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CLASSES OF CONVEX POLYHEDRA CLOSED UNDER MINKOWSKI ADDITION

Summary

In this paper we study classes of convex polyhedra with normal vectors belonging to a given finite set G of vectors. Since a Minkowski sum $A+B$ of two polyhedra may have normal vectors which are normal to no face of A or B, a given class of polyhedra may be not closed under Minkowski addition. The main result of this paper is a necessary and sufficient condition under which a class of convex polyhedra is closed with respect to Minkowski addition.

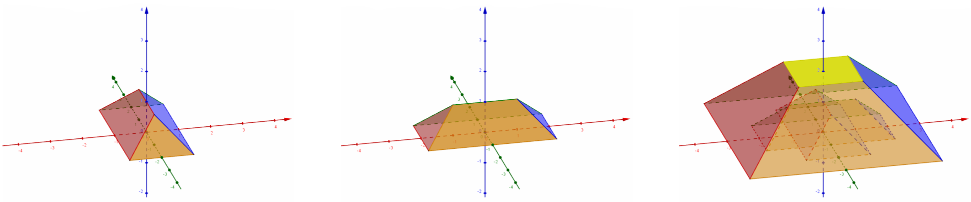
Keywords and phrases: crystal growth, abstract cone of convex polyhedra, Minkowski addition, convex polyhedron

1. Introduction

In nature monocrystals assume polyhedral shape. However, typically this shape has a number of defects. In our ideal model we identify monocrystals with convex polyhedra in three dimensional space. Growing a monocrystal can be identified with multifunction of a real variable and values in the family of all convex polyhedra. The growth of monocrystals can be described using vector addition also called Minkowski addition [5, 6, 16]. As we know the sum of two convex polyhedra is a convex polyhedron. However, crystal structure or the ordered arrangement of atoms limits the number of possible vectors normal to the faces of a given growing crystal. Therefore, it is profitable to restrict considerations to a smaller class of convex polyhedra with finite number of possible normal vectors.

The problem is that the sum of two convex polyhedra from such smaller class may not belong to this class. It can happen because the sum of two convex polyhedra with the same of normal vectors may have larger set of normal vectors. We show a simple example in Figure 1.

The polyhedron A is an elongated pyramid and polyhedron B is the same elongated pyramid rotated around vertical axis. On Figure 1 we can see two polyhedra and their Minkowski sum. Let first polyhedron be A, second be B and their Minkowski sum be $C = A + B$.



(a) Polyhedron A. (b) Polyhedron B. (c) Polyhedron $C = A + B$.

Fig. 1: Two polyhedra and their Minkowski sum.

Let us see that the set of all normal vectors to the faces of the polyhedron A is exactly the same set of all five normal vectors to the faces of the polyhedron B. This set is by necessity contained in the set of all normal vectors to the polyhedron C. However, the polyhedron C has an additional sixth face on top of it. Hence the class of all polyhedra with normal vectors contained in the set of five normal vectors to the faces of the polyhedron A is not closed under Minkowski addition.

This paper is dedicated to finding all finite sets of vectors belonging to the unit sphere, which define classes of convex polyhedra which are closed with respect to Minkowski addition. Further we show that such finite sets of vectors are very special but we should not neglect them, because the growth of crystals belonging to such classes can be easily described with the help of Minkowski addition.

2. Problem description

Let $X = (X, \tau)$ be a topological vector space. Let $\mathcal{B}(X)$ be the family of all nonempty bounded closed convex subsets of X and $\mathcal{K}(X)$ be the family of all compact sets from $\mathcal{B}(X)$. For $A, B \subset X$ we have $A + B = \{a + b \mid a \in A, b \in B\}$, which is called *Minkowski sum* of the sets A and B. Let $A \dot{+} B = \mathrm{cl}(A + B)$ and $\alpha A = \{\alpha a \mid a \in A, \alpha \in \mathbb{R}_+\}$. The triples $(\mathcal{K}(X), +, \cdot)$ and $(\mathcal{B}(X), \dot{+}, \cdot)$ are abstract convex cones studied in a number of papers [7], [10], [12] or [17].

Let $A \subset X$, then *convex hull* of A is a set $\mathrm{conv}A = \{\sum_{i=1}^{n} \alpha_i a_i \mid \alpha_i \in \mathbb{R}_+,\ a_i \in A,\ \sum_{i=1}^{n} \alpha_i = 1,\ n \in \mathbb{N}\}$. For $A \in \mathcal{B}(\mathbb{R}^3)$ we define a support function h_A on $\mathbb{R}^3$ by

$h_A(x) = \max\limits_{a \in X} \langle a, x \rangle$, where $\langle \cdot, \cdot \rangle$ is inner product. The correspondence between family $\mathcal{K}(\mathbb{R}^n)$ and the family of all positively homogeneous and convex functions on $\mathbb{R}^n$ is called *Minkowski duality*. Minkowski duality plays an important role in convex analysis and especially in the study of quasidifferential functions [1] and [2].

We define also a support set $A(z) = \{a \in A \mid \langle a, z \rangle = h_A(z)\}$ of the set A in the direction of a nonzero vector $z \in \mathbb{R}^3$. Let G be a finite set of vectors $\{z_i \in \mathbb{R}^3 \mid i \in \{1, \dots, n\}, \|z_i\| = 1\}$. So we denote

$$\mathcal{B}_G(\mathbb{R}^3) = \{A \in \mathcal{B}(\mathbb{R}^3) \mid A = \{x \in \mathbb{R}^3 \mid \langle x, z_i \rangle \le h_A(z_i), i = 1, \dots, m\}\}$$

and elements of this family are called *G-polyhedra*. In particular, a G-polyhedron can have an empty interior. Then it is a singleton, a segment or a polygon. A convex polyhedron (with nonempty interior) is a G-polyhedron if and only if all vectors normal to its faces belong to G.

