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THE BOUNDS OF SOME DETERMINANTS FOR FUNCTIONS OF BOUNDED TURNING OF ORDER ALPHA

Summary

In the present paper, the estimates of some determinants over the class $\mathcal{R}(\alpha)$, $0 \le \alpha < 1$, of analytic functions f standardly normalized such that

$$\mathrm{Re} f'(z) > \alpha, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

are shown.

Keywords and phrases: univalent functions, functions of bounded turning, functions of bounded turning of order α, Hankel determinant

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of f normalized by $f(0) := 0$ and $f'(0) := 1$, so of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Given $n, q \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathcal{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

where $a_1 := 1$. To find the growth of the Hankel determinant $H_{q,n}(f)$ dependent on q and n for the whole class $\mathcal{S} \subset \mathcal{A}$ of univalent functions as well as for its subclasses is one of the main problem to study. For the class $\mathcal{S}$ some important result was shown by Pommerenke [16]. For fixed q and n the growth problem is reduced to find the bound of the Hankel determinant over selected compact subclasses of $\mathcal{A}$. Recently many authors examined the Hankel determinant $H_{2,2}(f)$ of order 2 as well as the Hankel determinant $H_{3,1}(f)$ of order 3 (see e.g., [9], [14], [11], [2]). Note that $H_{2,1}(f) = a_3 - a_2^2$. Thus the Hankel determinant $H_{2,1}(f)$ of order 2 reduces to the well known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see e.g., [7, Vol. I, p. 35]).

Given $\alpha \in [0,1)$, by $\mathcal{R}(\alpha)$ we denote a subclass of $\mathcal{A}$ of functions f such that

$$\mathrm{Re} f'(z) > \alpha, \quad z \in \mathbb{D}. \tag{1.2}$$

Functions in $\mathcal{R}(\alpha)$ are called *of bounded turning of order* α and in $\mathcal{R} := \mathcal{R}(0)$ *of bounded turning* (see e.g., [7, Vol. I, p. 101]).

In this paper we found sharp estimates of the Hankel determinants $H_{2,2}(f)$, $H_{2,1}(f)$ and of the determinant

$$\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = a_4 - a_2 a_3 \tag{1.3}$$

over the class $\mathcal{R}(\alpha)$. Having these results, the Hankel determinant $H_{3,1}(f)$ can be estimated also, however far from sharpness. For the class $\mathcal{R}$ it was done in [1] with some correctness in [2]. This result for $\mathcal{R}$ was recently improved in [17].

Let $\mathcal{P}$ be the class of Carathéodory functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.4}$$

having a positive real part in $\mathbb{D}$. The results below for the class $\mathcal{P}$ will be used in further considerations.

Lemma 1.1. [6, p. 41] *If $p \in \mathcal{P}$ is of the form* (1.4)*, then*

$$|c_n| \le 2, \quad n \in \mathbb{D}. \tag{1.5}$$

The inequality is sharp and the equality holds for the function $p := L$, where

$$L(z) := \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}.$$

Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$.

Lemma 1.2. ([12], [13]) *If $p \in \mathcal{P}$ is of the form* (1.4) *with $c_1 > 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2) \tag{1.6}$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{1.7}$$

for some $\zeta \in \overline{\mathbb{D}}$ and $\eta \in \overline{\mathbb{D}}$.

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [5] (see also [15]). Define

$$Y(a,b,c) := \max_{z\in\overline{\mathbb{D}}} \left(|a+bz+cz^2| + 1 - |z|^2\right), \quad a,b,c \in \mathbb{R}.$$

Lemma 1.3. ([5]) *If $ac \ge 0$, then*

$$Y(a,b,c) = \begin{cases} |a|+|b|+|c|, & |b| \ge 2(1-|c|), \\ 1+|a|+\dfrac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|). \end{cases}$$

If $ac < 0$, then

$$Y(a,b,c)$$
$$= \begin{cases} 1-|a|+\dfrac{b^2}{4(1-|c|)}, & -4ac(c^{-2}-1) \le b^2 \wedge |b| < 2(1-|c|), \\ 1+|a|+\dfrac{b^2}{4(1+|c|)}, & b^2 < \min\left\{4(1+|c|)^2, -4ac(c^{-2}-1)\right\}, \\ R(a,b,c), & \text{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c|, & |c|(|b|+4|a|) \le |ab|, \\ -|a|+|b|+|c|, & |ab| \le |c|(|b|-4|a|), \\ (|c|+|a|)\sqrt{1-\dfrac{b^2}{4ac}}, & \text{otherwise}. \end{cases}$$

2. Main results

We will start with the determinant (1.3).

