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EFFECTIVE CALCULATION OF THE DEGREE OF $*$-COVERING

Summary

Let $f\colon \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n$ be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. Then the mapping $f\colon \mathbb{C}^n \to f(\mathbb{C}^n)$ is a $*$-covering (in the sense of [7]). In this paper we give an effective method of calculating the degree of this covering.

Keywords and phrases: multiplicity of mapping, intersection multiplicity, $*$-covering, local degree, effective formula

1. Introduction

R. Draper in [4] defined the multiplicity for proper intersection of analytic sets. Next, in the case of isolated intersection, R. Achilles, P. Tworzewski, T. Winiarski in [1] generalized this definition to improper intersection case. The above multiplicity leads to the definition of multiplicity $i_0(f)$ of a holomorphic mapping at a zero of this mapping (see [14]).

Let $\Omega \subset \mathbb{C}^n$ be a neighbourhood of the point $0 \in \mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}^m$, where $m \geq n$, be a holomorphic mapping such that 0 is an isolated point of the fiber $f^{-1}(0)$.

S. Spodzieja in [12] (see also [3], [13]) proved the following result which shows the connections between the multiplicity $i_0(f)$ and the degree of a $*$-covering (in the sense of [7]), inducted by f in a neighbourhood of 0.

Theorem 1. [12, Theorem 1.2] *There exists a neighbourhood $U \subset \Omega$ of the point 0 such that $f^{-1}(0) \cap U = \{0\}$, $f|_U : U \to f(U)$ is $*$-covering and*

$$i_0(f) = d(f|_U) \cdot \deg_0 f(U), \tag{1}$$

where $d(f|_U)$ *is the degree of the* $*$*-covering* $f|_U$ *and* $\deg_0 f(U)$ *is the degree of an analytic set* $f(U) \subset \mathbb{C}^m$ *at* $0 \in \mathbb{C}^n$*.*

In the paper we will give an effective method to calculate the degree of the $*$-covering $f\colon \mathbb{C}^n \to f(\mathbb{C}^n)$, provided f is a polynomial mapping. Theorem 1 will be the crucial tool in this calculations.

Let $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n$ be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. Denote by $\mathbb{L}(n, m)$ the set of all linear mappings $\mathbb{C}^n \to \mathbb{C}^m$.

We will effectively specify an open set $\hat{U} \subset \mathbb{L}(m, n)$ such that for any $p \in \hat{U}$ the mapping $\pi \circ (F^*, p)\colon \mathbb{C}^m \to \mathbb{C}^m$ is proper and $i_0(F^*, p) = m_0(\pi \circ (F^*, p))$ for the generic $\pi \in \mathbb{L}(s + n, m)$, where $F^*\colon \mathbb{C}^m \to \mathbb{C}^s$ is polynomial mapping such that $C_0(f(\mathbb{C}^n)) = (F^*)^{-1}(0)$, and $C_0(f(\mathbb{C}^n))$ is the tangent cone to $f(\mathbb{C}^n)$ at 0 in the sense of [15]. Next, fixing $p \in \hat{U}$, we describe a set $\mathcal{U} \subset \mathbb{L}(n, 1)$ of linear functions such that for any $l \in \mathcal{U}$ the function l is injective on the generic fiber of the mapping $p \circ f$. The main result of this paper is the following theorem.

Theorem 2. *Fix any* $p \in \hat{U}$*. There exist effectively computable: linear function* $l \in \mathcal{U}$*, non-zero irreducible polynomial of the form* $P_{p,l}(y, t) = \sum_{j=0}^{k} P_{p,l,j}(y)t^j$ *which vanishes on the image of the mapping* $(H_p, l)\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$*, where* $H_p(x) = (p \circ f)(x) + (x_1^{d^n+1}, \dots, x_n^{d^n+1})$*,* $d = \deg f$ *and non-zero irreducible polynomial* $P_p^*(\pi, N, w, t) = \sum_{i=0}^{l} P_{p,i}^*(\pi, N, w)t^i$ *which vanishes on the image of the mapping* $\Phi_p^*\colon \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m \to \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^{m+1}$ *of the form* $\Phi_p^*(\pi, N, y) = (\pi, N, H_{p,\pi}^*(y), N(y))$*,* $H_{p,\pi}^*(y) = (\pi \circ (F^*, p))(y) + (y_1^{d^{*m}+1}, ..., y_m^{d^{*m}+1})$*,* $d^* = \deg F^*$ *such that*

$$d(f) = \frac{\min\{j \in \{1, \dots, k\} : \mathrm{ord}_y P_{p,l,j} = 0\}}{\min\{i \in \{1, \dots, l\} : \mathrm{ord}_w P_{p,i}^* = 0\}}. \tag{2}$$

In the paper [10], there was presented a similar result concerning the calculation of the polynomial $P_{p,l}(y, t)$, not for fixed mappings p and l but for the linear mapping with variable coefficients. From the computational point of view, increasing the number of variables made the algorithm slower. The characterizations of the sets U and $\mathcal{U}^*$ are intended to make the algorithm faster.

