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Dedicated to the memory of
Professor Jan Kubarski (1950–2013)

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ON A STRUCTURE OF THE LIE ALGEBROID OF A VECTOR BUNDLE

Summary

The aim of the article is a construction of the Lie algebroid of a vector bundle without referring to geometrical interpretations which come from jet bundles or the Lie groupoid of linear isomorphisms between fibres of the given vector bundle.

Keywords and phrases: Lie algebroid, Lie algebroid of a vector bundle

1. Introduction

By a *Lie algebroid* we mean a vector bundle A over a manifold M with a homomorphism of vector bundles $\varrho_A : A \to TM$ called an *anchor*, and a real Lie algebra structure $(\Gamma(A), [\![\cdot,\cdot]\!])$ on the space $\Gamma(A)$ of the smooth global sections of A, such that $[\![a, f \cdot b]\!] = f \cdot [\![a,b]\!] + (\varrho_A \circ a)(f) \cdot b$ for all $a, b \in \Gamma(A)$, $f \in C^\infty(M)$. The Lie algebroid A is said to be *transitive* if ϱ_A is surjective. Lie algebroids were introduced by Jean Pradines in [8] as infinitesimal objects of Lie groupoids.

Now we briefly recall some approaches to the Lie algebroid of a vector bundle. To each Lie groupoid Φ on a manifold M with the source α, the vector bundle $A(\Phi)$ (called the Lie algebroid of Φ) of all α-vertical vectors on Φ tangent at units of Φ is associated [8], [9]. In the space of sections of $A(\Phi)$ we have a natural structure of a real Lie algebra. The construction of the vector bundle $A(\Phi)$ was based on some generalization of the fundamental relations between the Lie groupoid $\Pi^k M$ of all invertible k-jets of M and the vector bundle J^kTM of all k-jets of the tangent bundle TM (cf. [4]). The functor $\Phi \longmapsto A(\Phi)$ is called *the Lie functor* for Lie groupoids. Let E be any vector bundle and $GL(E)$ the Lie groupoid of all linear isomorphisms between fibres of E. The Lie algebra of sections of the Lie algebroid $A(GL(E))$ has an interesting interpretation discovered by Ngô Van Quê [7] which, in the language of exponential mappings for Lie groupoids (defined by Kumpera [3] and, next, thoroughly examined by Kubarski [1]) looks as follows: Let $\theta(E)$ be the space of all linear differential operators $L : \Gamma(E) \to \Gamma(E)$ satisfying the following condition:

- ▶ there exists a vector field X on M such that, for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$, we have $L(f \cdot \nu) = f \cdot L(\nu) + X(f) \cdot \nu$.

The rank of this operator is at most 1 and $\sigma(L) = X \otimes \mathrm{Id} \in \Gamma(TM \otimes E^* \otimes E)$ is its symbol. It is called, by Mackenzie [5], [6], a *covariant differential operator*. The space of all of them forms an $\mathbb{R}$-Lie algebra with respect to the natural commutator of differential operators. For $\Theta \in \Gamma(A(GL(E)))$, the formula

$$L(\Theta)(\nu)(x) = \left.\frac{d}{dt}\right|_0 [(\mathrm{Exp}\, t\Theta)(x)]^{-1} \cdot [\nu \circ \exp(tX)(x)]$$

determines a covariant differential operator, and the mapping

$$L : \Gamma(A(GL(E))) \longrightarrow \theta(E), \qquad \Theta \longmapsto L(\Theta)$$

is a $C^\infty(M)$-linear isomorphism of real Lie algebras. Following the above interpretations, Mackenzie gives an equivalent definition of the Lie algebroid $A(GL(E))$ as a subbundle $\mathrm{CDO}(E) \subset \mathrm{Hom}(J^1E, E)$ of the vector bundle of linear homomorphisms from the bundle of 1-jets of E to E containing elements $d \in \mathrm{Hom}(J^1E, E)_x$, such that the value $\sigma(d)$ of the symbol map $\sigma : \mathrm{Hom}(J^1E, E) \to \mathrm{Hom}(T^*M, \mathrm{End}(E)) = TM \otimes \mathrm{End}(E)$ is equal to $u \otimes \mathrm{Id}$ for some vector $u \in T_xM$ (see [7] for the symbol map). Therefore, $\mathrm{CDO}(E) = \sigma^{-1}[TM]$ with TM treated naturally is a subbundle of $\mathrm{Hom}(T^*M, \mathrm{End}(E))$. From the above interpretation one can notice that the fibre $A(GL(E))_x$ over $x \in M$ may be identified with the space of linear homomorphisms

$$l : \Gamma(E) \longrightarrow E_x$$

for which there exists a vector $u \in T_xM$ such that

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu_x \tag{1}$$

for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$. Following Kubarski, we will denote this space by $A(E)_x$, the total space by $A(E)$, and called the Lie algebroid $A(GL(E))$ by the *Lie algebroid of the vector bundle E*. Kubarski using Lie groupoids approach examining the representations of Lie algebroids actually found local trivializations of $A(GL(E))$ (cf. [2]).