Theorem 2.1. *Let $\alpha \in [0,1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$|a_2a_3 - a_4| \le \frac{1}{2}(1-\alpha). \tag{2.1}$$

The inequality is sharp and the equality holds for the function

$$f(z) := \int_0^z \frac{1+(1-2\alpha)u^3}{1-u^3}\mathrm{d}u, \quad z \in \mathbb{D}. \tag{2.2}$$

Proof. Fix $\alpha \in [0,1)$ and let $f \in \mathcal{R}(\alpha)$ be of the form (1.1). Then by (1.2) the function

$$p(z) := \frac{1}{1-\alpha}(f'(z)-\alpha), \quad z \in \mathbb{D}, \tag{2.3}$$

belongs to $\mathcal{P}$. Putting the series (1.1) and (1.4) into (2.3) by equating the coefficients we get

$$a_{n+1} = \frac{(1-\alpha)c_n}{n+1}, \quad n \in \mathbb{N}. \tag{2.4}$$

Hence

$$|a_2a_3 - a_4| = \frac{1}{12}(1-\alpha)|2(1-\alpha)c_1c_2 - 3c_3|. \tag{2.5}$$

Now by using (1.6) and (1.7) we have

$$\begin{aligned} &|a_2a_3 - a_4| \\ &= \frac{1}{48}(1-\alpha)|(1-4\alpha)c_1^3 + (4-c_1^2)[(-4\alpha-2)c_1\zeta + 3c_1\zeta^2 - 6(1-|\zeta|^2)\eta]|, \end{aligned} \tag{2.6}$$

where $(\zeta, \eta) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Since the class $\mathcal{R}(\alpha)$ is invariant under the rotations, by (1.5) we may assume that $c_1 =: t \in [0, 2]$.

Assume first that $t := 2$. Then $p = L$ in (2.3) (see e.g., [7, Vol. I, p. 100]), so

$$f(z) = \int_0^z \frac{1+(1-2\alpha)u}{1-u}\mathrm{d}u, \quad z \in \mathbb{D}, \tag{2.7}$$

with $a_2 = 1-\alpha,\ a_3 = 2(1-\alpha)/3$ and $a_4 = (1-\alpha)/2$. Hence we see that for $\alpha \in [0, 1)$ the inequality

$$|a_2a_3 - a_4| = \frac{1}{6}(1-\alpha)|1-4\alpha| \le \frac{1}{2}(1-\alpha),$$

is true, so the inequality (2.1) holds for the function (2.7).

Let now $t \in [0, 2)$. From (2.6) we have

$$\begin{aligned} |a_2a_3 - a_4| &\le \frac{1}{48}(1-\alpha) \\ &\times \left[\left|(1-4\alpha)t^3 + (4-t^2)[(-4\alpha-2)t\zeta + 3t\zeta^2\right| + 6(4-t^2)(1-|\zeta|^2)|\eta|\right] \\ &\le \frac{1}{8}(1-\alpha)(4-t^2)\left[\left|\frac{(1-4\alpha)t^3}{6(4-t^2)} - \frac{1}{3}(1+2\alpha)t\zeta + \frac{1}{2}t\zeta^2\right| + 1 - |\zeta|^2\right] \\ &= \frac{1}{8}(1-\alpha)(4-t^2)\left[|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2\right], \end{aligned}$$

where

$$A := \frac{(1-4\alpha)t^3}{6(4-t^2)}, \quad B := -\frac{1}{3}(1+2\alpha)t, \quad C := \frac{1}{2}t.$$

Thus to prove the theorem we will show that

$$(4-t^2)\left[|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2\right] \le 4. \tag{2.8}$$

I. Consider first the case $AC \ge 0$ which holds when $\alpha \in [0, 1/4]$.

1. Since $|B| \ge 2(1-|C|)$ holds when $t \in [3/(2+\alpha), 2)$, by Lemma 1.3 we have

$$\begin{aligned} &(4-t^2)\left[|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2\right] \\ &\le (4-t^2)(|A| + |B| + |C|) = (4-t^2) \cdot \frac{(1-4\alpha)t^3 + (4\alpha+5)(4-t^2)t}{6(4-t^2)} \\ &= -\frac{2}{3}(1+2\alpha)t^3 + \frac{2}{3}(4\alpha+5)t =: \gamma(t), \quad t \in [3/(2+\alpha), 2). \end{aligned} \tag{2.9}$$

