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k-th record estimator of the scale parameter of the *α*-stable distribution

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ABSTRACT

Various techniques of scale parameter estimation have been proposed in the case of alpha stable distributions. In the paper, the authors present an estimation technique that involves the k-th record theory. Although this theory is over 40 years old, its implementation in the classical extreme value theory – being the other cornerstone of the presented approach – is quite new, and tempting. Several theoretical properties of the introduced scale parameter estimators are presented. With the use of Monte Carlo methods, a comparative analysis is performed between the approach based on k-th records and approaches based on Hill's and Pickands' estimators. Additionally, the paper uses a real-life data set to illustrate how to effectively apply the k-th record estimator of the scale parameter. The research indicates several advantages of the k-th record approach over its other counterparts, especially when dealing with incomplete information about the underlying sample.

Key words: stable distribution, scale parameter estimator, k-th record values.

1. Introduction

Specificity of many financial data sets (regarded as proper time-series) imposes that the so-called heavy-tailed distributions constitute an attractive alternative way of modelling such data. Amongst these distributions, the class of α-stable ones gained one of prominent places.

There are several methods of estimation of stability index α. However, for a complete recognition of a theoretical α-stable distribution that approximates empirical data, it is necessary to estimate the other parameters of the distribution, including the scale parameter σ as well. For instance, this holistic look is the most appropriate approach when calculating risks measures such as VaR or CVaR (see, e.g. Stoyanov et al. 2006, Khindanova et al. 2001).

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Therefore, the present paper: 1) describes construction of k-th record estimators of parameter σ, in the case of stability index $\alpha < 2$, and 2) reveals some theoretical properties of the estimators introduced in the paper. The main goal of this article is to compare the quality of k-th record estimators of parameter σ with the two estimators of this parameter that are based on Hill's and Pickand's estimators. Such a comparative analysis is conducted by simulation research concerning some arbitrarily chosen range of α-stable distribution parameters ($1.8 \leq \alpha \leq 1.99$, $\beta = \mu = 0$, $0.01 \leq \sigma \leq 100$). Additionally, the paper is supplemented by an empirical example concerning energy prices quoted at the Nord Pool Spot.

The procedure for estimating the sigma parameter of the stable distribution described in this paper is part of a broader research trend that explores methods implementing the possibility of using k-th records in estimation. In the literature on the subject, one can find proposals for estimating the parameters of other distributions, such as: Gumbel's, Burr's, power, Weibull's, Rayleigh's, logistic or Pareto's ones (for instance see: Ahsanullah 1990, Malinowska et al. 2005). Moreover, k-th records, apart from the more classical approach, appear as a tool in Bayesian estimation (see: Malinowska and Szynal 2004).

2. Theoretical background

From now on, let $X_1, X_2, X_3, \ldots$ be *independent and identically distributed* (i.i.d.) random variables with a common *cumulative distribution function* (cdf) F. For any fixed $n \in \mathbb{N}_+$, the order statistics of a sample $X_1, X_2, \ldots, X_n$ are denoted by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$.

The main theorem of the *extreme value theory* (EVT) states that if there exist constants $a_n > 0$, b_n for $n \in \mathbb{N}_+$, and some non-degenerate distribution function G such that for all $x \in \mathbb{R}$ holds $\lim_{n\to\infty} \mathbb{P}\left(\frac{X_{n:n}-b_n}{a_n} \leq x\right) = G(x)$, then there exists a constant $\gamma \in \mathbb{R}$ such that the limit distribution G has the form:

$$G(x) = G_\gamma(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right) & 1+\gamma x > 0 \quad \gamma \neq 0 \\ \exp(-e^{-x}) & x \in \mathbb{R} \quad \gamma = 0 \end{cases}.$$

The parameter γ is called the *extreme value index* (EVI), and it impacts the right tail asymptotics of the common cdf F (e.g. see de Haan and Ferreira 2006).