The aim of this paper is to construct the vector bundle $A(E)$ on the basis of the local trivializations only and define the additional structure on it, giving the Lie algebroid structure in the sense of Pradines.

2. The structure of the Lie algebroid of a vector bundle

Let E be any real vector bundle of rank r over a base manifold M of dimension m with a vector space V as the typical fibre and the bundle projection $p : E \to M$. For $x \in M$, let $A(E)_x$ be the space of all linear homomorphism $l : \Gamma(E) \to E_x$ such that there exists a vector $u \in T_xM$ with the property (1). Notice that the vector $u \in T_xM$ is uniquely determined by $l \in A(E)_x$.

Define the disjoint union

$$A(E) = \bigsqcup_{x \in M} A(E)_x$$

and

$$\pi : A(E) \longrightarrow M$$

in such a way that $\pi(l) = x$ if and only if $l \in A(E)_x$.

Let $\psi : U \times V \to p^{-1}[U]$ be a local trivialization of the bundle E. For $\nu \in \Gamma(E)$, let ν_ψ denote the function $U \ni x \mapsto \psi_x^{-1}(\nu_x) \in V$. Let $x \in M$. Notice that if sections $\nu, \eta \in \Gamma(E)$ are equal on some open set $O \subset M$, and $x \in O$, then for any $l \in A(E)_x$, we have the equality $l(\nu) = l(\eta)$. For each $u \in T_xU$ and $a \in \mathrm{End}(V)$, define the function

$$\overline{\psi}(u, a) : \Gamma(E) \longrightarrow E_x \quad \text{by} \quad \overline{\psi}(u, a)(\nu) = \psi_x(u(\nu_\psi) + (a \circ \nu_\psi)(x)). \tag{2}$$

One can check that $\overline{\psi}(u, a) \in A(E)_x$. We thus obtain a well-defined map

$$\overline{\psi} : TU \times \mathrm{End}(V) \longrightarrow A(E)_U \quad \text{where} \quad A(E)_U := \pi^{-1}[U].$$

Using the identification $TU \cong U \times \mathbb{R}^m$, we will show below that $\overline{\psi}$ is a local trivialization of $A(E)$.

It is showed in [2] that for any $x \in M$ the mapping $\overline{\psi}_x : T_xU \times \mathrm{End}(V) \to A(E)_x$ given by $\overline{\psi}_x(u, a) = \overline{\psi}(u, a)$ for $u \in T_xU$, $a \in \mathrm{End}(V)$, is a linear isomorphism. However, for completeness, we recall the arguments here. Let $(u, a) \in T_xU \times \mathrm{End}(V)$ and $\overline{\psi}_x(u, a) = 0$. Then

$$0 = \overline{\psi}_x(u, a)(f \cdot \nu) = f(x) \cdot \overline{\psi}_x(u, a)(\nu) + u(f) \cdot \nu_x = u(f) \cdot \nu_x$$

for any $f \in C^\infty(M)$ and $\nu \in \Gamma(E)$. If we take $\nu \in \Gamma(E)$ such that $\nu_x \neq 0$, then we deduce immediately that u is the zero tangent vector. Consider the family $\{\xi^\omega \in \Gamma(E) \mid \xi^\omega(x) = \psi_x(\omega)\}_{\omega \in V}$ of global sections of E. On account of (2), we have $a(\omega) = \psi_x^{-1}(\overline{\psi}(0, a)(\xi^\omega))$ for each $\omega \in V$. Hence $a = 0$. Therefore the map $\overline{\psi}_x$ is a monomorphism.