Note that

$$\begin{aligned} \gamma'(t) &= -2(1+2\alpha)t^2 + \frac{2}{3}(4\alpha+5) \\ &= -2(1+2\alpha)(t+t_0)(t-t_0), \quad t \in [3/(2+\alpha), 2), \end{aligned}$$

where

$$t_0 := \sqrt{\frac{4\alpha+5}{3(1+2\alpha)}}.$$

But

$$t_0 < \frac{3}{2+\alpha}, \quad \alpha \in [0,1).$$

Indeed, the above inequality equivalently written as

$$\frac{4\alpha+5}{3(1+2\alpha)} < \frac{9}{(2+\alpha)^2}$$

is equivalent to the inequality

$$4\alpha^3 + 21\alpha^2 - 18\alpha - 7 = (\alpha-1)(4\alpha^2+25\alpha+7) < 0, \quad \alpha \in [0,1),$$

which clearly holds. Thus the function γ is decreasing, and moreover

$$\begin{aligned}\gamma(t) &\le \gamma\left(\frac{3}{2+\alpha}\right)\\ &= \frac{8\alpha^3+42\alpha^2+36\alpha+22}{(2+\alpha)^3} \le 4, \quad t \in [3/(2+\alpha),2).\end{aligned} \tag{2.10}$$

Indeed, the second inequality is equivalent to the inequality

$$2\alpha^3 + 9\alpha^2 - 6\alpha - 5 = (\alpha-1)(\alpha+5)(2\alpha+1) \le 0, \quad \alpha \in [0,1),$$

which clearly holds.

2. Since $|B| < 2(1-|C|)$ holds when $t \in [0, 3/(2+\alpha))$, by Lemma 1.3 we have

$$\begin{aligned}&(4-t^2)\left[|A+B\zeta+C\zeta^2| + 1 - |\zeta|^2\right]\\ &\le (4-t^2)\left(1+|A|+\frac{B^2}{4(1-|C|)}\right)\\ &= (4-t^2)\left(1+\frac{(1-4\alpha)t^3}{6(4-t^2)}+\frac{(1+2\alpha)^2t^2}{18(2-t)}\right)\\ &= (4-t^2)\cdot\frac{72+[2(1+2\alpha)^2-18]t^2+[3(1-4\alpha)+(1+2\alpha)^2]t^3}{18(4-t^2)} \qquad (2.11)\\ &= \frac{1}{18}\left(72+t^2[4(1-\alpha)^2t+8(\alpha-1)(\alpha+2)]\right)\\ &\le 4+\frac{1}{2(2+\alpha)^2}\left[4(1-\alpha)^2\cdot\frac{3}{2+\alpha}+8(\alpha-1)(\alpha+2)\right]\\ &= 4+\frac{2}{(2+\alpha)^3}(\alpha-1)(\alpha+5)(2\alpha+1) \le 4, \quad \alpha\in[0,1),\ t\in[0,3/(2+\alpha)).\end{aligned}$$

Summarizing, by (2.9) and (2.11) it follows that (2.8) is true for $\alpha \in [0, 1/4]$.

II. Consider the case $AC < 0$ which holds when $\alpha \in (1/4, 1)$.

1. Since

$$B^2C^2 + 4AC(1-C^2) = \frac{1}{9}(1-\alpha)^2t^4 \ge 0$$

and $|B| < 2(1-|C|)$ holds when $t < 3/(2+\alpha)$, by Lemma 1.3, repeating computing as in (2.11) we have

$$\begin{aligned}
&(4-t^2)\left[|A+B\zeta+C\zeta^2|+1-|\zeta|^2\right]\\
&\le (4-t^2)\left(1-|A|+\frac{B^2}{4(1-|C|)}\right) \qquad (2.12)\\
&= (4-t^2)\left(1+\frac{(1-4\alpha)t^3}{6(4-t^2)}+\frac{(1+2\alpha)^2t^2}{18(2-t)}\right)\le 4,\quad t\in[0,3/(2+\alpha)).
\end{aligned}$$

2. Since $4(1+|C|)^2 = (2+t)^2$ and $-4AC(C^{-2}-1) = (4\alpha-1)t^2/3$, so the inequality

$$\begin{aligned}
B^2 &= \frac{1}{9}(1+2\alpha)^2t^2\\
&< \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\} = \frac{1}{3}(4\alpha-1)t^2
\end{aligned}$$

is equivalent to the inequality $(\alpha-1)^2 < 0$ which clearly does not hold.