Classical estimators of EVI are based on upper order statistics. Among wide variety of such estimators, the most popular are Pickands' and Hill's ones (see Gomes et al. 2008), given respectively by formulas:

$$\hat{\gamma}_{\mathrm{P}}^k = \log_2 \frac{X_{n-k:n} - X_{n-2k:n}}{X_{n-2k:n} - X_{n-4k:n}}, \quad \hat{\gamma}_{\mathrm{H}}^k = \frac{1}{k}\sum_{i=0}^{k-1} \ln X_{n-i:n} - \ln X_{n-k:n}. \qquad \text{(1A, B)}$$

for any fixed $k \in \{1,2, \ldots [n/4]\}$ (case 1A), or $k \in \{1,2, \ldots, n-1\}$ (case 1B).

An alternative, proposed by Berred (1995), is based on the notion of k-th records, which were defined by Dziubdziela and Kopociński (1976). So for a fixed $k \in \mathbb{N}_+$, the k-th record times $\{T_n^{(k)}\}$, and the k-th record values $\{R_n^{(k)}\}$ are defined by recurrence relations:

$$T_1^{(k)} = k,\ T_n^{(k)} = \min\{\, j \in \mathbb{N} : j > T_{n-1}^{(k)}, X_j > X_{T_{n-1}^{(k)}-k+1:\, T_{n-1}^{(k)}}\}, \text{ for } n \geq 2,$$

$$R_n^{(k)} = X_{T_n^{(k)}-k+1:\, T_n^{(k)}}.$$

In other words, a sequence of k-th record values $R_1^{(k)} < R_2^{(k)} < R_3^{(k)} < \ldots$ is constructed by eliminating repetitions in the non-decreasing sequence of k-th order statistics $X_{1:k} \leq X_{2:k+1} \leq X_{3:k+2} \leq \ldots$, while $T_1^{(k)} < T_2^{(k)} < T_3^{(k)} < \ldots$ are the appearance numbers (the so-called *record times*) of the succeeding record values.

The original Berred's estimator based on the k-th record values is of the form:

$$\hat{\gamma}_{\mathrm{B}}^{k} = \ln \frac{R_{N(k,n)}^{(k)} - R_{N(k,n)-k}^{(k)}}{R_{N(k,n)-k}^{(k)} - R_{N(k,n)-2k}^{(k)}}, \tag{2}$$

where $N(k,n)$ is a random number of k-th records values in a sample of size n.

Pickands' and Berred's estimators are convenient for any real γ (these estimators are additionally invariant under any linear transformation – with a positive slope – of data, which is fully concordant with the linear transformation appearing in the main EVT theorem), while Hill's one is proper for $\gamma > 0$ only. Moreover, Berred's estimator value depends on sample order, which allows resampling, since i.i.d. property is assumed. (The mentioned resampling makes sense only if data do not represent any time series.)

We recall one of equivalent definitions of α-stable distribution in order to assume the parametrization we use. Thus, a random variable X has α-stable distribution (noted as: $X \sim S(\alpha, \beta, \mu, \sigma)$) if the logarithm of its characteristic function ϕ is given by the following formula:

$$\ln \phi(t) = \begin{cases} i\mu t - \sigma^{\alpha} |t|^{\alpha} \left(1 - i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2}\right), & \alpha \neq 1 \\ i\mu t - \sigma t \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(t) \ln|t|\right), & \alpha = 1 \end{cases},$$

where $\alpha \in (0, 2\rangle$ is the stability index, $\beta \in \langle -1,1 \rangle$ is the skewness parameter, $\sigma \in (0, \infty)$ is the scale parameter, $\mu \in \mathbb{R}$ is the location parameter. It should be also mentioned that in α-stable case the following relation holds: $\gamma = 1/\alpha$ for $\alpha \in (0, 2)$, and $\gamma = 0$ for $\alpha = 2$, which reveals discontinuous functional dependence of α-stable tails asymptotics on the stability parameter value. Thus, the tails of the stable distributions have a power decay (are the so-called "heavy tails") if they are distinct from normal distribution (see Nolan 2011, Weron 2001).