2. The Jelonek set of polynomial mapping

Let X and Y be locally compact topological spaces. The mapping $f : X \to Y$ is said to be *proper* if for any compact set $K \subset Y$ the set $f^{-1}(K)$ is a compact subset of X. We say that the mapping $f : X \to Y$ is *proper at point* $y \in Y$ if there exists an open neighbourhood D of y such that

$$f|_{f^{-1}(D)} : f^{-1}(D) \to D$$

is a proper mapping.

Proposition 3. [6, Remark 5.2] *Let X and Y be locally compact spaces. Then the mapping f is proper if and only if it is proper at every point $y \in Y$.*

The set of all points at which the mapping f is not proper is called the *set of non-properness of mapping f* (or the *Jelonek set* of f) and is denoted by S_f.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. The mapping f is said to be *dominant* if $\overline{f(\mathbb{C}^n)} = \mathbb{C}^n$.

The mapping f is said to be *finite* if it is proper and all its fibers are finite.

The following Proposition is well known.

Proposition 4. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a proper polynomial mapping. Then f is finite and surjective (hence dominant).*

Let us recall, after [6], the effective construction of the Jelonek set.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a dominant mapping such that $f(0) = 0$. Define mapping $\Phi_j : \mathbb{C}^n \to \mathbb{C}^{n+1}$ by

$$\Phi_j(x) = (f(x), x_j),$$

for any $j = 1, ..., n$, where $x = (x_1, ..., x_n)$. Since mapping f is dominant, hence $\overline{\Phi_j(\mathbb{C}^n)}$ is an algebraic set in $\mathbb{C}^{n+1}$ of dimension n, i.e. it is a hypersurface. There exists a polynomial $P_j \in \mathbb{C}[y, t]$, $y = (y_1, ..., y_n) \in \mathbb{C}^n$, $t \in \mathbb{C}$, irreducible and of minimal degree of the form

$$P_j(y, t) = P_{j,0}(y)t^{d_j} + ... + P_{j,d_j}(y), \quad P_{j,0} \neq 0, \quad j = 1, ..., n \tag{3}$$

such that $\overline{\Phi_j(\mathbb{C}^n)} = {P_j}^{-1}(0)$. We have

Lemma 5. [6, Proposition 7]

$$S_f = \{y \in \mathbb{C}^n \colon \prod_{j=1}^{n} P_{j,0}(y) = 0\}.$$

3. Effective Primitive Element Theorem

Now, we will give the effective method of finding the set of linear functions which separate points of fibers of polynomial mappings.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$, $f(0) = 0$, be a proper polynomial mapping (i.e. $S_f = \emptyset$).

Let $l \in \mathbb{L}(n, 1)$ be a linear function of the form $l(x) = a_1x_1 + ... + a_nx_n$, where $x = (x_1, ..., x_n) \in \mathbb{C}^n$, and $a = (a_1, ..., a_n) \in \mathbb{C}^n$. Define mapping

$$F : \mathbb{L}(n, 1) \times \mathbb{C}^n \to \mathbb{L}(n, 1) \times \mathbb{C}^n \times \mathbb{C}$$

by

$$F(l, x) = (l, f(x), l(x)).$$

Obviously the mapping F is proper. Then by the Chevalley Theorem $F(\mathbb{L}(n, 1) \times \mathbb{C}^n)$ is an algebraic irreducible set of dimension $2n$.

Therefore there exists an irreducible polynomial $P \in \mathbb{C}[l, y, t]$, $y \in \mathbb{C}^n$, $t \in \mathbb{C}$ of the form

$$P(l, y, t) = \sum_{j=0}^{d} P_j(l, y) t^j$$

such that $P_d \neq 0$ and $F(\mathbb{L}(n, 1) \times \mathbb{C}^n) = P^{-1}(0)$.

The polynomial P, which vanishes exactly on the image of the polynomial map F, could be effectively computed by means of Gröbner bases.

