH interval is the shortest. For the number of groups $g = 6$ and 10, the average length of the HK interval is shorter than the average length of the PB interval when $\rho < 0.5$. Conversely, the average length of the PB interval is shorter than the average length of the HK interval when $\rho \geq 0.5$.

5. Application

The numerical example from Brownlee (1965) is a study of the effects of environmental conditions on the measure of the ratio of electromagnetic and electrostatic units of electricity. The data set is shown in Table 2. Model (1) is used to des-cribe this data set, that is, $g = 5$, $n_i = (11, 8, 6, 24, 15)$, and $\Phi = 0.796$. Furthermore, a_i denote the random group effects of the environmental conditions and assume $a_i \sim N(0, \sigma_a^2)$, e_{ij} represent the random effect of the jth measure of electricity on the ith environmental condition and assume $e_{ij} \sim N(0, \sigma_e^2)$. Independence among a_i and e_{ij} is also assumed. The five confidence intervals for σ_a^2 based on the five methods in Section 3 are presented in Table 3. Table 3 shows that the PB method provides the shortest confidence interval for this data set.



(a) $g = 3$, $\rho < 0.5$

(b) $g = 3$, $\rho \geq 0.5$

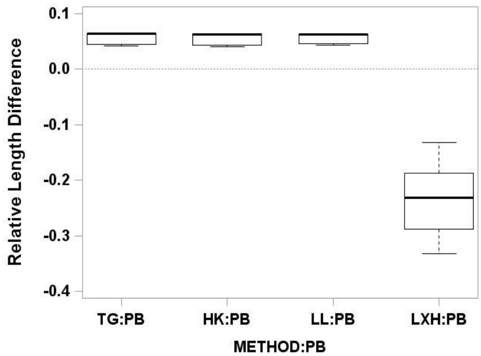
(c) $g = 6$, $\rho < 0.5$

(d) $g = 6$, $\rho \geq 0.5$

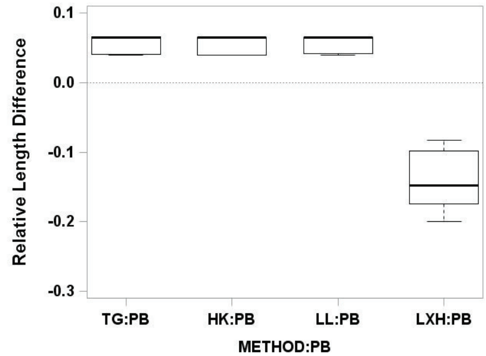
(e) $g = 10$, $\rho < 0.5$

(f) $g = 10$, $\rho \geq 0.5$

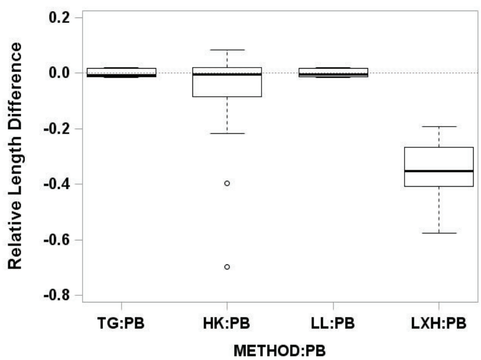
Figure 1: The empirical coverage probabilities of 95% confidence interval for σ_a^2



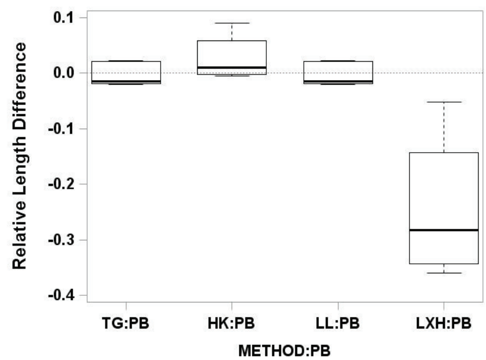
(a) $g = 3$, $\rho < 0.5$



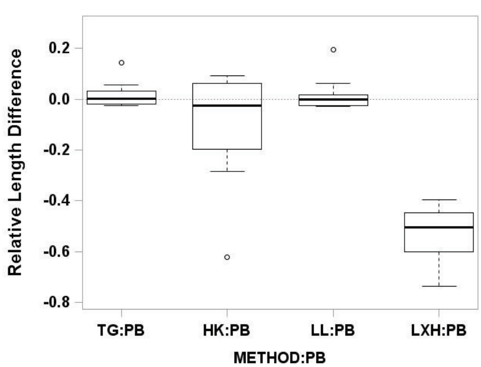
(b) $g = 3$, $\rho \geq 0.5$



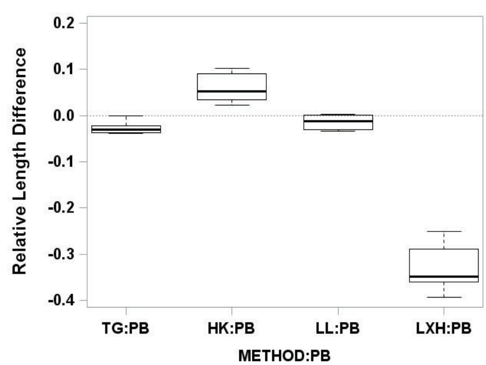
(c) $g = 6$, $\rho < 0.5$



(d) $g = 6$, $\rho \geq 0.5$



(e) $g = 10$, $\rho < 0.5$



(f) $g = 10$, $\rho \geq 0.5$

Figure 2: Relative difference of the average length of 95% confidence interval for σ_a^2

Table 2. The ratio of the electromagnetic to electrostatic units of electricity

Groups	Observations											
1	62	64	62	62	65	64	65	62	62	63	64	
2	65	64	63	62	65	63	64	63				
3	65	64	67	62	65	62						
4	62	66	64	64	63	62	64	64	66	64	66	63
	65	63	63	63	61	56	64	64	65	64	64	65
5	66	65	65	66	67	66	69	70	68	69	63	65
	64	65	64									

Table 3. Nominally 95% confidence interval for the data

Method	TG	HK	PB	LL	LXH
confidence interval	(0, 10.901)	(0, 11.311)	(0, 9.595)	(0, 10.836)	(0, 9.728)

6. Conclusion

This article studies the methods for constructing 95% confidence intervals for variance components in an unbalanced one-way random effects model. Simulation studies indicate that the TG procedure maintains the nominal confidence level for all situations except for the number of group $g = 10$, which is liberal when ρ is small. The HK procedure is conservative when ρ is large. On the contrary, when ρ is small, the HK procedure mostly maintains the nominal confidence level for the number group $g = 3$ and is liberal for the number of groups $g = 6$ and 10. The PB procedure maintains the nominal confidence level for all situations. The LL procedure is conservative for all situations. The LXH procedure is conservative for all ρ in the number of group $g = 3$. Nevertheless, for the number of groups $g = 6$ and 10, the LXH procedure does not adequately maintain the nominal confidence level. All of the average lengths of the confidence intervals behave similarly, but the average length of the LXH interval always has the shortest. Notice that the relative length values of the LXH method is negative.

In summary, the PB and LXH methods are recommended for the number of group $g = 3$. The PB and LL methods are recommended for the number of groups $g = 6$ and 10. The TG and HK methods are useful when ρ is large. Furthermore, the LXH method is preferred in a very unbalanced design.

Acknowledgements

The authors thank the reviewers for their valuable comments and suggestions which substantially improved the quality of the article. This work is supported by the Science Achievement Scholarship of Thailand (SAST).

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