Let $Z = |X|$ for $X \sim S(\alpha, \beta, \mu, \sigma)$, and let G and Q be the cdf, and quantile function of variable Z, respectively. Moreover, let $k = k_n$ be an increasing sequence of natural numbers such that the following condition is fulfilled:

$$k_n \to \infty \quad \text{and} \quad k_n/n \to 0, \quad \text{as } n \to \infty.$$

Basic properties of α-stable distribution tail yield that:

$$1 - G(x) \sim C_\alpha \sigma^\alpha x^{-\alpha}, \quad \text{as } x \to \infty, \tag{3}$$

for the constant $C_\alpha = \frac{2}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$, and the gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ (see Samorodnitsky and Taqqu 1994, Nolan 2011). As a consequence (see Meraghni and Necir 2007) we obtain:

$$Q\left(1 - \frac{k}{n}\right)\left(\frac{k}{n}\frac{\pi}{2\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}\right)^{1/\alpha} \to \sigma, \quad \text{as } n \to \infty. \tag{4}$$

The last convergence enables straightforward construction of estimators in the following manner. For a given sample Z_1, Z_2, …, Z_n of independent copies of Z, an unknown quantile $Q\left(1 - \frac{k}{n}\right)$ may be substituted by the appropriate order statistic $Z_{n-k:n}$ taken out of that sample. Additionally, if parameter α is also unknown, it may be substituted by any of its estimators, let us say $\hat{\alpha}^{n,k}_{(\mathrm{E})}$. It may be Hill's, Pickand's, Dekkers-Einmahl-de Haan's one (e.g. see de Haan and Ferreira 2006), to mention but a few. In the α-stable case, owing to the formula (3), these estimators may be applied to sample $Z_1, Z_2, \ldots, Z_n$ instead of sample $X_1, X_2, \ldots, X_n$.

Therefore, the estimator of the scale parameter takes the form:

$$\hat{\sigma}^{n,k}_{(\mathrm{E})} = Z_{n-k:n}\left(\frac{k\pi}{2n\Gamma\left(\hat{\alpha}^{n,k}_{(\mathrm{E})}\right)\sin\frac{\pi\hat{\alpha}^{n,k}_{(\mathrm{E})}}{2}}\right)^{1/\hat{\alpha}^{n,k}_{(\mathrm{E})}}, \tag{5}$$

which is quite general, but limited for 'order statistics' case.

It occurs that k-th records may be applied in the convergence (4), which leads to the following estimator:

$$\hat{\sigma}^{n,k}_{(\mathrm{R})} = R^{(k)}_{N(n,k)}\left(\frac{k\pi}{2n\Gamma\left(\hat{\alpha}^{n,k}_{(\mathrm{R})}\right)\sin\frac{\pi\hat{\alpha}^{n,k}_{(\mathrm{R})}}{2}}\right)^{1/\hat{\alpha}^{n,k}_{(\mathrm{R})}}, \tag{6}$$

as the ideas from original proofs of Meraghni and Necir (2007), concerning properties of the estimator (5), may be straightforwardly adapted to the 'k-th records' case.

To do this, it suffices to notice that $R^{(k)}_{N(n,k)} =^d Z_{n-k+1:n}$ (as $n \to \infty$) for any continuous probability distribution, where '$=^d$' designates equality in distribution (see Wodecka 2016, Lemma 2.18). Moreover, the key is a direct consequence of the formula (3) that $Q\left(1-\frac{k-1}{n}\right)/Q\left(1-\frac{k}{n}\right) \to 1$ (as $n \to \infty$) for variable $Z = |X|$ defined above herein. So, the replacement of a proper order statistic $Z_{n-k:n}$ in (5) by one of its nearest neighbour $R^{(k)}_{N(n,k)}$ creates the formula (6).