Denote by Δ the discriminant of P i.e.

$$\Delta(l, y) = \mathrm{Res}_t \left(P, \frac{\partial P}{\partial t} \right)(l, y), \quad \text{for any} \quad (l, y) \in \mathbb{L}(n, 1) \times \mathbb{C}^n.$$

Put

$$\mathcal{W} = \{(l, y) \in \mathbb{L}(n, 1) \times \mathbb{C}^n : \Delta(l, y) = 0\}.$$

and

$$\mathcal{V} = \{l \in \mathbb{L}(n, 1) : \Delta(l, y) = 0 \quad \text{for any} \quad y \in \mathbb{C}^n\}.$$

Put now

$$\mathcal{U} = \mathbb{L}(n, 1) \setminus \mathcal{V}.$$

For any fixed $l \in \mathcal{U}$ denote $\Delta_l(y) = \Delta(l, y)$, and $V_l = \{y \in \mathbb{C}^n : \Delta_l(y) = 0\}$.

We have

Proposition 6. *For any $l \in \mathcal{U}$ there exists $y \in \mathbb{C}^n \setminus V_l$ such that the restriction*

$$l|_{f^{-1}(y)} : f^{-1}(y) \to \mathbb{C}$$

is injective.

Proof. Let us fix arbitrary $l \in \mathcal{U}$. Then there exists $y \in \mathbb{C}^n$ such that $\Delta_{l,y} := \Delta_l(y) \neq 0$. Thus, for this y, the polynomial

$$P_{l,y}(t) = P(l, y, t) \in \mathbb{C}[t]$$

has no double roots. Let $t_1, ..., t_d$ be all roots of $P_{l,y}$. Then

$$P_{l,y}(t) = P_{l,y,0} \cdot \prod_{j=1}^{d} (t - t_j),$$

where $P_{l,y,0} \in \mathbb{C} \setminus \{0\}$ and

$$0 \neq \Delta_{l,y} = \mathrm{Res}(P_{l,y}, P'{}_{l,y}) = P_{l,y,0}^{2d-2} \cdot \prod_{i<j} (t_i - t_j)^2. \tag{4}$$

Since f is proper and polynomial mapping, then by Proposition 4 there exist a finite set $\{x^1, ..., x^k\} \subset \mathbb{C}^n \setminus f^{-1}(V_l)$, $x^i \neq x^j$, such that $f^{-1}(y) = \{x^1, ..., x^k\}$.

Hence, for any $x^i \in f^{-1}(y)$ we have

$$0 = P_l(f(x^i), l(x^i)) = P_l(y, l(x^i)) = P_{l,y}(l(x^i)).$$

Therefore $t_i = l(x^i)$, for $i = 1, ..., k$ (after an eventual permutation). Since all t_j are different, therefore $k = d$, and by (4), we have

$$P^{2d-2}_{l,y,0} \cdot \prod_{i<j} (l(x^i) - l(x^j))^2 \neq 0.$$

This ends the proof. □

As an immediate consequence of Proposition 6, we get

Corrolary 7. *For any $l \in \mathcal{U}$, putting $t = l(x_1, \ldots, x_n)$, we have that t is the primitive element of the extension $\mathbb{C}(f_1, \ldots, f_n) \subset \mathbb{C}(x_1, \ldots, x_n)$ i.e.*

$$\mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(f_1, \ldots, f_n)(t).$$

4. Index of overdetermined mapping

Let us recall (see [7]) the definition of the multiplicity of a holomorphic mapping.

Let M and N be complex manifolds of the same dimension $n > 0$, and let $f : M \to N$ be a holomorphic mapping. Let $a \in M$ be an isolated point of its fiber $f^{-1}(f(a))$. We define the *multiplicity of the mapping f at the point a* by

$$m_a(f) = \sup\{\#(f|_U)^{-1}(y) : y \in D\},$$

where U and D are sufficiently small neighbourhoods of the points a and $f(a)$, respectively.

Let us recall [2] (see also [4]) the definition of the multiplicity of the proper, isolated intersection of the analytic sets.

Let $X_1, \ldots, X_k$ be analytic sets in a domain $D \in \mathbb{C}^n$ of pure dimensions $p_1, \ldots, p_k$, respectively. Assume that

$$0 = \dim \bigcap_{j=1}^{k} X_j = \sum_{j=1}^{k} p_j - (k-1)n,$$

i.e. the intersection $\bigcap_{j=1}^{k} X_j$ is proper and isolated. Denote by

$$\Delta = \{(x^1, \ldots, x^{kn}) \in \mathbb{C}^{kn} : x^1 = ... = x^{kn}\}$$

the diagonal set in $\mathbb{C}^{kn}$.

