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Fixed point theorems of generalized contractions and their applications*

Abstract. The paper presents some theorems about fixed points of non-expansive mappings of certain convex subsets of Banach spaces, as well as examples of applications of these theorems, among others, to justify the correctness of the square roots approximation process in Heron's algorithm (from the 1st century AD).

1. Introduction

The aim of the paper is to present some theorems about fixed points of non-expansive mappings, in particular to discuss the fixed point theorems about contractive mappings of certain one-dimensional and multidimensional sets. Three examples of the application of theorems of this type are also included in it. These examples illustrate how certain problems can be approached from the perspective of fixed point theory. In one of them there is a justification for the correctness of the approximation process of square roots in Heron's algorithm. We also discuss various methods of approximating square roots. Previous (Barcz, 2021; Barcz, 2020; Barcz, 2019) papers have presented applications of the Edelstein and Banach fixed point theorems to the golden number φ approximation. It can be seen that the presented work is a development of (Barcz, 2021; Barcz, 2020; Barcz, 2019).

2. Introduction to some fixed point theorems of non-expansive mappings

We will introduce the definitions of the concepts that will be used latter and give a useful fact about the relationship between the so-called ε-fixed point and the fixed point.

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Let (X, d) be a metric space and let $f : X \to X$.
A *fixed point* of the mapping $f : X \to X$ is called a point $\hat{x} \in X$ such that $f(\hat{x}) = \hat{x}$. We say that the mapping f satisfies the *Lipschitz condition* on X if there is a constant $k \geqslant 0$ such that the inequality

$$d(f(x), f(y)) \leqslant kd(x, y)$$

holds for all $x, y \in X$; the smallest such k is called the *Lipschitz constant* $k(f)$ of f. If $k(f) < 1$, the mapping f is called the *contraction*; if $k(f) = 1$, the mapping f is said to be *non-expansive*. We say f is a *contractive* if for all $x, y \in X$ such that $x \neq y$ we have

$$d(f(x), f(y)) < d(x, y).$$

Let A be a subset of X and $f : A \to X$. Given an $\varepsilon > 0$, any point $a \in A$ with $d(a, f(a)) < \varepsilon$ is called an ε- *fixed point* for f.

Let $A \subset X$ be a bounded and closed set. A continuous mapping $f : A \to X$ is called a *compact mapping* if $\overline{f(A)}$ is a compact set.
We have the following

FACT 1 *Let $f : A \to X$ be a compact mapping. If f has an ε-fixed point for every $\varepsilon > 0$, then it has a fixed point.*

Indeed, let x_n be a $\left(\frac{1}{n}\right)$-fixed point for f, $n = 1, 2, \ldots$, this is $d(x_n, f(x_n)) < \frac{1}{n}$. Since f is compact, we may assume that $f(x_n) \to x_0 \in \overline{f(A)}$. It follows that $x_n \to x_0$ (because $d(x_n, x_0) \leqslant d(x_n, f(x_n)) + d(f(x_n), x_0) \to 0$). Since A is closed, it follows that $x_0 \in A$. By the continuity of f, we have $f(x_n) \to f(x_0)$. Since $d(x_0, f(x_0)) \leqslant d(x_0, f(x_n)) + d(f(x_n), f(x_0)) \to 0$, f has a fixed point x_0.

3. Some of the fixed point theorems of non-expansive mappings

Lemma 1 and Theorems 1-5 are classic results. We add proofs to them (except Theorem 4) for the completeness of considerations and to highlight the connections between them. For example, in the proof of Theorem 5, Theorem 3 is used.

THEOREM 1
(Banach Contraction Principle) Let $(X, \|\cdot\|)$ be a Banach space, and $D \subset X$ closed and $f : D \to D$ a contraction. Then f has a unique fixed point $u \in D$, and $f^n(x_0) \to u$ for each $x_0 \in D$

Proof. First we show that f can have at most one fixed point. Then we construct a sequence which converges and we show that its limit is a fixed point of f.