A nonempty intersection of finite number of closed half-spaces is called a *general polyhedron*. A bounded general polyhedron is a convex hull of finite number of points. A bounded general polyhedron with nonempty interior is called a *convex polyhedron*.

The family $\mathcal{B}_G(\mathbb{R}^3)$ is nonempty if and only if the origin is an interior point of a convex hull of G. If G is contained in a unit sphere S^2 then G is contained in no hemisphere. Every normal vector of a G-polyhedron A belongs to G. Unfortunately if A, B are G-polyhedra we cannot deduce that the polyhedron $A + B$ is a G-polyhedron. See for example Figure 1. However, every face, edge and vertex of a given polyhedron A is a support set $A(z)$ of A in some direction z. Obviously, a support set of $A + B$ is a Minkowski sum of respective support sets of A and B, namely $(A + B)(z) = A(z) + B(z)$. Therefore, *if a support set $(A + B)(z)$ is a face of $A + B$ i.e. it is a two-dimensional figure then the support set $A(z)$ is a face of A or $B(z)$ is a face of B or both $A(z)$ and $B(z)$ are non-parallel edges of A and B.* If we consider polyhedra with nonempty interior then every edge is an intersection of two faces. The third possibility shows that (in general) a sum of two G-polyhedra is not a G-polyhedron.

Our purpose is to find all sets G such that for any two G-polyhedra A and B if the support set $(A + B)(z)$ is two-dimensional then $z \in G$. In other words, we want to find which classes $\mathcal{B}_G(\mathbb{R}^3)$ are closed under Minkowski addition.

In three-dimensional Euclidean space we consider a unit sphere of radius 1 and center 0. The unit sphere is described by a quadric equation $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Let π be a plane passing through the center 0 of the sphere S^2. The section $\pi \cap S^2$ is called a *great circle*. Two points $x, y \in S^2$ of the sphere are called *antipodal* if $y = -x$. The shortest path $[x, y]_S$ between two non-antipodal points x, y on a sphere is the shorter segment of a great circle passing through x and y and it is called a *great circle segment* joining x and y.

If an edge of a polyhedron is an intersection of two faces $A(x)$ and $A(y)$ then this edge is equal to $A(z)$, where z is any point belonging to the great circle segment $[x, y]_S$ joining x and y. So in the case of a polyhedron with nonempty interior each edge corresponds to a great circle segment joining points corresponding to the faces of a polyhedron containing this edge.

If there exists a face of $(A + B)(z)$ with normal vector z which is not normal to any face of A or B then this face is equal to the sum $A(z) + B(z)$ of two edges where z lays on the intersection of two great circle segments $[x, y]_S$ and $[u, w]_S$ such that $A(x)$ and $A(y)$ are faces of A containing the edge $A(z)$ and $B(u)$ and $B(w)$ are faces of B containing the edge $B(z)$.

The following theorem reduces our problem to a geometric problem for the three-dimensional space $\mathbb{R}^3$.

Theorem 2.1. *The family $\mathcal{B}_G(\mathbb{R}^3)$ of all G-polyhedra is closed with respect to Minkowski addition if and only if every intersection of two great circle segments with endpoints in G belongs to G.*

Proof. Sufficiency follows directly from previous considerations.

Necessity. Let $x, y, u, w \in G$ and $[x, y]_S \cap [u, w]_S = \{z\}$. Let us define $A = \{a \in \mathbb{R}^3 \mid \langle a, v\rangle \leq 1 \text{ for all } v \in G, \langle a, x\rangle \leq \epsilon, \langle a, y\rangle \leq \epsilon\}$. The set A is a polyhedron and if ϵ is sufficiently small then the faces $A(x)$ and $A(y)$ intersect along the edge $A(z)$. Similarly if $B = \{b \in \mathbb{R}^3 \mid \langle b, v\rangle \leq 1 \text{ for all } v \in G, \langle b, u\rangle \leq \epsilon, \langle b, w\rangle \leq \epsilon\}$, then the set B is a polyhedron and if ϵ is sufficiently small then the faces $B(u)$ and $B(w)$ intersect along the edge $B(z)$. The edges $A(z)$ and $B(z)$ are nonparallel, and the face $(A + B)(z)$ is a parallelogram. Since $A + B$ belongs to $\mathcal{B}_G(\mathbb{R}^3)$, the normal vector z to the faces $(A + B)(z)$ belongs to G. $\square$

In section 4 we characterize all finite subsets G of unit sphere such that every point intersection of two great circle segments with endpoints in G belongs to G.

3. Planar sets with internally intersecting skeleton

Before characterizing finite subsets of unit sphere we characterize in this section similar subsets of the plane. Solution of a planar problem will make the spherical one clearer.

Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^2$. The set $A \veebar A = \bigcup_{a,b \in A} [a, b]$, i.e. the union of line segments with endpoints in A is called a *skeleton* of A. We say, that a set A has *internally intersecting skeleton*, when every point intersection of two line segments from the skeleton $A \veebar A$ belongs to the set A. In this section, we aim to answer the following question: What is a necessary and sufficient condition for the set A to have an internally intersecting skeleton?

First, we define a forbidden domain of a triangle. Let a, b, c be vertices of triangle. An open unbounded set $F(c, [a, b]) = \{\gamma(\alpha a + (1 - \alpha)b) + (1 - \gamma)c \mid 0 < \alpha < 1 < \gamma\}$ is called *a forbidden domain with respect to the point c and the segment $[a, b]$* (see FIGURE 2). Vertices and sides of any triangle generate three forbidden domains with respect to a vertex and an opposite side. The union of these three disjoint domains is called *a forbidden domain of a triangle.*

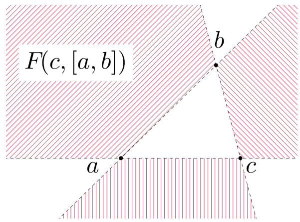


Fig. 2: Forbidden domain of a triangle $\Delta(a, b, c)$.