If $a \in \mathbb{C}^n$ is an isolated point of $\bigcap_{j=1}^{k} X_j$, then $(a)^k := (a, \ldots, a) \in \mathbb{C}^{kn}$ is the isolated point of $(X_1 \times ... \times X_k) \cap \Delta$. Hence the projection

$$\pi_\Delta|_{X_1 \times \ldots \times X_k} : X_1 \times ... \times X_k \to \Delta^\perp \subset \mathbb{C}^{kn}$$

along Δ is an analytic cover in some neighbourhood of $(a)^k$.

The multiplicity

$$\mu_{(a)^k}(\pi_\Delta|_{X_1 \times \ldots \times X_k})$$

of the projection $\pi_\Delta|_{X_1\times\ldots\times X_k}$ at the point $(a)^k \in \mathbb{C}^{kn}$ is called the *intersection multiplicity* or *intersection index* of the sets $X_1,\ldots,X_k$ at a point $a \in \bigcap_{j=1}^k X_j$ and we denoted it by $i(X_1\cdot\ldots\cdot X_k;a)$. For $a \in D\setminus\bigcap_{j=1}^k X_j$ we put $i(X_1\cdot\ldots\cdot X_k;a)=0$.

Recall after [1] some fact concerning the improper intersection of the analytic sets.

Let X be a pure k-dimensional analytic subset of a complex manifold M of dimension m. Let N be a submanifold of M of dimension n such that N intersects X at an isolated point $a \in M$. We denote by $\mathcal{F}_a(X,N)$ the set of all locally analytic subsets V of M satisfying the following conditions:

(i) V has pure dimension $m-k$,

(ii) $N_a \subset V_a$, where N_a, V_a denote the germs at a of N and V, respectively,

(iii) a is an isolated point of $V\cap X$.

We define the *multiplicity of improper isolated intersection the analytic set X with submanifold N at the point a* by

$$\tilde{i}(X\cdot N;a)=\min\{i(X\cdot V;a): V\in\mathcal{F}_a(X,N)\}.$$

By *index of a holomorphic mapping $f\colon D\to\mathbb{C}^m$ at* 0, where $D\subset\mathbb{C}^n$ is a neighbourhood of 0 such that 0 is an isolated zero of f, we mean the multiplicity

$$\tilde{i}(\Gamma_f\cdot(\mathbb{C}^n\times\{0\}),(0,0))$$

of improper intersection of the graph Γ_f (i.e $\Gamma_f=\{(x,f(x)):x\in D\}$) of f and the set $\mathbb{C}^n\times\{0\}\subset\mathbb{C}^n\times\mathbb{C}^m$ at the point $(0,0)\in\mathbb{C}^n\times\mathbb{C}^m$ and denote it by $i_0(f)$.

Let $D\subset\mathbb{C}^n$ be an open neighbourhood of the point $0\in\mathbb{C}^n$ and let $f: D\to\mathbb{C}^m$ be a holomorphic mapping such that $f(0)=0$ and 0 is an isolated point of the set $f^{-1}(0)$.

Theorem 8. [12, Theorem 1.1]*For any $p\in\mathbb{L}(m,n)$ such that the point 0 is an isolated zero of $p\circ f$ we have*

$$i_0(f)\le m_0(p\circ f).$$

Moreover, for the generic $p\in\mathbb{L}(m,n)$, the point 0 is an isolated zero of $p\circ f$ and

$$i_0(f)=m_0(p\circ f). \tag{5}$$

Let $f:\mathbb{C}^n\to\mathbb{C}^m$, $m\ge n$ be a polynomial mapping such that 0 is an isolated point of the set $f^{-1}(0)$ and $f(0)=0$. Then there exists a neighbourhood U of the point 0 such that $f(U)\ni 0$ is an algebraic set in $\mathbb{C}^m$ of the pure dimension n. Let $f|_U\colon U\to f(U)$. Put

$$\tilde{\mathcal{U}}=\{p\in\mathbb{L}(m,n): i_0(f)=m_0(p\circ f)\} \tag{6}$$

and

$$\mathcal{U}'=\{p\in\mathbb{L}(m,n): C_0(f(U))\cap\ker p=\{0\}\}, \tag{7}$$

where $C_0(f(U))$ is a tangent cone of $f(U)$ at the point 0 (in the sense of [15]).