3.(a) Note that the inequality $|C|(|B|+4|A|)\le |AB|$, i.e., the inequality

$$\frac{t}{2}\left(\frac{1}{3}(1+2\alpha)t-\frac{2(1-4\alpha)t^3}{3(4-t^2)}\right)\le\frac{(4\alpha-1)(1+2\alpha)t^4}{18(4-t^2)}$$

which after computing is equivalent to the inequality

$$t\ge\frac{1}{1-\alpha}\sqrt{\frac{3}{2}(1+2\alpha)}$$

does not hold since the inequality

$$\frac{1}{1-\alpha}\sqrt{\frac{3}{2}(1+2\alpha)}>2,\quad \alpha\in(1/4,1),$$

equivalently written as

$$8\alpha^2-22\alpha+5<0,\quad \alpha\in(1/4,1),$$

is clearly true.

(b) Observe that the inequality $|AB|\le |C|(|B|-4|A|)$, i.e., the inequality

$$\frac{(4\alpha-1)(1+2\alpha)t^4}{18(4-t^2)}\le\frac{t}{2}\left(\frac{1}{3}(1+2\alpha)t-\frac{2(4\alpha-1)t^3}{3(4-t^2)}\right)$$

for $t\in[3/(2+\alpha),2)$, is equivalent to the inequality

$$\frac{3}{2+\alpha}\le t\le\sqrt{\frac{3(1+2\alpha)}{2\alpha^2+8\alpha-1}}. \qquad (2.13)$$

Note first that $2\alpha^2+8\alpha-1>0$ for $\alpha\in(1/4,1)$. Moreover

$$\sqrt{\frac{3(1+2\alpha)}{2\alpha^2+8\alpha-1}}>\frac{3}{2+\alpha}.$$

Indeed, by squaring both sides of above inequality and by simple computing we equivalently get the inequality

$$2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = (1-\alpha)^2(2\alpha+7) > 0, \quad \alpha \in (1/4, 1),$$

which clearly holds. We also see that that the inequality

$$\sqrt{\frac{3(1+2\alpha)}{2\alpha^2+8\alpha-1}} < 2$$

holds, since by squaring and further computing it is equivalent to the true inequality

$$8\alpha^2 + 26\alpha - 7 = 8\left(\alpha + \frac{7}{2}\right)\left(\alpha - \frac{1}{4}\right) > 0, \quad \alpha \in (1/4, 1).$$

Arguing now exactly as of Part I.1 for t satisfying (2.13) we get

$$\begin{aligned} &(4-t^2)\left[|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2\right] \\ &\le (4-t^2)(-|A| + |B| + |C|) \qquad (2.14) \\ &= -\frac{2}{3}(1+2\alpha)t^3 + \frac{2}{3}(4\alpha+5)t = \gamma(t) \le 4. \end{aligned}$$

(c) It remains to consider

$$\sqrt{\frac{3(1+2\alpha)}{2\alpha^2+8\alpha-1}} < t < 2. \qquad (2.15)$$

Then by by Lemma 1.3 we have

$$\begin{aligned} &(4-t^2)\left[|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2\right] \\ &\le (4-t^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} \\ &= \left(4 - \frac{4}{3}(1-\alpha)t^2\right)\sqrt{\frac{(1-\alpha)^2t^2 - (1+2\alpha)^2}{3(1-4\alpha)}} =: \varphi(t), \end{aligned}$$

where t satisfies (2.15). Hence $\varphi'(t) = 0$ iff

$$-\frac{8}{3}(1-\alpha)t \cdot \frac{(1-\alpha)^2t^2 - (1+2\alpha)^2}{3(1-4\alpha)} + \frac{4}{3}\left(1 - \frac{1}{3}(1-\alpha)t^2\right)\frac{(1-\alpha)^2 t}{1-4\alpha} = 0$$

i.e., after simplifying iff

$$3(1-\alpha)^2t^2 = 8\alpha^2 + 5\alpha + 5,$$

and hence iff

$$t = \pm\frac{1}{\sqrt{3}(1-\alpha)}\sqrt{8\alpha^2+5\alpha+5} =: \pm t_1,$$

where $\alpha \in (1/4, 1)$ and t satisfies (2.15). But $t_1 > 2$. Indeed, this inequality after simplifying is equivalent to the inequality