Now, we formulate a theorem, which presents a solution to the planar problem.

Theorem 3.1. *Let A be a finite subset of $\mathbb{R}^2$. The following conditions are equivalent*:
(i) *The set A has an internally intersecting skeleton.*
(ii) *For any $a, b, c \in A$ if the triangle $\Delta(a, b, c)$ contains no other point of A, then the forbidden domain of this triangle contains no points of A.*

Proof. (ii) $\Rightarrow$ (i) Proof by contradiction. Let us assume that the set A does not have an internally intersecting skeleton. It implies, that there exist four points $a_1, a_2, a_3, a_4 \in A$ such that an intersection of line segments $[a_1, a_2]$ and $[a_3, a_4]$ is a singleton $\{b\}$ and the point b does not belong to A. We can assume, that between a_1 and a_2 there are no other points belonging to the set A. In a similar way we can assume, that there are no points of A between a_3 and a_4.

Let $a_5 \in A$ be a point of the triangle $\Delta(a_2, a_3, a_4)$ other than a_3 and a_4 such that the distance of a_5 from a straight line a_3a_4 is the shortest. Let us notice that the triangle $\Delta(a_5, a_3, a_4)$ does not contain any point of A other than its vertices. It can happen that $a_5 = a_2$. But a_1 belongs to $F(a_2, [a_3, a_4]) \subset F(a_5, [a_3, a_4])$ and $F(a_5, [a_3, a_4])$ is the forbidden domain designated by the triangle $\Delta(a_5, a_3, a_4)$, a contradiction.
(i) $\Rightarrow$ (ii) Proof by contradiction. Let a triangle $\Delta(a_1,\ a_2,\ a_3)$, $a_1, a_2, a_3 \in A$ contain no other point of A and let $a_4 \in A$ lie in the forbidden domain designated by this triangle. Without loss of generality we can assume, that $a_4 \in F(a_3, [a_1, a_2])$ (see FIGURE 4). Then two line segments $[a_1, a_2]$ and $[a_3, a_4]$ intersect in their relative interiors at some point a_5 not belonging to A. Hence the set A does not have an internally intersecting skeleton. $\square$

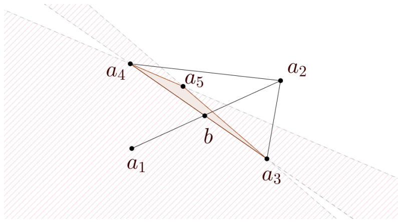


Fig. 3: Point a_1 lies in the forbidden domain $F(a_5, [a_3, a_4])$.

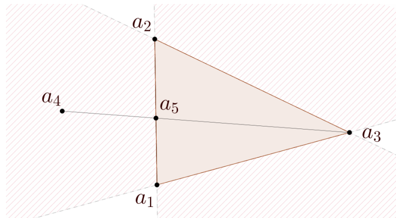


Fig. 4: Point $a_5 \notin A$ of intersection of segments $[a_1, a_2]$ and $[a_3, a_4]$.

The following theorem characterizes more precisely the sets with an internally intersecting skeleton.

Theorem 3.2. *A finite set $A \subset \mathbb{R}^2$ has an internally intersecting skeleton if and only if there exists a straight line l such that one of the following conditions holds true*:
(a) *The set A is contained in l.*
(b) *The set $A \setminus l$ is a singleton.*
(c) *We have $A \setminus l = \{a, a'\}$, points a, a' lie on the opposite sides of the line l and the point of intersection of l and the line aa' belongs to A.*
(d) *We have $A \setminus l = \{a, a'\}$, points a, a' lie on the opposite sides of the line l then the set A lies on one side of the line aa'.*
(e) *We have $A = \{a_1, a_2, \ldots, a_6\}$, $a_2, a_3, a_4 \in l$, $a_3 \in [a_2, a_4]$, $a_1 \in [a_3, a_5]$ and $a_2 \in [a_1, a_6]$.*

The theorem states that a set A has an internally intersecting skeleton if and only if one of the five following cases (see Figure 5) holds true. (a) The set A is a set of collinear points. In particular, a singleton and a pair of points are sets of collinear points. (b) The set A except one point is a set of collinear points. (c) The set A except two points, let us say a and a', is a subset of a straight line l and some point of A lies between a and a'. Obviously, that point lies on the line l. (d) This case is similar to (c). The difference is that no point of A lies between a and a', but all the set A lies on one side of the straight line aa'. (e) The set A has six points $a_1, \ldots, a_6$. Points a_1, a_2 and a_3 are vertices of triangle $\Delta(a_1, a_2, a_3)$ which contains no other point of A. The point a_4 lies on ray a_2a_3, a_5 lies on ray a_3a_1 and a_6 lies on ray a_1a_2.