We have the following

Proposition 9. [11, Theorem 2.2]

$$\mathcal{U}' \subset \tilde{\mathcal{U}}. \tag{8}$$

The set $\mathcal{U}'$ can be effectively calculated (see [11] or Proposition 11).

5. Local degree of algebraic set

Let X be a pure s-dimensional analytic set in $\mathbb{C}^n$ and $a \in X$. Denote by $G(n-s,n)$ the set of all $(n-s)$-dimensional linear subspaces of $\mathbb{C}^n$. Let $L \in G(n-s,n)$ be such that a is an isolated point of the set $X \cap (a+L)$. It is well known, that there exists a neighbourhood $U \subset \mathbb{C}^n$ of the point a such that $X \cap U \cap (a+L) = \{a\}$, and such that the projection

$$\pi_L : X \cap U \to U' \subset L^{\perp}$$

along L is a k-sheeted analytic cover, for some $k \in \mathbb{N}$, where $L^{\perp}$ is subspace of $\mathbb{C}^n$ orthogonal to $L \subset \mathbb{C}^n$. This number k is called the *multiplicity of the projection* $\pi_L|X$ at the point a, and is denoted by $\mu_a(\pi_L|X)$.

We put $\mu_a(\pi_L|X) = +\infty$ if there exists $b \in U'$ such that $a \in (\pi_L)^{-1}(b)$ and $\dim(\pi_L)^{-1}(b) > 0$.

We put $\mu_a(\pi_L|X) = 0$ if $a \notin X$.

Denote by $\mathcal{G}(n-s,n)$ the set of all linear subspaces $L \in G(n-s,n)$, $L \ni 0$ such that a is an isolated point of the set $X \cap (a+L)$. Then for every $L \in \mathcal{G}(n-s,n)$ the multiplicity of the projection, $\mu_a(\pi_L|X)$, is finite. The number

$$\min\{\mu_a(\pi_L|X) : L \in \mathcal{G}(n-s,n)\}$$

is said to be the *local degree of* X at the point a and is denoted by $\deg_a(X)$.

Proposition 10. [2, Proposition 11.2] *Let X be a pure s-dimensional analytic set in a neighbourhood of* 0 *in $\mathbb{C}^n$, $0 \in X$, and let $L \in G(n-s,n)$. Then*

$$\mu_0(\pi_L|X) = \deg_0(X) \iff L \cap C_0(X) = \{0\},$$

where $C_0(X)$ is a tangent cone of X at the point 0 (in the sense of [15]).

6. Effective calculations of local degree

Now, we will present the effective procedure of calculation of the local degree of an algebraic set.

Let $F = (f_1, ..., f_n) : \mathbb{C}^m \to \mathbb{C}^n$, be a polynomial mapping such that $F(0) = 0$. Assume that $\dim F^{-1}(0) = k$, $0 < k < m$.

Let $I \subset \mathbb{C}[x_1, \dots, x_m]$ be the ideal generated by F i.e. $I = \langle f_1, \dots, f_n \rangle$ and let $B = \left\{ \tilde{f}_1, \dots, \tilde{f}_s \right\}$ be the standard base of I. Then, by [5, Lemma 5.5.11]

$$\operatorname{in} I = \left\langle \operatorname{in} \tilde{f}_1, \dots, \operatorname{in} \tilde{f}_s \right\rangle,$$

where $\operatorname{in} \tilde{f}_j$ denotes the initial form of $\tilde{f}_j$, for $j = 1, \dots, s$. Put

$$F^* = (\operatorname{in} \tilde{f}_1, \dots, \operatorname{in} \tilde{f}_s) \colon \mathbb{C}^m \to \mathbb{C}^s.$$

Then, by [15, Theorem 10.6] we have

$$C_0(F^{-1}(0)) = (F^*)^{-1}(0),$$

and by [15, Lemma 8.11] we get that

$$\dim (F^*)^{-1}(0) = \dim F^{-1}(0) = k.$$

Define mapping

$$\tilde{F}^* : \mathbb{C}^m \times \mathbb{L}(m,k) \to \mathbb{C}^{s+k} \times \mathbb{L}(m,k)$$

by

$$\tilde{F}^*(x,p) = (F^*(x), p(x), p).$$

Observe that for the generic $p \in \mathbb{L}(m,k)$, denoting $\tilde{F}^*_p(x) = \tilde{F}^*(x,p)$, we have

$$\dim(\tilde{F}^*_p)^{-1}(0) = 0.$$

Hence $s + k \geq m$. Next, define

$$F' : \mathbb{C}^m \times \mathbb{L}(m,k) \times \mathbb{L}(s+k,m) \to \mathbb{C}^m \times \mathbb{L}(m,k) \times \mathbb{L}(s+k,m)$$