Now take $l \in A(E)_x$, and let $u \in T_xM$ be the tangent vector which satisfies condition (1) for l. The element $\psi_x^{-1}(l(\sigma)) - u(\sigma_\psi)$ of V depends only on the value of a section $\sigma \in \Gamma(E)$ at x. In fact, let $\varepsilon \in \Gamma(E)$ and $\varepsilon_x = \sigma_x$. There exist functions $f^1, \dots, f^k \in C^\infty(M)$ and sections $\eta_1, \dots \eta_k \in \Gamma(E)$ such that $f^j(x) = 0$ for any j and

$$(\varepsilon - \sigma)\big|_O = \left(\sum_{j=1}^{k} f^j \cdot \eta_j\right)\Bigg|_O$$

for a certain neighbourhood $O \subset M$ of x. We thus obtain

$$\psi_x^{-1}(l(\varepsilon)) - u(\varepsilon_\psi) = \psi_x^{-1}(l(\sigma)) - u(\sigma_\psi).$$

For $\omega \in V$, we will denote by $a(\omega)$ the element of the form $\psi_x^{-1}(l(\tau)) - u(\tau_\psi)$ where $\tau \in \Gamma(E)$ is an arbitrary taken section such that $\tau(x) = \psi_x(\omega)$. Now define a mapping $a : V \to V$ by $\omega \mapsto a(\omega)$. Clearly, a is linear. According to the obvious equality $\nu_x = \psi_x(\nu_\psi(x))$, we see that

$$\overline{\psi}_x(u, a)(\nu) = \psi_x(u(\nu_\psi) + \psi_x^{-1}(l(\nu)) - u(\nu_\psi)) = l(\nu).$$

We have thus proved that the map $\overline{\psi}_x$ is an epimorphism.

Theorem 2.1. *Let $\psi : U_1 \times V \to E_{U_1}$, $\varphi : U_2 \times V \to E_{U_2}$ be local trivializations of E with $U_1 \cap U_2 \neq \varnothing$, and let $\overline{\psi} : TU_1 \times \mathrm{End}(V) \to A(E)_{U_1}$, $\overline{\varphi} : TU_2 \times \mathrm{End}(V) \to A(E)_{U_2}$ be determined by ψ and φ, respectively, via* (2)*. Moreover, let $\lambda : U_1 \cap U_2 \to GL(V)$ be the mapping given by $y \mapsto \psi_y^{-1} \circ \varphi_y$. Then*

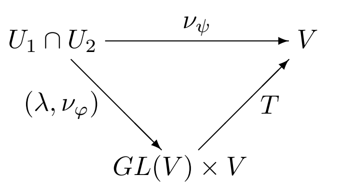
$$\left(\overline{\varphi}_x^{\,-1} \circ \overline{\psi}_x\right)(u, a) = \left(u, \lambda(x)^{-1} \circ (\lambda_{*x}(u) + a \circ \lambda(x))\right) \tag{3}$$

for $u \in T_xU$, $a \in \mathrm{End}(V)$.

Proof. Let $x \in U_1 \cap U_2$, $\nu \in \Gamma(E)$, $u \in T_xM$, $a \in \mathrm{End}(V)$. By definition, we can write

$$\nu_\psi(x) = \psi_x^{-1}(\nu_x) = \left(\psi_x^{-1} \circ \varphi_x\right)\left(\varphi_x^{-1}(\nu_x)\right) = \lambda(x)(\nu_\varphi(x)).$$

Let $T : GL(V) \times V \to V$ be the canonical left action of the Lie group $GL(V)$ on a vector space V, defined by $(L, z) \mapsto L(z)$. One can see that the diagram



commutes. Clearly, $T(\cdot, z) : GL(V) \to V$ is a restriction of the linear mapping $\widetilde{T}(\cdot, z) : \mathrm{End}(V) \to V$, $\widetilde{T}(\cdot, z)(L) = L(z)$. Since the derivative of a linear mapping between finite dimensional vector spaces at a given point can be treated as the same mapping, we get

$$\begin{aligned} u(\nu_\psi) &= (\nu_\psi)_x(u) = (T \circ (\lambda, \nu_\varphi))_x\,(u) \\ &= T(\cdot, \nu_\varphi(x))_{*\lambda(x)}(\lambda_{*x}(u)) + T(\lambda(x), \cdot)_{*\nu_\varphi(x)}(u(\nu_\varphi)) \\ &= \widetilde{T}(\lambda_{*x}(u), \nu_\varphi(x)) + \widetilde{T}(\lambda(x), u(\nu_\varphi)) \\ &= \lambda_{*x}(u)(\nu_\varphi(x)) + \lambda(x)(u(\nu_\varphi)). \end{aligned}$$