As a result, the estimator $\hat{\sigma}^{n,k}_{(\mathrm{R})}$ is consistent if $k = k_n \sim dn^{\theta}$, as $n \to \infty$, for some constants $d > 0$ and $\theta \in (0,1)$ (see Wodecka 2016, Theorem 2.19). Additionally, $\frac{\sqrt{k}}{\log\frac{k}{n}}\left(\log \hat{\sigma}^{n,k}_{(\mathrm{R})} - \log \sigma\right) \to^{\mathcal{D}} \mathcal{N}\left(0, \frac{e^{2/\alpha}+1}{\left(e^{1/\alpha}-1\right)^2\alpha^2}\right)$, as $n \to \infty$, where $\to^{\mathcal{D}}$ stands for convergence in distribution, which means that the estimator $\hat{\sigma}^{n,k}_{(\mathrm{R})}$ has asymptotically log-normal property (see Wodecka 2016, Theorem 2.20).

Moreover, in contrast to the order statistics, the formula (6) allows to estimate σ even in case of unknown sample size. For this purpose, one may use, for instance, the following estimators of the sample size:

$$\hat{n}_{\psi} = \psi^{-1}\left(\frac{N(k,n)}{k} + \psi(k)\right) - 1, \; \hat{n}_{l} = k\, exp\left(\frac{N(k,n)}{k}\right) - 1. \qquad (7A, B)$$

The above holds, since: a) $\mathbb{E}\big(N(k,n)\big) = k\sum_{i=k}^{n}\frac{1}{i} = k\big(\psi(n+1) - \psi(k)\big)$, where ψ is the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$, and it has logarithmic asymptotics in infinity, and b) $Var\big(N(k,n)\big) = k\sum_{i=k}^{n}\frac{1}{i} - k^2\sum_{i=k}^{n}\frac{1}{i^2}$ is relatively small, as: $0 < Var\big(N(k,n)\big) < \mathbb{E}\big(N(k,n)\big)$.

3. Study of the quality of estimators

3.1. Comparing estimators

For a while, let us consider quite general perspective, and let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of an unknown parameter θ. We assume that we wish to assess which of these estimators is "better" than the other. One of the criteria for solving this question is the *Pitman nearness measure* (see Pitman 1937) given as:

$$\mathbf{P}\big(\hat{\theta}_1, \hat{\theta}_2 \big| \theta\big) = \mathbb{P}\big(\big|\hat{\theta}_1 - \theta\big| < \big|\hat{\theta}_2 - \theta\big| \big),$$

which indicates that $\hat{\theta}_1$ is *Pitman-closer estimator* than $\hat{\theta}_2$, if $\mathbb{P}\big(\hat{\theta}_1 = \hat{\theta}_2\big) = 0$, and $\mathbf{P}\big(\hat{\theta}_1, \hat{\theta}_2 \big| \theta\big) > \frac{1}{2}$. The measure is very natural and intuitive, and additionally – as it preserves bivariate relation of both estimators, regarded as joint vector $(\hat{\theta}_1, \hat{\theta}_2)$ – it is very advisable, in contrast to such measures that rely only on univariate (marginal) distributions of both compared estimators.

Therefore, in the sequel we select Pitman nearness criterion as the main one, and we use it in every case that provides large enough bivariate sample size, in a sense of pairwise completeness. Otherwise, we decide to employ an analogue of the commonly known mean square error. The chosen measure is given by $\eta_i = \left|\mathrm{Me}(\hat{\theta}_i) - \theta\right| + \frac{\mathrm{IQR}(\hat{\theta}_i)}{2}$ for $i \in \{1,2\}$, and we say that $\hat{\theta}_1$ is better than $\hat{\theta}_2$, providing that $\eta_1 < \eta_2$. We prefer the positional measure to classical ones since this approach is unlimited by incongruity to any theoretical assumptions including existence of the high order moments of distribution, as long as we consider the α-stable case (see: Stachura 2017).

3.2. Simulation study

In order to compare estimates of parameter σ based on record value theory with several selected estimates based on classical order statistic approach, simulation research is executed as follows, in the case of stability index $\alpha < 2$. (The simulation research, and additionally all the calculations and plots presented hereunder, are accomplished in R environment (R Core Team 2018).)