Proof. ($\Leftarrow$) It is easy to see, that the set described in the theorem has an internally intersecting skeleton.
($\Rightarrow$) We assume that the set A has an internally intersecting skeleton. We may also assume that not all points of A are collinear. Let $a_1, a_2, a_3 \in A$ be vertices of a triangle $\Delta(a_1, a_2, a_3)$ with the minimal surface among all triangles with vertices belonging to

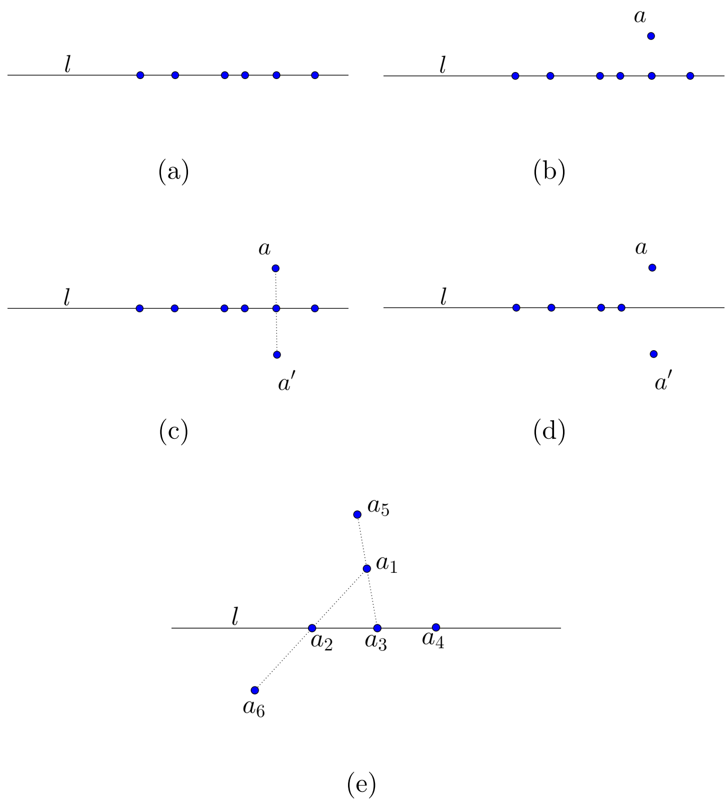
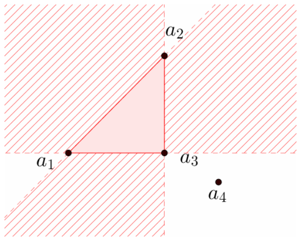


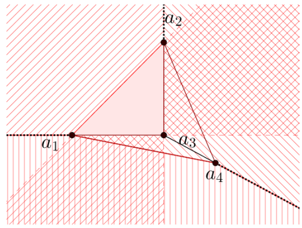
Fig. 5: Five possible structures of a set A.

A. It implies that in the triangle $\Delta(a_1, a_2, a_3)$, inside and on its sides, there are no other points of A. Otherwise, we could choose a triangle with smaller surface. From Theorem 3.1 we know, that no point of A belongs to the forbidden domain of the triangle $\Delta(a_1, a_2, a_3)$.

Let us consider a case when neither straight line a_1a_2 nor a_2a_3, nor a_1a_3 contains another point of A. If the set A contains only a_1, a_2 and a_3 then we obtain the situation from Figure 5b. Otherwise, let $a_4 \in A$ belong to one of vertically opposite angles to the angles of the triangle $\Delta(a_1, a_2, a_3)$ (see Figure 6a). Without loss of generality we can assume, that a_4 lies at the vertically opposite angle to the angle $\measuredangle a_1a_3a_2$ and that no other point of A lying at this angle is closer to the line a_1a_2. Notice, that neither the triangle $\Delta(a_1, a_3, a_4)$ nor $\Delta(a_2, a_3, a_4)$ contains other points of A. Analysing union of all the forbidden domains of the triangles $\Delta(a_1, a_2, a_3)$, $\Delta(a_1, a_3, a_4)$ and $\Delta(a_2, a_3, a_4)$ we see that other points of A may lie only on three rays opposite to the ray a_1a_3, the ray a_2a_3 and the ray a_4a_3 (see FIGURE 6b). However, by the case assumption only the ray opposite to the ray a_4a_3 may contain points from A. This case leads us to the situation from FIGURE 5d.



(a) Point $a_4 \in A$ belongs to the vertically opposite angle to the angle $\measuredangle a_1a_3a_2$.



(b) Three rays (dotted) with initial points a_1, a_2, a_4.

Fig. 6: Neither straight line a_1a_2 nor a_2a_3, nor a_1a_3 contains another point of A.

Let us consider a complementary case when at least one of the lines a_1a_2, a_2a_3 or a_1a_3 contains some other point of A. Without loss of generality we can assume that $a_4 \in$ line a_2a_3. We have two possibilities.

First, the line a_2a_3 contains more than three points of A.
Let $a_2, \ldots, a_n \in A$, $n \geqslant 5$ be all points of A lying on the line a_2a_3. Let $a_k, a_l \in A$ be such that $a_2, \ldots a_n$ lie between a_k and a_l. If $A = \{a_1, a_2, \ldots, a_n\}$ then we obtain a situation from FIGURE 5b. Otherwise, let $a_{n+1} \in A$ be a point, which is not belonging to the line a_2a_3. The point a_{n+1} does not belong to the union of forbidden domains of all triangles $\Delta(a_1, a_i, a_j)$, such that between a_i and a_j there is no points of A. If the point a_{n+1} lies on the line a_1a_i, $i = 2, \ldots, n$, then we obtain the situation from FIGURE 5c. Otherwise, the point a_{n+1} lies at one of vertically opposite angles to the angle $\measuredangle a_1a_la_k$ or $\measuredangle a_1a_ka_l$ (see FIGURE 7). This case leads us to the situation from FIGURE 5d.