(a) Let $u, u' \in D$. Suppose u and u' are fixed points of f. Then $\|u - u'\| = \|f(u) - f(u')\| \leqslant q\|u - u'\|$. Since $q < 1$, this implies that $\|u - u'\| = 0$, i.e. $u = u'$.

(b) Let $x_0 \in D$ be any element, and define an iterative sequence (x_n) by putting $x_{n+1} = f(x_n)$ $(x_n = f^n(x_0))$, $n = 0, 1, 2, \ldots$ Note that for a fixed $p \in \mathbb{N}$ and any $n \in \mathbb{N}$:

$$\begin{aligned}\|f^n(x_0) - f^{n+p}(x_0)\| =& \|f^n(x_0) - f^n(f^p(x_0))\| \leqslant q^n \|x_0 - f^p(x_0)\| \leqslant \\ \leqslant& q^n \left(\|x_0 - f(x_0)\| + \cdots + \|f^{p-1}(x_0) - f^p(x_0)\|\right) \leqslant \\ \leqslant& q^n \left(1 + q + q^2 + \cdots + q^{p-1}\right) \|x_0 - f(x_0)\| \to 0\end{aligned}$$

as $n \to \infty$, because $q < 1$.

This shows that (x_n) is a Cauchy sequence in D, and D is complete. Hence (x_n) must be convergent, say $\lim_{n\to\infty} x_n = u \in D$. Since f is continuous, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(u).$$

Thus u it is a (unique) fixed point.

Remark 1
The above proof can be simplified when additionally D is a bounded set; because then

$$\begin{aligned}\|f^n(x_0) - f^{n+p}(x_0)\| =& \|f^n(x_0) - f^n(f^p(x_0))\| \leqslant q^n \|x_0 - f^p(x_0)\| \leqslant \\ \leqslant& q^n diam(D) \to 0 \text{ as } n \to \infty.\end{aligned}$$

From now on X will be a Banach space with the norm $\|\cdot\|$.

Lemma 1 *If C is a closed, bounded and convex subset of X and if $f : C \to C$ is a non-expansive, i.e. $\|f(x) - f(y)\| \leqslant \|x - y\|$ for all $x, y \in C$, then for any $\varepsilon > 0$ there is a contraction $f_\varepsilon : C \to C$ such that for all $x \in C$ $\|f(x) - f_\varepsilon(x)\| < \varepsilon$.*

Proof. Let's take any point $z \in C$. Take arbitrary $\varepsilon > 0$. Let $q \in (0,1)$ be a real number such that $q < \frac{\varepsilon}{\mathrm{diam} C}$ and

$$f_q(x) = qz + (1-q)f(x),\ x \in C.$$

The mapping f_q is a contraction on the set C: for any $x, y \in C$ we have

$$\|f_q(x) - f_q(y)\| = (1-q)\|f(x) - f(y)\| \leqslant (1-q)\|x - y\|.$$

On the basis of Banach Contraction Principle there is such point $x_q \in C$ that $f_q(x_q) = x_q$. Also for any $x \in C$, we have

$$\|f_q(x) - f(x)\| = \|qz + (1-q)f(x) - f(x)\| = q\|z - f(x)\| \leqslant q \ \mathrm{diam} C,$$

$$\|f_q(x) - f(x)\| < \frac{\varepsilon}{\mathrm{diam} C} \cdot \mathrm{diam} C = \varepsilon \tag{$*$}$$

Therefore from $(*)$ we have

$$\|x_q - f(x_q)\| = \|f_q(x_q) - f(x_q)\| \leqslant q \operatorname{diam} C < \varepsilon. \qquad (**)$$

Since a positive number ε is arbitrary, so

$$\inf_{x \in C} \{\|x - f(x)\|\} = 0.$$

In view of this result obtained from Lemma 1, we can formulate the following

Theorem 2
If C is a closed, bounded and convex subset of X, $f : C \to C$ is a non-expansive mapping, then

$$\inf_{x \in C} \|x - f(x)\| = 0.$$

The above result does not guarantee the existence of fixed point, and it shows that there are points with can be considered as "arbitrarily little moved".