Consequently,

$$\begin{aligned} \overline{\psi}(u,a)(\nu) &= \psi_x(u(\nu_\psi) + (a \circ \nu_\psi)(x)) \\ &= \psi_x(\lambda_{*x}(u)(\nu_\varphi(x)) + \lambda(x)(u(\nu_\varphi)) + (a \circ \nu_\psi)(x))) \\ &= \psi_x\left(\lambda(x)(u(\nu_\varphi)) + (\lambda_{*x}(u) + a \circ \lambda(x))\,(\nu_\varphi(x))\right) \\ &= \varphi_x(u(\nu_\varphi)) + \varphi_x\big(\big(\lambda(x)^{-1} \circ (\lambda_{*x}(u) + a \circ \lambda(x))\big)(\nu_\varphi(x))\big) \\ &= \varphi_x\big(u(\nu_\varphi) + \big(\lambda(x)^{-1} \circ (\lambda_{*x}(u) + a \circ \lambda(x))\big)(\nu_\varphi(x))\big)\,. \end{aligned}$$

We have thus proved (3). □

Let $\psi : U_1 \times V \to E_{U_1}$, $\varphi : U_2 \times V \to E_{U_2}$ be local trivializations of the bundle E with $U_1 \cap U_2 \neq \varnothing$. From (3) in Theorem 2.1 we conclude that the map

$$\overline{\varphi}^{-1} \circ \overline{\psi} : T(U_1 \cap U_2) \times \operatorname{End}(V) \longrightarrow T(U_1 \cap U_2) \times \operatorname{End}(V)$$

is smooth. Besides, from the theorem on the construction of a vector bundle, we have that $A(E)$ is a vector bundle with the typical fibre $\mathbb{R}^m \oplus \operatorname{End}(V)$ and the projection π. Moreover, $\overline{\psi} : TU \times \operatorname{End}(V) \to A(E)_U$ is an isomorphism of vector bundles.

It remains to introduce in $A(E)$ the structure of a Lie algebroid.

Let $l \in A(E)$. There exists a point $x \in M$ such that a linear homomorphism $l : \Gamma(E) \to E_x$ is an element of the vector space $A(E)_x$. Denote by u^l (determined by l uniquely) a tangent vector to the manifold M at x, satisfying (1). We define a mapping

$$\varrho : A(E) \longrightarrow TM, \quad l \longmapsto u^l. \tag{4}$$

Certainly, the maps $\varrho_x : A(E)_x \to T_xM$ $(x \in M)$ are linear. Moreover, since the diagrams

$$\begin{array}{ccc} A(E) & \xrightarrow{\ \varrho\ } & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ M & \xrightarrow[\ \mathrm{id}_M\ ]{} & M \end{array} \qquad\qquad \begin{array}{ccc} TU \times \operatorname{End}(V) & \xrightarrow{\ \overline{\psi}\ } & A(E)_U \\ {\scriptstyle\mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle\varrho|_{A(E)_U}} \\ TU & \xrightarrow{\ \iota\ } & TM \end{array}$$

commute, ϱ is an epimorphism of vector bundles.

Lemma 2.2. *Let $\mathcal{L} : M \to A(E)$ be a function such that $\pi \circ \mathcal{L} = \mathrm{id}_M$. If, for each $\nu \in \Gamma(E)$, the mapping $\zeta_\nu : M \to E$, $\zeta_\nu(x) = \mathcal{L}_x(\nu)$, is a smooth section of the vector bundle E, then $\mathcal{L}$ is a smooth section of the vector bundle $A(E)$.*

Proof. The problem is local. For this reason, we need only to consider a trivial vector bundle $E = M \times V$. Then $TM \times \mathrm{End}(V) \cong A(E)$ via the isomorphism $\overline{\psi}$ defined for the identical trivialization $\psi = \mathrm{id}_{M\times V}$. Let $\nu \in \Gamma(M \times V)$, $x \in M$, and take $\widetilde{\nu} = \mathrm{pr}_2 \circ \nu \in C^\infty(M; V)$. There exist functions $X : M \to TM$ and $\sigma : M \to \mathrm{End}(V)$ (whose smoothness is to be proved) for which

$$\begin{aligned}\zeta_\nu(x) &= \mathcal{L}_x(\nu)\\ &= \overline{\mathrm{id}_{M\times V}}(X_x, \sigma(x))\\ &= \big(x, X_x(\nu_{\mathrm{id}_{M\times V}}) + \sigma(x)(\nu_{\mathrm{id}_{M\times V}}(x))\big)\\ &= (x, X_x(\widetilde{\nu}) + \sigma(x)(\widetilde{\nu}(x)))\\ &= \Big(\mathrm{id}_M, X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu})\Big)(x).\end{aligned}$$

Hence the map $X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu})$ is smooth for arbitrary taken $\nu \in \Gamma(M \times V)$.

