DOI: 10.26485/0459-6854/2017/67.2/3 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017 Vol. LXVII

Recherches sur les déformations no. 2

pp. 31–33

Andrei V. Pokrovskii

INSPIRATION COMING FROM A PAPER OF A. BOUTET DE MONVEL, J. JANAS, AND S. NABOKO

Summary

We present our talk at the round table discussion at the Hypercomplex Seminar 2016 (Będlewo, Poland). The note describes an inspiration related with elementary models of unbounded Jacobi martices with with a few bounded gaps.

Keywords and phrases: Jacobi matrix, gap of a matrix

The main result of the paper under discussion [1] is a constructive example of an unbounded Jacobi operator J acting in the space $l_0^2(\mathbb{N})$ of sequences with a finite number of nonzero terms, such that J admits a unique self-adjoint extension in $l^2(\mathbb{N})$ and its essential spectrum $\sigma_{\mathrm{ess}}(J)$ has an arbitrary finite number of gaps. Moreover, the authors constructed such an example in a very special class of Jacobi operators, which are compact perturbations of operators given by block matrices with increasing sizes of the blocks such that the entries of these matrices satisfy some periodicity condition. This is a remarkable and important feature of this construction because such Jacobi operators are of theoretical and applied interest and their block matrices generating the essential spectrum are defined actually only by a finite number of parameters, which make the construction transparent as well as nontrivial (the authors notice that in the class of arbitrary Jacobi operators, for any closed set X in $\mathbb{R}$, one can trivially construct an unbounded operator whose essential spectrum coincides with X).

I think that it makes sense to look at this result from the point of view of approximation theory. I mean the following problem posed by Stechkin [2], which is popular among mathematicians from the former Soviet Union [3]. Let A be an unbounded linear operator in a Banach space H with domain of definition $D(A) \subset H$ and let $K \subset D(A)$. For $N > 0$ denote by $L(N)$ the set of all bounded linear operators $B : H \to H$ with $\|B\| \le N$, where $\|B\| := \sup_{x \in H \setminus \{0\}} \|Bx\|_H / \|x\|_H$, and define

$$E_N(A, K) := \inf_{B \in L(N)} \sup_{x \in K} \|Ax - Bx\|_H.$$

The problem is to investigate the behaviour of the value $E_N(A, K)$ as $N \to \infty$. It is clear that this problem makes sense if $E_N(A, K)$ is finite for some $N > 0$. Stechkin established some general results concerning the estimation of $E_N(A, K)$ and found the values $E_N(A, K)$ for the case in which A is the differentiation operator on I, where I is the real axis or the positive semiaxis, H is the space of all bounded real-valued continuous functions on I with uniform norm, and K is the class of all functions differentiable $r - 1$ times on I whose derivatives of order $r - 1$ satisfy the Lipschitz condition with constant 1 on I, $r = 2, 3, \ldots$. In particular, it follows from Stechkin's results that in this case $E_N(A, K) \to 0$ as $N \to \infty$, i.e., the differentiation operator on I is approximated by bounded linear operators.

Recently, Babenko and Bilichenko [4] considered the problem on the approximation of an unbounded self-adjoint operator A in a Hilbert space H by bounded operators on the set $K_r = \{x \in D(A^r) : \|A^r x\|_H \le 1\}$, where $r \in \{2, 3, \ldots\}$. They proved, among other things, that

$$E_N(A, K_r) \le \frac{1}{N^{r-1} r} \left(1 - \frac{1}{r}\right)^{r-1},$$

and this estimation is sharp if

$$(E_s - E_t) D(A^{2r}) \ne \{0\}, \quad \forall s, t : 0 \le s < t \le +\infty,$$

where $\{E_t : -\infty \le t \le +\infty\}$ is the partition of unit from the spectral theorem for the operator A. It is not difficult to see that the last condition fails for the Jacobi operator $A = J$ constructed in [1] ($H = l^2(\mathbb{N})$) whence the question on the behaviour of the value $E_N(J, K_r)$ as $N \to \infty$ is arising. In particular, is it true that $E_N(J, K_r) = o(N^{1-r})$ as $N \to \infty$? Babenko and Bilichenko also found extremal approximating operators realizing $E_N(A, K_r)$ which are not Jacobi operators in the case $H = l^2(\mathbb{N})$ and $A = J$. Hence it is natural to modify the definition of $E_N(J, K_r)$ by taking into consideration only approximations of J by bounded Jacobi operators whose norms do not exceed N.

Acknowledgement

The author was supported by the Polish-Ukrainain grant No. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and the National Academy of Sciences of Ukraine.

References

[1] A. Boutet de Monvel, J. Janas, and S. Naboko, *Elementary models of unbounded Jacobi matrices with a few bounded gaps*, Operators and Matrices **6** (2012), 543–565.

[2] S. B. Stechkin, *Best approximation of linear operators*, Math. Notes **1** (1967), 91–99.

[3] V. V. Arestov, *Approximation of unbounded operators by bounded operators and related extremal problems*, Russian Math. Surveys **51** (1996), 1093–1126.

[4] V. F. Babenko and R. O. Bilichenko, *Approximation of unbounded operators by bounded operators in a Hilbert space*, Ukr. Math. J. **61** (2009), 179–187.

Institute of Mathematics
National Academy of Sciences of Ukraine
Tereshchenkivska str. 3, UA-01004 Kyiv
Ukraine
E-mail: pokrovsk@imath.kiev.ua

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 3, 2016.

INSPIRACJA BADAWCZA WYNIKAJĄCA Z NIEDAWNEJ PRACY A. BOUTET DE MONVEL, J. JANASA I A. NABOKO

S t r e s z c z e n i e

Niniejsza nota opisuje inspirację badawczą związaną z elementarnymi modelami nieograniczonych macierzy Jacobiego z nielicznymi ograniczonymi lukami.

Słowa kluczowe: macierz Jacobiego, luka macierzy