Firstly, for a fixed pair of parameters α and σ – taken from arbitrarily chosen ranges $\alpha \in \{1.8, 1.82, 1.84, 1.86, 1.88, 1.9, 1.92, 1.94, 1.96, 1.98, 1.99\}$, $\sigma \in \{0.01, 0.1, 1, 10, 100\}$ – and $\beta = 0$, $\mu = 0$, and for a fixed n – out of $\{50, 80, 110\}$ – pseudo-random i.i.d. sample of size n is generated (with the use of the R's package stabledist by Wuertz et al. 2016). The choice of α's range is motivated by the reason that the research by Stachura and Wodecka (2016), and Wodecka (2016) – including α's from 0.1 to 1.9 by 0.1 step – showed that the values of estimates were alarmingly discrepant near $\alpha = 2$, so the authors decided to examine the case of $\alpha \geq 1.8$ far more accurately, taking a tiny step 0.02. Besides, this new range integrates with α's detected in empirical research in financial data (just about 1.6 – 1.9 see e.g. Weron 2004). Next,

a. with respect to formulas (1A), (1A), (2), and the relation $\alpha = 1/\gamma$, estimators $\hat{\alpha}_{\mathrm{H}}^{k}$, $\hat{\alpha}_{\mathrm{P}}^{k}$, $\hat{\alpha}_{\mathrm{B}}^{k}$ are calculated on the basis of absolute values of a sample, for each possible k, which means $k \in K_n = \{1, 2, \ldots, [n/4] - 1\}$ (k-th records – necessary for $\hat{\alpha}_{\mathrm{B}}^{k}$ – are calculated in the R's package Records by Chrapek 2012)
b. each estimate $\hat{\alpha}_{\mathrm{H}}^{k}$, $\hat{\alpha}_{\mathrm{P}}^{k}$, $\hat{\alpha}_{\mathrm{B}}^{k}$ that is beyond the interval $(0, 2)$, is rejected and, as a consequence, omitted in the sequel (this is the reason why we deal with the already mentioned meaningful pairwise incompleteness of bivariate samples of estimates),
c. for all the other cases – based on formulas (5), (6) – estimates $\hat{\sigma}_{\mathrm{H}}^{k}$, $\hat{\sigma}_{\mathrm{P}}^{k}$, $\hat{\sigma}_{\mathrm{B}}^{k}$ are computed based on known sample size n,
d. concurrently with $\hat{\sigma}_{\mathrm{B}}^{k}$, considering formulas (7A, B), two additional estimates $\hat{\sigma}_{\psi}^{k}$, $\hat{\sigma}_{l}^{k}$ are calculated as if a sample size n was unknown.

Secondly, the previous step is replicated $J = 10000$ times independently, so that for any set of given α, σ, n, k we get five sequences $\hat{\sigma}_{\mathrm{H}}^{k}$, $\hat{\sigma}_{\mathrm{P}}^{k}$, $\hat{\sigma}_{\mathrm{B}}^{k}$, $\hat{\sigma}_{\psi}^{k}$, $\hat{\sigma}_{l}^{k}$ of sizes at most J (because of marginal incompleteness).

Thirdly, we perform "internal" comparative analysis of estimators $\hat{\sigma}_{\mathrm{H}}^{k}$, $\hat{\sigma}_{\mathrm{P}}^{k}$, $\hat{\sigma}_{\mathrm{B}}^{k}$, $\hat{\sigma}_{\psi}^{k}$, $\hat{\sigma}_{l}^{k}$ (within these 5 types of estimators separately) in order to indicate the best k for any given n. The Pitman nearness measure is evaluated for any pair of distinct $k_1, k_2 \in K_n$, given a set of α, σ, and n. A demonstrative Table 1 presents values of Pitman nearness measure for estimates based on Berred's approach with known sample size ($\hat{\sigma}_{\mathrm{B}}^{k}$), with fixed $n = 50$, $\alpha = 1.9$, $\sigma = 1$. Tables of Pitman nearness measure for other estimators and other values of α, σ, n provide quite similar tables of matrices, whose dimensions vary depending on sample sizes.