Second, the line a_2a_3 contains exactly three points of A.
Without loss of generality we can assume that $a_3 \in [a_2, a_4]$. If the set A contains no other points, then we obtain the situation from FIGURE 5b. Otherwise, we have four possibilities:
(1) The points a_5 and a_6 belong to the rays a_3a_1 and a_1a_2. This case is the situation from FIGURE 5e. (2) The point a_5 belongs to the line a_1a_3 and the point a_6 does not belong to the ray a_1a_2. This case leads us to the situation from FIGURE 5c, where the line a_1a_3 is our line l. Then all other possible points of A belong to l. (3) We have $A = \{a_1, \ldots, a_5\}$ and the point a_5 lies on the line a_1a_2 or a_1a_4. Then we also obtain the situation from FIGURE 5c and the line a_2a_3 is our line l. (4) We have

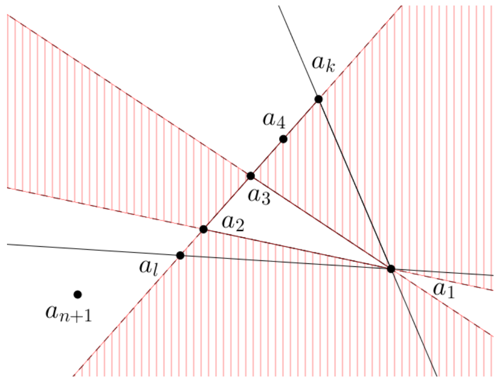


Fig. 7: Point a_{n+1} lies at one of vertically opposite angles to the angle $\measuredangle a_1 a_l a_k$.

$A = \{a_1, \ldots, a_5\}$ and the point a_5 lies at the vertically opposite angle to the angle $\measuredangle a_1 a_2 a_3$ or $\measuredangle a_1 a_4 a_3$. This case leads us to the situation from FIGURE 5d. □

Having solved our planar problem we move to solve a similar spherical problem.

4. Spherical sets with internally intersecting skeleton

In this section our considerations are similar to those concerning the planar case. Let G be a finite subset of a unit sphere $S^2 \subset \mathbb{R}^3$. Let vectors $g_1, g_2, g_3 \in G$ be linearly independent and let $[g_1, g_2]_S, [g_1, g_3]_S, [g_2, g_3]_S$ be great circle segments. The union of these segments is a boundary of exactly two closed and simply-connected subsets of the unit sphere. The smaller of these sets is called a *spherical triangle* $\Delta(g_1, g_2, g_3)_S$. The spherical triangle $\Delta(g_1, g_2, g_3)_S$ is the intersection of three spherical biangles C_1, C_2, C_3. Let a biangle C_1 (see FIGURE 8) have vertices g_1 and $-g_1$, where $-g_1$ is an antipodal point to g_1 and let g_2, g_3 belong to the sides of the biangle C_1. Let biangles C_2, C_3 have respectively vertices g_2, $-g_2$ and g_3, $-g_3$. The interior of a set $(C_1 \cup C_2 \cup C_3) \setminus (C_1 \cap C_2 \cap C_3)$ (see FIGURE 9) is called the *forbidden domain of the triangle* $\Delta(g_1, g_2, g_3)_S$. In fact, the forbidden domain is a union of the interiors of three spherical triangles $\Delta(-g_1, g_2, g_3)_S$, $\Delta(g_1, -g_2, g_3)_S$ and $\Delta(g_1, g_2, -g_3)_S$.

The set $G \veebar G = \bigcup\limits_{\substack{g, g' \in G \\ g \neq -g'}} [g, g']_S$ is called a *spherical skeleton of* G. Now, we say that the set G *has an internally intersecting skeleton* if every point intersection of two great circle segments from the skeleton $G \veebar G$ is contained in the set G.

We formulate in the spherical situation a theorem analogous to Theorem 3.1.

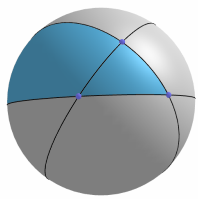


Fig. 8: Biangle C_1.

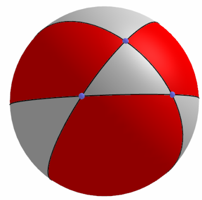


Fig. 9: Forbidden domain of $\Delta(g_1, g_2, g_3)_S$

Theorem 4.1. *Let G be a finite subset of S^2. The following conditions are equivalent:*
(i) *The set G has an internally intersecting skeleton.*
(ii) *For any $g_1, g_2, g_3 \in G$ if the triangle $\Delta(g_1, g_2, g_3)_S$ contains no other point of G, then the forbidden domain of this triangle contains no points of G.*

Proof. (i) $\Rightarrow$ (ii) Let $\Delta(g_1, g_2, g_3)_S$, $g_1, g_2, g_3 \in G$ be a triangle and let no other point of G belong to the triangle. If a point $g_4 \in G$ lies in the forbidden domain of the triangle then without loss of generality we can assume that $g_4 \in \operatorname{int} \Delta(g_1, g_2, -g_3)_S$ (see FIGURE 10). Thus, by condition (i), a singleton $\{g_5\} = [g_1, g_2]_S \cap [g_3, g_4]_S$ is contained in G. Then $g_5 \in [g_1, g_2]_S \subset \Delta(g_1, g_2, g_3)_S$, which contradicts the assumption of condition (ii).

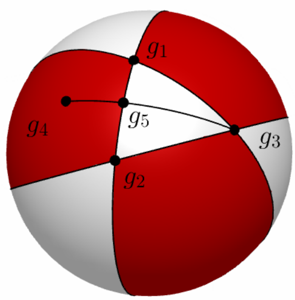


Fig. 10: Point intersection $\{g_5\} = [g_1, g_2]_S \cap [g_3, g_4]_S$.

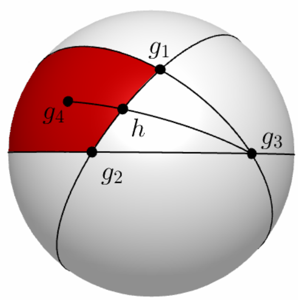


Fig. 11: Point g_4 belongs to forbidden domain of the triangle $\Delta(g_1, g_2, g_3)_S$.