If C is additionally a compact set, so f is a compact mapping, then from the inequality $(**)$ on the basis of Fact 1 we conclude that the mapping f has a fixed point. This justifies the truth of the following theorem:

Theorem 3
If C is a compact and convex subset of (the Banach space) X, and $f : C \to C$ is a non-expansive mapping, then f has a fixed point in the set C.

Theorem 4
Each contractive mapping $f : [a, b] \to [a, b]$ of the interval $[a, b] \subset \mathbb{R}$ into itself has a unique fixed point $\hat{x}$, and $f^n(x) \to \hat{x}$ for each $x \in [a, b]$.

Below we consider the space $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$.
Let C be the Cartesian product of closed intervals $[a_i, b_i] \subset \mathbb{R},\ i = 1, 2, \ldots, n$, i.e. $C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.
The generalization of Theorem 4 is the following

Theorem 5
If $f : C \to C$ is a contractive mapping of the set C into itself, then f has a unique fixed point $u \in C$, and $f^n(x_0) \to u$ for each $x_0 \in C$.

Proof. Obviously $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space and C is a compact subset of $\mathbb{R}^n$. Moreover, it is a convex set. Of course f is a non-expansive mapping. Therefore on the basis of Theorem 3 f has a fixed point $u \in C$. We get the uniqueness of the fixed point from the contradiction:

$$\|u - u'\| = \|f(u) - f(u')\| < \|u - u'\|$$

obtained when $f(u) = u \neq u' = f(u')$.
Now let's take any $x_0 \in C$ and let's define $\varrho_n = d(f^n(x_0), u) = \|f^n(x_0) - u\|$. The sequence of non-negative numbers ϱ_n is non-increasing, and therefore convergent. Let $c = \lim_{n \to \infty} \varrho_n$, so $c \geqslant 0$.

Since C is a compact set, so the subsequence $(f^{n_k}(x_0))$ of $(f^n(x_0))$ converges to some $y \in C$. Therefore $\|y-u\| = c$, and putting $a_{n_k} = \|f^{n_k}(x_0) - u\|$ in the face of a contradiction:

$$c = \lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \|f^{n_k}(x_0) - u\| = \lim_{k\to\infty} \|f^{1+n_k}(x_0) - u\| = \|f(y) - u\| =$$
$$= \|f(y) - f(u)\| < \|y-u\| = c$$

obtained when $c > 0$, we have $c = 0$. Therefore $y = u$. Since all subsequences $(f^{n_k}(x_0))$ converge to u, we get $f^n(x_0) \to u$ as $n \to \infty$.

Remark 2
Theorems 4 and 5 are versions of Edelstein's fixed point theorem. The proof of Theorem 4 as analogous to the proof of Theorem 5 has been omitted.

4. Applications

Example 1 *Let us consider the sequence (x_n) given by conditions:*

$$x_{n+1} = 1 + \frac{1}{1+x_n}, \; x_1 = 1.$$

We will show its convergence to $\sqrt{2}$. For this purpose, let f be a function defined on the interval $\langle 1,2\rangle$ by the formula $f(x) = 1 + \frac{1}{1+x}$. Since $f(1) = 1\frac{1}{2}$, $f(2) = 1\frac{1}{3}$ and f is a decreasing function, so $f(\langle 1,2\rangle) \subset \langle 1,2\rangle$. Let's note that for any $x, x' \in \langle 1,2\rangle$ we have

$$|f(x) - f(x')| = \frac{|1+x-1-x'|}{(1+x)(1+x')} \leqslant \frac{|x-x'|}{4},$$

so for any $x, x' \in \langle 1,2\rangle$ such that $x \neq x'$

$$|f(x) - f(x')| < |x-x'|,$$

and $f : \langle 1,2\rangle \to \langle 1,2\rangle$ is a contractive mapping.

By Theorem 4, we conclude that there is only one number $u \in \langle 1,2\rangle$ such that $u = f(u)$. We also obtain the convergence the sequence (x_n), $x_n = f^{n-1}(1)$ to $\sqrt{2}$.