Next, all the indications of which k provides better (in the sense of being Pitman-closer estimate) within the same sample size, against other k's, are counted up. This procedure leads to optimal choices of k's for a given estimator type and sample size n, which is presented in Table 2.

Table 1. Pitman-closer measures, comparing all possible k's (order k_1 is assigned to the first of compared estimators) – selected case of for $\hat{\sigma}_{\mathrm{B}}^{k}$, $n = 50$ $\alpha = 1.9$, $\sigma = 1$.

k_1 \ k_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	-	0.250	0.197	0.170	0.175	0.199	0.247	0.245	0.226	0.406	0.250
2	0.750	-	0.300	0.295	0.239	0.255	0.290	0.242	0.262	0.346	0.400
3	0.803	0.700	-	0.323	0.329	0.268	0.444	0.315	0.333	0.438	0.556
4	0.830	0.705	0.677	-	0.362	0.345	0.399	0.300	0.244	0.474	0.429
5	0.825	0.761	0.671	0.638	-	0.369	0.414	0.400	0.388	0.280	0.600
6	0.801	0.745	0.732	0.655	0.631	-	0.517	0.458	0.412	0.478	0.875
7	0.753	0.710	0.556	0.601	0.586	0.483	-	0.463	0.476	0.382	0.357
8	0.755	0.758	0.685	0.700	0.600	0.542	0.537	-	0.478	0.444	0.438
9	0.774	0.738	0.667	0.756	0.612	0.588	0.524	0.522	-	0.500	0.522
10	0.594	0.654	0.563	0.526	0.720	0.522	0.618	0.556	0.500	-	0.588
11	0.750	0.600	0.444	0.571	0.400	0.125	0.643	0.563	0.478	0.412	-

Source: own study.

Table 2. Optimal choices of k's for 5 types of estimators.

$\boldsymbol{n}$	**50**	**80**	**110**
$\hat{\sigma}_{\mathrm{H}}$	5	6	7
$\hat{\sigma}_{\mathrm{P}}$	10	13	15
$\hat{\sigma}_{\mathrm{B}}$	8	10	12
$\hat{\sigma}_{\psi}$	10	13	14
$\hat{\sigma}_{l}$	10	13	14

Source: own study.

Fourthly, we perform "external" comparative analysis of the best estimators of each type. In contrast to the "internal" case, we are forced to rely on the previously introduced measure η. For a given sample size, and values of both parameters α and σ, measures η of each estimator are calculated, and then ranked. Demonstrative values of these measures, for fixed $n = 50$, $\alpha = 1.9$ and all five types of estimators, are gathered in Table 3, while Table 4 includes their corresponding ranks (from 0 – the best to 4 – the worst).

Table 3. Values of measure η – case of $n = 50$ $\alpha = 1.9$, all σ's.

σ	0.01	0.1	1	10	100
$\hat{\sigma}_H$	0.00897	0.08565	1.0553	8.634	85.68
$\hat{\sigma}_P$	0.00666	0.05361	0.6323	6.975	70.52
$\hat{\sigma}_B$	0.00467	0.06508	0.5912	5.935	68.1
$\hat{\sigma}_\psi$	0.00558	0.05985	0.5223	4.615	59.62
$\hat{\sigma}_l$	0.00541	0.05804	0.5032	4.393	57.66

Source: own study.

Next, within the type of estimator, single ranks are summed up in the whole range of σ's and the sums are ranked ("combined ranks") as the preconceived approach to estimation assumes naturally that the value of parameter σ is unknown – see the two last columns of demonstrative Table 4.

Finally, within the type of estimator, "combined ranks" are summed up simultaneously in the range of all sample sizes and all values of parameter α (partial illustration of this procedure is contained in Table 5). Again, the sums obtained in this way are ranked ("total ranks").

Table 4. Ranks of η values – case of $n = 50$ $\alpha = 1.9$, all σ's.