(ii) $\Rightarrow$ (i) Let us assume that the set G does not have an internally intersecting skeleton. It implies that there exist four points $g_1, g_2, g_3, g_4 \in G$ such that $\{h\} = [g_1, g_2]_S \cap [g_3, g_4]_S$, $h \notin G$ (see FIGURE 11). Since G is finite, we may assume that the triangle $\Delta(g_1, g_2, g_3)_S$ does not contain any point of G other than its vertices.

Then the point g_4 belongs to the forbidden domain of the triangle $\Delta(g_1, g_2, g_3)_S$, a contradiction. □

The next theorem is a spherical version of Theorem 3.2.

Theorem 4.2. *A finite set $G \subset S^2$ has an internally intersecting skeleton if and only if there exists a great circle l such that one of the following conditions holds true*:
(a) *The set G is contained in l.*
(b) *The set $G \setminus l$ is a singleton.*
(c) *We have $G \setminus l = \{g, g'\}$, points g, g' lie on the complementary hemispheres determined by l and the point of intersection of l and the great circle segment $[g, g']_S$ belongs to G.*
(c′) *We have $G \setminus l = \{g, g'\}$ and the points g and g' are antipodal.*
(d) *We have $G \setminus l = \{g, g'\}$, the points g and g' lie on the complementary hemispheres determined by l and $G \cap l$ is contained in some halfcircle which is disjointed with the great circle segment $[g, g']_S$.*
(e) *We have $G = \{g_1, g_2, \ldots, g_6\}$, $g_2, g_3, g_4 \in l$, $g_3 \in [g_2, g_4]_S$, $g_1 \in [g_3, g_5]_S$ and $g_2 \in [g_1, g_6]_S$.*

The theorem states that the set G has an internally intersecting skeleton if and only if one of the four following conditions holds true. (a) The set G is contained in some great circle (see Figure 12a). In particular, a singleton and a pair of points are sets contained in some great circle. (b) The set G, except one point, is contained in some great circle (see Figure 12b). (c) All points of the set G except two points belong to some great circle l and some point h of G lies between g and g' (see Figure 12c). (c′) If $g = -g'$ then the great circle segment $[g, g']_S$ does not exist, but Figure 12c well illustrates this case. (d) No point of G lies between g and g', all other points of the set G are contained in some half circle of l and this half circle does not contain the point h of intersection of $[g, g']_S$ and l (see Figure 12d). (e) The set G has six points $g_1, \ldots, g_6$. Points g_1, g_2 and g_3 are vertices of spherical triangle $\Delta(g_1, g_2, g_3)_S$ which contains no other point of G. The point g_4 lies on the great circle segment $[g_3, -g_2]_S$, g_5 lies on the great circle segment $[g_1, -g_3]_S$ and g_6 lies on the great circle segment $[g_2, -g_1]_S$ (see Figure 12e).

Proof. (⇐) It is easy to see, that a skeleton of a set contained in a great circle is contained in this circle. If we add to this set one or two points in the way described by the theorem then, obviously, the set has an internally intersecting skeleton.

(⇒) We assume that the set G has an internally intersecting skeleton. We may also assume that not all points of G belong to some great circle. Let $\Delta(g_1, g_2, g_3)_S$, $g_1, g_2, g_3 \in G$ be a spherical triangle with minimal surface among all spherical triangles with vertices belonging to G. It implies that the closed triangle $\Delta(g_1, g_2, g_3)_S$

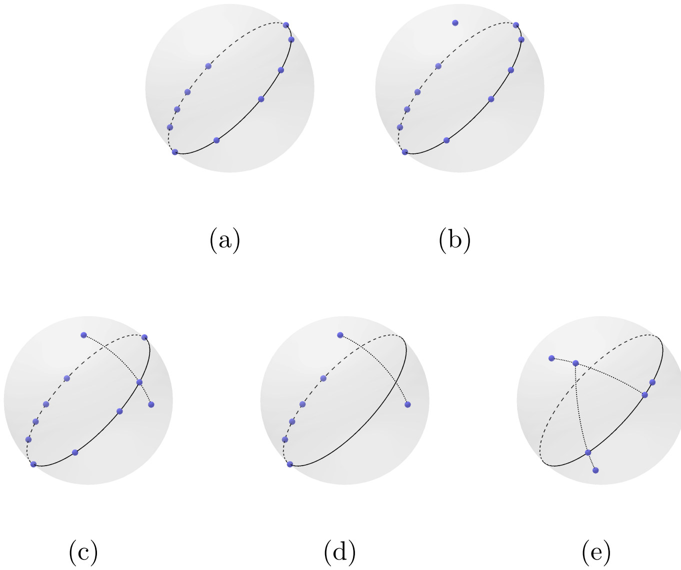


Fig. 12: Five possible structures of the set G.

contains no other points of G. By Theorem 4.1, no point of G belongs to the forbidden domain of the triangle $\Delta(g_1, g_2, g_3)_S$.

(I) Let us consider a case when no one of the great circles g_1g_2, g_2g_3, g_1g_3 contains another point of G. If the set $G = \{g_1, g_2, g_3\}$ then we obtain the situation from Figure 12b. Otherwise, $g_4 \in \operatorname{int} \Delta(g_3, -g_1, -g_2)_S \cup \operatorname{int} \Delta(g_2, -g_1, -g_3)_S \cup$
$\cup \operatorname{int} \Delta(g_1, -g_2, -g_3)_S \cup \operatorname{int} \Delta(-g_1, -g_2, -g_3)_S$.