by

$$F'(x,p,\pi) = (\pi \circ \tilde{F}^*(x,p), \pi)$$

and mapping

$$\tilde{\Phi}_j \colon \mathbb{C}^m \times \mathbb{L}(m,k) \times \mathbb{L}(s+k,m) \to \mathbb{C}^m \times \mathbb{L}(m,k) \times \mathbb{L}(s+k,m) \times \mathbb{C}$$

by

$$\tilde{\Phi}_j(x,p,\pi) = (F'(x,p,\pi), x_j),$$

for $j = 1, \dots, m$. Since the mapping F' is dominant for the generic (p,π), there exists a polynomial $P^*_j \in \mathbb{C}[y,p,\pi,t]$ of the form

$$P^*_j(y,p,\pi,t) = P^*_{j,0}(p,\pi)t^{d_j} + P^*_{j,1}(y,p,\pi)t^{d_j-1} + \dots + P^*_{j,d_j}(y,p,\pi),$$

such that

$$\overline{\tilde{\Phi}_j(\mathbb{C}^m \times \mathbb{L}(m,k) \times \mathbb{L}(s+k,m))} = (P^*_j)^{-1}(0).$$

Similarly as in Section 3, each of the polynomial P^*_j, $j = 1, \dots, m$ can be effectively calculated.

Let

$$\mathcal{V}^* = \left\{ (p,\pi) \in \mathbb{L}(m,k) \times \mathbb{L}(s+k,m) \colon \prod_{j=1}^{m} P^*_{j,0}(p,\pi) = 0 \right\}.$$

Put $\mathcal{U}^* = (\mathbb{L}(m,k) \times \mathbb{L}(s+k,m)) \setminus \mathcal{V}^*$. Denote

$$\hat{\mathcal{U}} = \{p \in \mathbb{L}(m,k)\colon \prod_{j=1}^{m} P^*_{j,0}(p,\pi) \neq 0 \text{ for some } \pi \in \mathbb{L}(s+k,m)\}.$$

We have the following

Proposition 11. *For any $p \in \hat{\mathcal{U}}$ putting $L = \ker p$ we have*

$$\mu_0(\pi_L | F^{-1}(0)) = \deg_0(F^{-1}(0)).$$

Proof. Let us fix arbitrary $p \in \hat{\mathcal{U}}$. Then there exists $\pi \in \mathbb{L}(s+k,m)$ such that $(p,\pi) \in \mathcal{U}^*$. Hence

$$P^*_{j,0}(p,\pi) \neq 0,$$

for each $j = 1,\dots,m$. Therefore the mapping $\pi \circ (F^*, p)$ is proper. This means that the fiber $(\pi \circ (F^*,p))^{-1}(0)$ is finite. Since

$$(\pi \circ (F^*,p))^{-1}(0) \supset (F^*,p)^{-1}(0),$$

hence the fiber $(F^*,p)^{-1}(0)$ is also finite. But the mapping F^* is homogeneous. Therefore $(F^*,p)^{-1}(0) = \{0\}$. On the other hand we have

$$(F^*,p)^{-1}(0) = (F^*)^{-1}(0) \cap p^{-1}(0) = C_0(F^{-1}(0)) \cap p^{-1}(0).$$

Putting $L = \ker p$ we obtain

$$C_0(F^{-1}(0)) \cap L = \{0\}. \tag{9}$$

This together with the Proposition 10 gives the assertion. □

Let us fix $p \in \hat{\mathcal{U}}$. We will calculate $i_0(F^*,p)$. Define mapping $H^*_{p,\pi}\colon \mathbb{C}^m \to \mathbb{C}^m$ by

$$H^*_{p,\pi}(y) = (\pi \circ (F^*,p))(y) + (y_1^{d^{*m}+1}, \dots, y_m^{d^{*m}+1}),$$

where $d^* = \max\{\deg F_1^*, \dots, \deg F_s^*\}$, $\pi \in \mathbb{L}(s+n,m)$ and $N \in \mathbb{L}(m,1)$. Next define mapping $\Phi^*_p\colon \mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^m \to \mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^{m+1}$ by

$$\Phi^*_p(\pi, N, y) = (\pi, N, H^*_{p,\pi}(y), N(y)).$$

Since the mapping Φ^*_p is proper we can find an irreducible polynomial

$$P^*_p \in \mathbb{C}[\pi, N, w, t]$$

of the form $P^*_p(\pi,N,w,t) = \sum_{i=0}^{l} P^*_{p,i}(\pi,N,w)t^i$ such that $P^*_{p,l} \neq 0$ and

$$\Phi^*_p(\mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^m) = {P^*_p}^{-1}(0).$$