σ	0.01	0.1	1	10	100	sums of ranks	combined ranks
$\hat{\sigma}_H$	4	4	4	4	4	20	4
$\hat{\sigma}_P$	3	0	3	3	3	12	3
$\hat{\sigma}_B$	0	3	2	2	2	9	2
$\hat{\sigma}_\psi$	2	2	1	1	1	7	1
$\hat{\sigma}_l$	1	1	0	0	0	2	0

Source: own study.

Table 5. Total ranks of η values.

σ	$n = 50$ $\alpha = 1.8$	$n = 50$ $\alpha = 1.82$	…	$n = 50$ $\alpha = 1.9$	…	$n = 80$ $\alpha = 1.9$	…	$n = 110$ $\alpha = 1.98$	$n = 110$ $\alpha = 1.99$	sums of combined ranks	total ranks
$\hat{\sigma}_H$	4	4	…	4	…	4	…	4	4	130	4
$\hat{\sigma}_P$	3	2	…	3	…	3	…	3	3	82	3
$\hat{\sigma}_B$	2	3	…	2	…	1.5	…	2	2	69.5	2
$\hat{\sigma}_\psi$	1	1	…	1	…	0	…	0	0	15	0
$\hat{\sigma}_l$	0	0	…	0	…	1.5	…	1	1	33.5	1

Source: own study.

The reported procedure of making rankings shows that the 'k-th-record' estimators of the scale parameter are appraised to be the best ones. Interestingly, regardless of quite small discrepancies in the values of measure η, especially good performance characterises estimators assuming unknown sample size.

Moreover, quite similar indications may be noticed globally, with the use of scaled measure η. To do so, every value of η is divided by the adequate σ, which allows to carry out a comparative analysis of estimates that are obtained for different σ's. (It is imposed by a simple fact that $\sigma X \sim S(\alpha, 0,0,\sigma)$ for $X \sim S(\alpha, 0,0,1)$, and any $\sigma > 0$). Within almost all values of α, the best estimates of the scale parameter are those based on k-th-records (see Table 6). Additionally, the total sums of the scaled measure η confirm previous insights, and accentuate approximate quality level of estimation based on Pickands' and Berred's approaches.

Table 6. Summed values of measure η scaled by σ – within groups of α's.

$\boldsymbol{\alpha}$	$\hat{\boldsymbol{\sigma}}_{\mathbf{H}}$	$\hat{\boldsymbol{\sigma}}_{\mathbf{P}}$	$\hat{\boldsymbol{\sigma}}_{\mathbf{B}}$	$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\psi}}$	$\hat{\boldsymbol{\sigma}}_{\boldsymbol{l}}$
1.8	11.41176	10.69419	10.63437	9.02193	9.29989
1.82	11.34495	10.10213	10.58824	9.28663	9.57081
1.84	12.10306	10.89211	10.28613	8.56202	8.77059
1.86	12.48158	10.64509	10.39769	8.55502	8.68862
1.88	13.09939	10.42322	9.78006	7.89665	8.18492
1.9	13.92617	10.54751	10.10855	9.11400	9.33091
1.92	13.89563	10.12217	9.98363	9.08145	9.29733
1.94	13.91820	10.44230	9.94078	8.01853	8.22405
1.96	14.83072	10.07188	9.39138	8.93096	9.15942
1.98	15.78266	10.75358	9.93393	8.53742	8.62024
1.99	19.33139	10.30731	10.04061	8.06486	8.21329
Total	152.12551	115.00149	111.08538	95.06947	97.36007

Source: own study.

4. Empirical example

To illustrate how the introduced estimation works in practice we consider electric energy prices in Finland quoted in euro at the Nord Pool Spot (www.nordpoolgroup.com). The chosen time series represents weekly prices from the 10$^{\text{th}}$ week of 2018 to the 9$^{\text{th}}$ week of 2020, which makes the sample size to be $n = 104$ (time span of two years). Figure 1 illustrates the mentioned data, and suggests SARMA-GARCH approach as an appropriate way to model the series. Such types of models are effectively applied for electricity market data (see for instance Aiube et al. 2013, Stachura and Wodecka 2016).