(IA) Let $g_4 \in \operatorname{int} \Delta(g_3, -g_1, -g_2)_S$. We can choose g_4 in such a way that the triangle $\Delta(g_1, g_2, g_4)_S$, which contains the triangle $\Delta(g_1, g_2, g_3)_S$, does not contain points of G other than g_1, g_2, g_3, g_4. Notice, that none of the triangles $\Delta(g_1, g_2, g_4)_S$, $\Delta(g_1, g_3, g_4)_S$ and $\Delta(g_2, g_3, g_4)_S$ contains points of G other than the vertices. The union of all forbidden domains of the triangles $\Delta(g_1, g_2, g_3)_S$, $\Delta(g_1, g_3, g_4)_S$ and $\Delta(g_2, g_3, g_4)_S$ contains all the exterior of the triangle $\Delta(g_1, g_2, g_4)_S$ except three great circle segments $[g_1, -g_3]_S$, $[g_2, -g_3]_S$ and $[g_4, -g_3]_S$. The great circle segments $[g_1, -g_3]_S$ and $[g_2, -g_3]_S$ are subsets of the great circles g_1g_3 and g_2g_3, respectively. Only the segment $[g_4, -g_3]_S$ may contain other points from G. This case leads us to the situation from Figure 12d. If $g_4 \in \Delta(g_1, -g_2, -g_3)_S$ or $\Delta(g_2, -g_1, -g_3)_S$ we obtain analogous cases.

(IB) No point of G belongs to the union
$\operatorname{int} \Delta(g_3, -g_1, -g_2)_S \cup \operatorname{int} \Delta(g_2, -g_1, -g_3)_S \cup \operatorname{int} \Delta(g_1, -g_2, -g_3)_S$ and
$g_4 \in \Delta(-g_1, -g_2, -g_3)_S$. Since no triangle with vertices in G is contained in the

antipodal triangle $\Delta(-g_1,-g_2,-g_3)_S$ then all points $g_4,\dots,g_n$ of G belong to one of three great semicircles $[g_1,g_4]_S \cup [g_4,-g_1]_S$, $[g_2,g_4]_S \cup [g_4,-g_2]_S$ or $[g_3,g_4]_S \cup [g_4,-g_3]_S$. This case leads us to the situation from FIGURE 12d.

(II) One of the great circles g_1g_2, g_2g_3 or g_1g_3 contains exactly three points of G. Without loss of generality we can assume that $g_4 \in$ great circle g_2g_3. Since the great circle g_2g_3 is divided by points g_2, $-g_2$, g_3 and $-g_3$ into four great circle segments $[g_2,g_3]_S$, $[-g_2,g_3]_S$, $[g_2,-g_3]_S$ and $[-g_2,-g_3]_S$, the following subcases depend on the location of g_4.

(IIA) Let $g_4=-g_2$. If the set G has only four points then we obtain the situation from FIGURE 12b. Otherwise, a point $g_5 \in G$ may belong to one of the following subsets of the unit sphere.

(IIA1) Let $g_5 \in \operatorname{int} \Delta(g_1,-g_3,g_4)_S$. The point intersection $[g_3,g_5]_S \cap [g_1,g_4]_S$ is contained in the set G. This case leads us to the situation from FIGURE 12c where the great circle g_1g_2 is our great circle l.

(IIA2) Let $g_5 \in \Delta(-g_1,g_2,g_3)_S \cup \Delta(-g_1,-g_2,-g_3)_S$ (biangle with vertices g_2 and $-g_2=g_4$). The great circle g_1g_2 is our great circle l. If $[g_3,g_5]_S \cap l \in G$ then we obtain the situation from FIGURE 12c. Otherwise, this case leads us to the situation from FIGURE 12d.

(IIA3) Let $g_5 \in$ great circle g_1g_3. This case leads us to the situation from FIGURE 12c. If the great circle g_1g_3 contains no other points of G then one of the great circle g_1g_2 or g_2g_3 is our great circle l. Otherwise, the great circle g_1g_3 is our great circle l.

The situation $g_4=-g_3$ is analogous to the situation of $g_4=-g_2$.

(IIB) Let $g_4 \in \operatorname{relint} [-g_2,g_3]_S$. The triangle $\Delta(g_1,g_3,g_4)_S$ contains no points of G except its vertices. If the set G contains no other points, then we obtain the situation from FIGURE 12b. Otherwise, we have the three following possibilities.

(IIB1) The points g_5 and g_6 belong to spherical great circle segments $[g_1,-g_3]_S$ and $[g_2,-g_1]_S$. Then cardinality $|G|$ is equal to six and we obtain the situation from FIGURE 12e.

(IIB2) The point $g_5 \in G$ belongs to the great circle g_1g_3. This case leads us to the situation from FIGURE 12c, where the great circle g_1g_3 is our great circle l.

(IIB3) The point g_5 lies on a great circle g_1g_2 or g_1g_4, then we also obtain the situation from FIGURE 12c and the great circle g_2g_3 is our great circle l. In this case the set G has exactly five points.

(IIB4) The point g_5 lies in the triangle $\Delta(g_4,-g_1,-g_3)_S$ or triangle $\Delta(g_2,-g_1,-g_3)_S$, but not on the great circle g_1g_3. This case leads us to the situation from FIGURE 12d. Also in this case the set G has exactly five points.

The situation $g_4 \in \operatorname{relint} [g_2,-g_3]_S$ is analogous to the situation of $g_4 \in \operatorname{relint} [-g_2,g_3]_S$.

(IIC) Let $g_4 \in \operatorname{relint} [-g_2,-g_3]_S$. If the set G contains no other points, then we obtain the situation from FIGURE 12b. Otherwise, since G has internally intersecting skeleton, the points $g_5, \ldots, g_n, n = |G|$ belong to $[g_2, -g_1]_S$, $[g_3, -g_1]_S$ or $[g_1, g_4]_S \cup [g_4, -g_1]_S$. These cases are represented by FIGURE 12d, where l is, respectively, the great circle g_1g_2, g_1g_3 or g_1g_4.