Example 2 *Suppose that $c > 0, x_0 > \sqrt{c}$ and $x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right)$ for $n \geqslant 1$. We will show that $\lim_{n\to\infty} x_n = \sqrt{c}$ justifying the correctness of calculations in Heron's algorithm (from the first century AD) defined by the above conditions. Let us consider a function $f(x) = \frac{1}{2}\left(x + \frac{c}{x}\right)$ defined on interval $C = \langle\sqrt{c}, c\rangle$. Note that $f(\sqrt{c}) = \sqrt{c}$ and $f(c) < c$ (since it is easy to see that $f(x) < x$ for $x > \sqrt{c}$), furthermore f is an increasing function. Therefore $f(C) \subset C$.*

For each $x \in C$ we have $f'(x) = \frac{1}{2}(1 - \frac{c}{x^2})$. Because $0 \leqslant f'(x) < \frac{1}{2}$ for each $x \in C$ (comp. Remark 3), $f : C \to C$ is a contractive mapping.

Therefore by Theorem 4 there is a unique $u \in C$ such that $u = f(u)$. In addition $\lim_{n\to\infty} x_n = \sqrt{c}$.

The above convergence can be obtained without using the fixed point theorems. Lets us note that, since $x_n > 0$ we have

$$x_n^2 = \frac{1}{4}\left(x_{n-1} - \frac{c}{x_{n-1}}\right)^2 + c \geqslant c \text{ for } n = 1, 2, 3, \ldots,$$

so $x_n \geqslant \sqrt{c}$ for $n = 0, 1, 2, 3, \ldots$
Whereas

$$x_{n+1} - x_n = \frac{1}{2}\left(\frac{c}{x_n} - x_n\right) = \frac{c - x_n^2}{2x_n} \leqslant 0$$

Then there is a limit $g = \lim_{n\to\infty} x_n \geqslant \sqrt{c}$. The limit g satisfies the equation $g = \frac{1}{2}\left(g + \frac{c}{g}\right)$, therefore $g = \sqrt{c}$.

Remark 3
In the above example we additionally assumed that $x_0 \leqslant c$. We can consider the more general situation $x_0 \in [\sqrt{c}, \infty)$. Then considering $f : [\sqrt{c}, \infty) \to [\sqrt{c}, \infty)$ you can see that f is a contraction ($0 \leqslant f'(x) < \frac{1}{2}$ for each $x \geqslant \sqrt{c}$), so by Lagrange's theorem $|f(x) - f(x')| \leqslant$ $\leqslant \frac{1}{2}|x - x'|$ for every $x, x' \in C$). By Banach Contraction Principle there is (one) fixed point u and for any $x_0 > \sqrt{c}$ $x_n = f^n(x_0) \to \sqrt{c}$.

Example 3 *In the paper (Barcz, 2019) there is an equality*

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} \qquad (***)$$

which can be obtained by the sequence (a_n) defined by the conditions

$$a_0 = \sqrt{1}, a_{n+1} = \sqrt{1 + a_n},$$

noting that it has a limit a as an increasing and bounded sequence. This limit satisfies the equation $a^2 = 1 + a$, of course $a = \frac{1+\sqrt{5}}{2} = \varphi$ (because $a > 0$). Writing this limit in the form $\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$, we have the equality (***).
Below we will present a way to obtain this equality using Banach Contraction Principle (comp. (Barcz, 2020)). Let $f(x) = \sqrt{1 + x}$ be a function defined on $C = \langle 0, 3\rangle$.
Since $f(0) = 1$, $f(3) = 2$ and f is an increasing function, so $f(C) \subset C$. For any $x, x' \in C$ we have

$$|f(x) - f(x')| = |\sqrt{1 + x} - \sqrt{1 + x'}| = \frac{|x - x'|}{\sqrt{1 + x} + \sqrt{1 + x'}} \leqslant \frac{1}{2}|x - x'|.$$

Therefore $f : C \to C$ is a contraction, and by Banach Contraction Principle f has a unique fixed point $u = \sqrt{1 + u}$, that is $u = \varphi$. Moreover, $a_n = f^n\left(\sqrt{1}\right) \to \varphi$, and by defining $\lim_{n\to\infty} a_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$ we get (***).

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