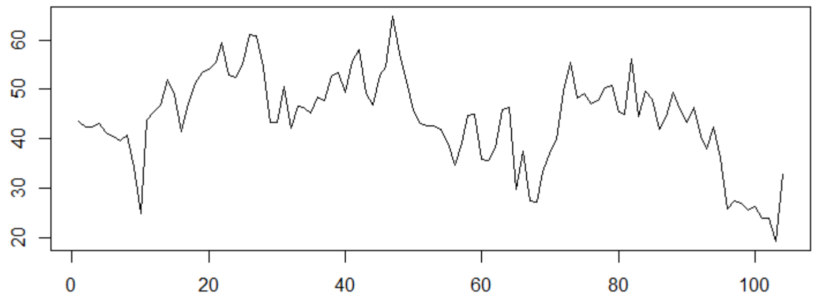

Figure 1. Week electricity prices in Finland.

Source: own study.

Amongst several initially estimated models (the R's package fGarch by Wuertz et al. 2020), the $\mathrm{SARMA}(1{,}1)_{13} \times \mathrm{GARCH}(1{,}1)$ occurred to be the best. Its residuals may be recognized as a random sample (Wald-Wolfowitz runs test p-value $= 0.9147$ – calculated with use of the R's package randtests by Caeiro and Mateus 2014) taken from normal distribution (Jarque-Bera test p-value $= 0.9201$, Shapiro-Wilk test p-value $=$ 0.9945 – both calculated with the use of the R's package fGarch by Wuertz et al. 2020). The detected normality may as well indicate that residuals' distribution is α-stable with the stability parameter close to 2.

We decide to approximate the distribution of the residuals with a stable distribution $S(\alpha, 0, 0, \sigma)$. To do so, we fix record order $k = 3$. As formula (3) holds, we use formula (2) for absolute values of the residuals, obtaining $\hat{\alpha}_{\mathrm{B}}^{k} = 1.812339$. Then, formula (6) yields $\hat{\sigma}_{\mathrm{B}}^{k} = 0.895889$. It occurs that such gained approximation is accepted in view of two goodness of fit tests (Anderson-Darling test p-value $= 0.1051$ – calculated with use of the R's package goftest by Faraway et al. 2019, Kolmogorov-Smirnov test p-value $=$ 0.3532).

5. Conclusions

The presentation of simulation research results gives some straightforward conclusions, which are as follows:

- 'k-th record' approach to estimation of the scale parameter σ is at least as good as the other classical methods presented herein (also, or even especially, assuming unknown sample size).
- 'k-th record' approach gives globally quite comparable results to Pickands' approach, which should not be surprising, as Berred's estimator is an analogue of Pickands' one.

- Estimation of σ based on Hill's estimator is distinctly characterized by the lowest stability in the sense that scale parameter estimates become more and more biased as stability index α tends to 2.
- 'k-th record' approach seems to be "unbeatable" in the region of stability index α very close to 2.

Concluding in general, it must be also remarked that the insights, hereinbefore specified, should be perceived essentially as the advantages of the 'k-th record' approach over the others presented, since the Berred's estimator, and the scale parameter estimator based on it, may be employed in cases of incomplete information about an underlying sample.

On the one hand, this incompleteness may be very useful if an analysed database must stay undisclosed, even for a researcher/statistician working on it, or more, the data are only partially recorded (i.e. record values of a proper order or several orders). On the other hand, if in contrary an analysed database is absolutely fulfilled and disclosed, the 'k-th record' approach opens up opportunities to make use of permutation methods in order to make repeated estimation that leads to much more precise results. Obviously, the key to success in the latter case is that the data correspond to i.i.d. random sample.

However, it should be pointed out that the 'k-th record' approach still requires a complete recognition of theoretical properties of the 'k-th record' estimator of the scale parameter, at least in a range of enhancing the results of Wodecka (2016) in the context of how fast is the 'k-th record' estimators' convergence.

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