$$4\alpha^2 - 29\alpha + 7 = 4\left(\alpha - \frac{1}{4}\right)(\alpha - 7) < 0, \quad \alpha \in (1/4, 1),$$

which is obviously true. Consequently $\varphi'(t) > 0$ iff

$$-3(1-\alpha)^2t^2 + 8\alpha^2 + 5\alpha + 5 < 0$$

which holds for t satisfying (2.15). Thus the function φ is increasing and hence

$$\begin{aligned}\varphi(t) \le \varphi(2) &= \left(4 - \frac{16}{3}(1-\alpha)\right)\sqrt{\frac{4(1-\alpha)^2-(1+2\alpha)^2}{3(1-4\alpha)}} \\ &= \frac{4}{3}(4\alpha-1) \le 4, \quad \alpha \in (1/4, 1).\end{aligned} \tag{2.16}$$

Summarizing, by (2.14), (2.15) and (2.16) it follows that (2.8) is true for $\alpha \in (1/4, 1)$. In this way it was shown that the inequality (2.8) is true which ends the proof of the inequality (2.1).

To show the sharpness, fix $\alpha \in [0, 1)$ and take the function f given by (2.2). Clearly, $f \in \mathcal{R}(\alpha)$ with $a_2 = a_3 = 0$ and $a_4 = (1-\alpha)/2$ which make the equality in (2.1). □

For $\alpha := 0$, i.e., for the class $\mathcal{R}$ the above theorem reduces to Theorem 2.1 of [2].

Remark 2.2. The proof of Theorem 2.1 was based on Lemma 1.3 which is useful for such computing. This lemma was also applied in [3] and [4] to find sharp bounds analogous as in Theorem 2.1 for the class of starlike functions of order α and of strongly starlike functions, respectively. Let now remark that the result of Theorem 2.1 can be achieved in a simple way by using Theorem 2.4 of [8] which particularly produces the following result: if $p \in \mathcal{P}$ is of the form (1.4) and $\mu \in [0, 1]$, then for $n, m \in \mathbb{N},\ m < n$, the following sharp estimate holds

$$|c_n - \mu c_{n-m}c_m| \le 2. \tag{2.17}$$

Hence, from (2.5) and since $\mu := 2(1-\alpha)/3 \le 1$ for $\alpha \in [0, 1)$, we have

$$|a_2a_3 - a_4| = \frac{1}{4}(1-\alpha)\left|c_3 - \frac{2}{3}(1-\alpha)c_1c_2\right| \le \frac{1}{2}(1-\alpha).$$

Theorem 2.4 can be found in [14] as Corollary 3.2. We reprove it by using the result below for the class $\mathcal{R}$ shown in [9].

Lemma 2.3. *If $f \in \mathcal{R}$ is of the form (1.1), then*

$$|a_2a_4 - a_3^2| \le \frac{4}{9}. \tag{2.18}$$

The inequality is sharp and the equality holds for the function

$$f(z) := \int_0^z \frac{1+u^2}{1-u^2}\mathrm{d}u = -z + \log\frac{1+z}{1-z}, \quad z \in \mathbb{D},\ \log 1 := 0. \tag{2.19}$$

Theorem 2.4. *Let $\alpha \in [0,1)$. If $f \in \mathcal{R}(\alpha)$ is the form* (1.1)*, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}(1-\alpha)^2. \tag{2.20}$$

The inequality is sharp and the equality holds for the function

$$\begin{aligned} f(z) &:= \int_0^z \frac{1+(1-2\alpha)u^2}{1-u^2}\mathrm{d}u \\ &= (-1+2\alpha)z + (1-\alpha)\log\frac{1+z}{1-z}, \quad z \in \mathbb{D},\ \log 1 := 0. \end{aligned} \tag{2.21}$$

Proof. Fix $\alpha \in [0,1)$ and let $f \in \mathcal{R}(\alpha)$. Define

$$g(z) := \frac{1}{1-\alpha}(f(z) - \alpha z), \quad z \in \mathbb{D}. \tag{2.22}$$

Assuming that

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2.23}$$

and using (1.1), from (2.22) by comparing the coefficients of the series, we get

$$a_n = (1-\alpha)b_n, \quad n \in \mathbb{N} \setminus \{1\}. \tag{2.24}$$

Since $g \in \mathcal{R}$, by applying the inequality (2.18) we get

$$|a_2a_4 - a_3^2| = (1-\alpha)^2|b_2b_4 - b_3^2| \leq \frac{4}{9}(1-\alpha)^2.$$

Since the equality in (2.18) holds for the function given by (2.19), it follows from (2.22) that the equality in (2.20) holds for the function given by (2.21). □

To prove Theorem 2.6 we use the following result related to the Fekete-Szegö functional for the class $\mathcal{R}$ (see e.g., [10, Corollary 2.7]).