Then by [9, Theorem 7] there exists $r^* \in \mathbb{N}$, $0 \le r^* < l$ such that

$$\operatorname{ord}_w P^*_{p,i} > 0 \text{ for } i = 0,\dots,r^* \text{ and } \operatorname{ord}_w P^*_{p,r^*+1} = 0.$$

Hence and by [10, Theorem 4] we have

Corrolary 12.

$$\deg_0(F^{-1}(0)) = i_0(F^*, p) = \min\{i \in \{1, \dots, l\} : \operatorname{ord}_w P^*_{p,i} = 0\}.$$

7. Proof of the main theorem

Let M and N be arbitrary complex manifolds and $X \subset M$ and $Y \subset N$ be non-empty analytic subsets. Let $f : X \to Y$ be a proper holomorphic mapping such that its fibres are finite.

We say that the mapping f is $*$-*covering* if there exist nowhere dense analytic sets $Z \subset X$ and $\Sigma \subset Y$ such that $X \setminus Z$ and $Y \setminus \Sigma$ are manifolds, $Y \setminus Z$ is connected, $f(X \setminus Z) \subset Y \setminus \Sigma$ and restriction

$$f_{X \setminus f^{-1}(Z)} : X \setminus f^{-1}(Z) \to Y \setminus Z \tag{10}$$

is a finite covering. The degree of the covering (10) is called the *degree* of the $*$-covering and denoted by $d(f)$.

Let $f = (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n$, be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. By Theorem 1 we have that

$$i_0(f) = d(f) \cdot \deg_0 f(\mathbb{C}^n). \tag{11}$$

In this Section we give effective method to calculate the number $d(f)$ in (11).

Let I_f be the ideal of the graph of the mapping f, i.e.

$$I_f = \langle y_1 - f_1, \dots, y_m - f_m \rangle,$$

where $y_1, \dots, y_m$ are new variables. Equip $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ with an eliminating local ordering $<$ for $x_1, \dots, x_n$. Let G be the Gröbner basis of the ideal I_f with respect to this ordering. Such base can always be effectively computed. Let $J_f = I_f \cap \mathbb{C}[y_1, \dots, y_m]$. Then the ideal J_f is generated by a set $G' = G \cap \mathbb{C}[y_1, \dots, y_m]$. Let $G' = \{g_1, \dots, g_u\}$, $u \geq m - n$. Then $J_f = \langle g_1, \dots, g_u \rangle$. Let $G = (g_1, \dots, g_u)\colon \mathbb{C}^m \to \mathbb{C}^u$. Then

$$f(\mathbb{C}^n) = G^{-1}(0).$$

Let $\{h_1, \dots, h_s\}$ be the standard basis of ideal J_f with respect to the degree ordering given by $<$. Then by [5, Lemma 5.5.11]

$$\operatorname{in}(J_f) = \langle \operatorname{in}(h_1), \dots, \operatorname{in}(h_s) \rangle, \quad s \geq m - n,$$

where $\operatorname{in}(h_j)$ denotes the initial form of h_j, $j = 1, \dots, s$. Set

$$F^* = (F^*_1, \dots, F^*_s)\colon \mathbb{C}^m \to \mathbb{C}^s,$$

where $F^*_j = \operatorname{in}(h_j)$, for $j = 1, \dots, s$. By [15, Theorem 10.6] we have

$$(F^*)^{-1}(0) = C_0(f(\mathbb{C}^n)).$$

Repeating argument used in Section 5 we can find open sets $\hat{U} \subset \mathbb{L}(m,n)$ such that for any $p \in \hat{U} \subset \mathbb{L}(m,n)$ we have

$$(F^*)^{-1}(0) \cap \ker p = \{0\}.$$

Let us fix $p \in \hat{U}$. Define mapping $H^*_{p,\pi} \colon \mathbb{C}^m \to \mathbb{C}^m$ by

$$H^*_{p,\pi}(y) = (\pi \circ (F^*, p))(y) + (y_1^{d^{*m}+1}, \ldots, y_m^{d^{*m}+1}),$$

where $d^* = \max\{\deg F_1^*, \ldots, \deg F_s^*\}$, $\pi \in \mathbb{L}(s+n, m)$ and $N \in \mathbb{L}(m,1)$. Next define mapping $\Phi_p^* \colon \mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^m \to \mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^{m+1}$ by