Let $(e_j)_{j=1}^r$ be a base of the linear space V and let $\nu_j \in \Gamma(M \times V)$ be constant sections of the vector bundle $M \times V$, defined by $\nu_j(x) = (x, e_j)$. Since there exist functions $\beta_j^i \in C^\infty(M)$ such that, for $x \in M$, we have

$$\Big(X(\widetilde{\nu}_j) + \widetilde{T} \circ (\sigma, \widetilde{\nu}_j)\Big)(x) = \sigma(x)(e_j) = \sum_{i=1}^r \beta_j^i(x) e_i,$$

it follows that

$$\sigma(x) = \sum_{i,j=1}^r \beta_j^i(x)\gamma_i^j,$$

where the maps γ_i^j form a base of the space $\mathrm{End}(V)$, determined by $(e_j)_{j=1}^r$ in such a way that $\gamma_i^j(e_k) = \delta_k^j e_i$. Hence it appears that σ is smooth.

Since the mappings $X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu}), \widetilde{T}, \widetilde{\nu}$ and σ are smooth, it follows that $X(\widetilde{\nu}) \in C^\infty(M; V)$. From this we conclude that X is a smooth vector field on M. □

A section $\mathcal{L}$ of the vector bundle $A(E)$ determines a covariant differential operator $L_{\mathcal{L}} : \Gamma(E) \to \Gamma(E)$ by $L_{\mathcal{L}}(\nu)(x) = \mathcal{L}_x(\nu)$ for $x \in M$, $\nu \in \Gamma(E)$. Besides, each covariant differential operator in E is of the form $L_{\mathcal{L}}$ for exactly one section $\mathcal{L}$ of the vector bundle $A(E)$. Indeed, a covariant differential operator L is equal to $L_{\mathcal{L}}$ for $\mathcal{L}_x(\nu) = (L(\nu))(x)$, $x \in M$, $\nu \in \Gamma(E)$. The smoothness of $\mathcal{L}$ now follows from Lemma 2.2.

The Lie bracket $[\![\cdot,\cdot]\!]$ in $\Gamma(A(E))$ is defined in the classical way, like that for differential operators. For $\mathcal{K}, \mathcal{L} \in \Gamma(A(E))$ we define $[\![\mathcal{K}, \mathcal{L}]\!] \in \Gamma(A(E))$ in such a way that $L_{[\![\mathcal{K},\mathcal{L}]\!]} = L_{\mathcal{K}} \circ L_{\mathcal{L}} - L_{\mathcal{L}} \circ L_{\mathcal{K}}$ noticing that the right-hand side of the last formula is a covariant differential operator. This also shows that $\mathrm{Sec}\,\varrho : \Gamma(A(E)) \to \mathfrak{X}(M)$,

$\mathcal{L} \mapsto \varrho \circ \mathcal{L}$, is a homomorphism of Lie algebras. Moreover,

$$[\![\mathcal{K}, f \cdot \mathcal{L}]\!]_x(\nu) = f(x) \cdot [\![\mathcal{K}, \mathcal{L}]\!]_x(\nu) + (\varrho \circ \mathcal{K})_x(f) \cdot \mathcal{L}_x(\nu)$$

for any $\mathcal{K}, \mathcal{L} \in \Gamma(A(E))$, $f \in C^\infty(M)$, $x \in M$, $\nu \in \Gamma(E)$.

Thus we have demonstrated that in the vector bundle $A(E)$ we have the structure of a transitive Lie algebroid with the above given Lie bracket $[\![\cdot, \cdot]\!]$ on the space of global sections of $A(E)$ and the anchor ϱ defined by (4).

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O STRUKTURZE ALGEBROIDU LIEGO WIĄZKI WEKTOROWEJ

S t r e s z c z e n i e

W artykule przedstawiono konstrukcję algebroidu Liego wiązki wektorowej bez odnoszenia się do wiązek dżetów czy grupoidu Liego liniowych izomorfizmów włókien danej wiązki wektorowej.

Słowa kluczowe: algebroid Liego, algebroid Liego wiązki wektorowej