Theorem 2.5. *Let $\lambda \in [0,4/3]$. If $f \in \mathcal{R}$ is of the form* (1.1)*, then*

$$|a_3 - \lambda a_2^2| \leq \frac{2}{3}. \tag{2.25}$$

The inequality is sharp and for each $\lambda \in [0,4/3]$ the equality holds for the function (2.19)*.*

Theorem 2.6. *Let $\alpha \in [0,1)$. If $f \in \mathcal{R}(\alpha)$ is the form* (1.1)*, then*

$$|a_3 - a_2^2| \leq \frac{2}{3}(1-\alpha). \tag{2.26}$$

The inequality is sharp and the equality holds for the function (2.21)*.*

Proof. Let $\alpha \in [0,1)$ and $f \in \mathcal{R}(\alpha)$. Let g be a function defined by (2.22) of the form (2.23). Since $g \in \mathcal{R}$ by (2.24) and by applying the inequality (2.25) with $\lambda := 1-\alpha \in [0,1]$, we have

$$|a_3 - a_2^2| = (1-\alpha)|b_3 - (1-\alpha)b_2| \leq \frac{2}{3}(1-\alpha).$$

Since the equality in (2.25) holds for the function given by (2.19), it follows from (2.22) that the equality in (2.26) holds for the function (2.21).

The inequality (2.26) follows also from the inequality (2.17). Indeed, by using (2.17), (2.4) and since $\mu := 3(1-\alpha)/4 \leq 1$ for $\alpha \in [0,1)$, we have

$$|a_3 - a_2^2| = \tfrac{1}{3}(1-\alpha)\left|c_2 - \tfrac{3}{4}(1-\alpha)c_1^2\right| \leq \tfrac{2}{3}(1-\alpha). \qquad \square$$

Let $\alpha \in [0,1)$ and let f of the form (1.1) belong to the class $\mathcal{R}(\alpha)$. Then the function p given by (2.3) is in $\mathcal{P}$. Thus by (2.4) and (1.5) we have the following well known result.

Theorem 2.7. *Let $\alpha \in [0,1)$. If $f \in \mathcal{R}(\alpha)$ is the form* (1.1)*, then*

$$|a_n| \leq \frac{2(1-\alpha)}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{2.27}$$

The inequality (2.27) *is sharp and the equality holds for the function*

$$f(z) := (-1+2\alpha)z - 2(1-\alpha)\log(1-z), \quad z \in \mathbb{D}, \ \log 1 := 0.$$

Since for $f \in \mathcal{A}$,

$$|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|,$$

by using (2.1), (2.20), (2.26) and (2.27) we have

Theorem 2.8. *Let $\alpha \in [0,1)$. If $f \in \mathcal{R}(\alpha)$ is the form* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{1}{540}(1-\alpha)^2(439-160\alpha).$$

Particularly, for $\alpha := 0$ we get Theorem 2.2 of [2].

Corollary 2.9. *If $f \in \mathcal{R}$ is the form* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{439}{540} \approx 0.813.$$

Recent result [17] improves the above one.

Corollary 2.10. *If $f \in \mathcal{R}$ is the form* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{41}{60} \approx 0.683.$$

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OSZACOWANIA PEWNYCH WYZNACZNIKÓW DLA FUNKCJI O OGRANICZONYM OBROCIE RZĘDU ALFA

S t r e s z c z e n i e

W pracy podane są oszacowania pewnych wyznaczników w klasie $\mathcal{R}(\alpha)$, $0 \le \alpha < 1$, funkcji analitycznych f standardowo unormowanych takich, że

$$\mathrm{Re} f'(z) > \alpha, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Słowa kluczowe: funkcje jednolistne, funkcje o ograniczonym obrocie, funkcje o ograniczonym obrocie rzędu alpha, wyznacznik Hankela