$$\Phi_p^*(\pi, N, y) = (\pi, N, H^*_{p,\pi}(y), N(y)).$$

Since the mapping Φ_p^* is proper we can find an irreducible polynomial

$$P_p^* \in \mathbb{C}[\pi, N, w, t]$$

of the form $P_p^*(\pi, N, w, t) = \sum_{i=0}^{l} P^*_{p,i}(\pi, N, w)t^i$ such that $P^*_{p,l} \neq 0$ and

$$\Phi_p^*(\mathbb{L}(s+n,m) \times \mathbb{L}(m,1) \times \mathbb{C}^m) = P_p^{*-1}(0).$$

Again by [9, Theorem 7] there exists $r^* \in \mathbb{N}$, $0 \leq r^* < l$ such that

$$\operatorname{ord}_w P^*_{p,i} > 0 \text{ for } i = 0, \ldots, r^* \text{ and } \operatorname{ord}_w P^*_{p,r^*+1} = 0.$$

and by [10, Theorem 4]

$$\deg_0 f(\mathbb{C}^n) = \deg_0(F^{-1}(0)) = \min\{i \in \{1, \ldots, l\} : \operatorname{ord}_w P^*_{p,i} = 0\}.$$

This gives the effective calculation of the number $\deg_0 f(\mathbb{C}^n)$.

On the other hand, by the choice of $p \in \hat{U}$, and by Proposition 9, $i_0(f) = m_0(p \circ f)$. Then the mapping $H_p \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$H_p(x) = (p \circ f)(x) + (x_1^{d^n+1}, \ldots, x_n^{d^n+1}),$$

where $d = \max\{\deg f_1, \ldots, \deg f_m\}$, is proper (i.e. $S_{H_p} = \emptyset$) and $i_0(f) = m_0(H_p)$. Therefore, repeating argument used in Section 3 for mapping H_p, we can find the open set $\mathcal{U} \subset \mathbb{L}(n,1)$ of linear functions which separate the fibers of the mapping H_p. Let us fix such $l \in \mathcal{U}$ and define

$$\Phi_{p,l} \colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}$$

by

$$\Phi_{p,l}(x) = (H_p(x), l(x)).$$

The mapping $\Phi_{p,l}$ is proper and consequently $\Phi_{p,l}(\mathbb{C}^n)$ is an algebraic set of pure dimension n. So, there exists an irreducible polynomial $P_{p,l} \in \mathbb{C}[y,t]$, where $y = (y_1, \ldots, y_n)$ and $y_1, \ldots, y_n, t$ are independent variables, of the form

$$P_{p,l}(y,t) = \sum_{j=0}^{k} P_{p,l,j}(y)t^j \tag{12}$$

such that $P_{p,l,k} \neq 0$ and $\Phi_{p,l}(\mathbb{C}^n) = P_{p,l}^{-1}(0)$. Hence, by [9, Theorem 7] we have that there exist $r \in \mathbb{N}$ with $0 \leq r < k$ such that

$$\operatorname{ord}_y P_{p,l,j} > 0 \quad \text{for} \quad j = 0, \ldots, r \quad \text{and} \quad \operatorname{ord}_y P_{p,l,r+1} = 0. \tag{13}$$

Hence and by [10, Theorem 4] we have

$$i_0(f) = m_0(H_p) = \min\{j \in \{1, \dots, k\} : \mathrm{ord}_y P_{p,l,j} = 0\}.$$

The above proposition gives an effective calculation method of the number $i_0(f)$.

In summary, we can find effectively $i_0(f)$ and $\deg_0 f(\mathbb{C}^n)$ in (11). Hence the third number in (11) can be effectively determined. The above considerations and Theorem 1 ends the proof of Theorem 2.

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EFEKTYWNE WYLICZANIE STOPNIA ∗-NAKRYCIA

S t r e s z c z e n i e

Niech $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n$ będzie odwzorowaniem wielomianowym takim, że $f(0) = 0$ oraz 0 jest punktem izolowanym zbioru $f^{-1}(0)$. Wówczas odwzorowanie $f|_D \colon D \to f(D)$ jest $*$-nakryciem (w rozumieniu definicji [7]) w pewnym otoczeniu D punktu $0 \in \mathbb{C}^n$.
W pracy podajemy efektywną metodę wyliczania stopnia tego nakrycia.

Słowa kluczowe: krotność odwzorowania, krotność przecięcia, ∗-nakrycie, efektywny wzór