(III) The great circle g_2g_3 contains exactly four consecutive points g_2, g_3, g_4, g_5 and neither great circle g_1g_2 nor g_1g_3 contain exactly three points of G. These circles may contain two, four or more points of G.

(IIIA) The skeleton of $\{g_2, g_3, g_4, g_5\}$ is equal to the great circle g_2g_3.

(IIIA1) If $g_4 = -g_2$ and $g_5 = -g_3$ then the point $g_6 = -g_1$ belongs to G. Other points of G may belong to one of thegreat circles g_1g_2 or g_1g_3. Hence we obtain the situation from FIGURE 12c, where the great circle l is either great circle g_1g_2 or g_1g_3.

(IIIA2) If $g_4 = -g_2$ and $g_5 \neq -g_3$ then the great circle g_1g_2 contains some point g_6 of G other than g_1, g_2 or g_4. This case leads us to the situation from FIGURE 12c, where a great circle g_1g_2 is a great circle l.

(IIIA3) If $g_4 \neq -g_2$ and $g_5 = -g_3$ then we have a case analogous to the previous one.

(IIIA4) Let $g_4 \neq -g_2$ and $g_5 \neq -g_3$. By $H(l, g)$ denote the hemisphere which contains g and has great circle l as a boundary. Analysing forbidden domains we obtain that the hemisphere $H(\text{great circle } g_2g_3, g_1)$ has no other points of G. The set G may contain no more points (situation from FIGURE 12b) or one more point g_6 which lies on the segment $[g_4, -g_1]_S$ or $[g_5, -g_1]_S$ (situation from FIGURE 12c).

(IIIB) The skeleton of $\{g_2, g_3, g_4, g_5\}$ is a proper subset of the great circle g_2g_3.

(IIIB1) If the set G has only five points then we obtain the situation from FIGURE 12b.

(IIIB2) Let g_6 belong to one of the great circle segments $[g_2, -g_1]_S$, $[g_3, -g_1]_S$, $[g_4, -g_1]_S$ or $[g_5, -g_1]_S$. If $g_6 \in [g_2, -g_1]_S$ or $g_6 \in [g_3, -g_1]_S$ then the point g_4 or g_5 has to be equal to, respectively, $-g_3$ or $-g_2$. In either case, we obtain the situation from FIGURE 12c.

(IIIB3) If g_6 does not belong to the segments mentioned above then the intersection of the skeleton of $\{g_2, g_3, g_4, g_5\}$ and the great circle segment $[g_1, g_6]_S$ is empty. We obtain the situation from FIGURE 12d.

(IV) The great circle g_2g_3 contains consecutive points $g_2, g_3, g_4, \ldots, g_n$, $n > 5$ of G and neither great circle g_1g_2 nor g_1g_3 contains exactly three or four points of G. Then the great circle g_2g_3 is our great circle l. If the set $|G| = n$ then we obtain the situation from FIGURE 12b. Otherwise, $|G| > n$ and we have one of the following subcases.

(IVA) The skeleton of $\{g_2, g_3, g_4, \ldots, g_n\}$ is equal to the great circle g_2g_3. Then the point g_{n+1} belongs to one of great circle segments $[g_i, -g_1]_S$, $i = 4, \ldots, n$. This case leads us to the situation from FIGURE 12c.

(IVB) The skeleton of $\{g_2, g_3, g_4, \ldots, g_n\}$ is a proper subset of the great circle g_2g_3.

(IVB1) If the point g_{n+1} belongs to one of great circle segments $[g_i, -g_1]_S$, $i = 4, \dots, n$ then we obtain the situation from FIGURE 12c.

(IVB2) If g_{n+1} does not belong to the segments mentioned above then the intersection of the skeleton of $\{g_2, g_3, g_4, g_5\}$ and the great circle segment $[g_1, g_{n+1}]_S$ is empty. We obtain the situation from FIGURE 12d. □

Theorem 4.2 fully characterizes finite subsets G of unit sphere such that a family of general polyhedra with normal vectors belonging to G is closed with respect to Minkowski addition. Now we are going to characterize, more precisely than in Theorem 2.1, the set G such that the family $\mathcal{B}_G(\mathbb{R}^3)$ of all bounded intersections of half-spaces with normal vectors belonging to G is closed with respect to Minkowski addition.

Theorem 4.3. *The family $\mathcal{B}_G(\mathbb{R}^3)$ of all bounded intersections of half-spaces with normal vectors belonging to G is closed with respect to Minkowski addition if and only if for the finite set G and some great circle l one of the following conditions (compare with Theorem 4.2) holds true*:
(c) *We have $G \setminus l = \{g, g'\}$, points g, g' lie on the complementary hemispheres determined by l and the point of intersection of l and the great circle segment $[g, g']_S$ belongs to G. Moreover, $G \cap l$ is contained in no semicircle.*
(c′) *We have $G \setminus l = \{g, g'\}$ and the points g and g' are antipodal. Moreover, $G \cap l$ is contained in no semicircle.*
(d) *We have $G \setminus l = \{g, g'\}$, the points g and g' lie on the complementary hemispheres determined by l and $G \cap l$ is contained in some semicircle which is disjointed with the great circle segment $[g, g']_S$. Moreover, no semicircle containing $G \cap l$ contains the singleton $l \cap [g, g']_S$.*
(e) *We have $G = \{g_1, g_2, \dots, g_6\}$, $g_2, g_3, g_4 \in l$, $g_3 \in [g_2, g_4]_S$, $g_1 \in [g_3, g_5]_S$ and $g_2 \in [g_1, g_6]_S$. Moreover, the spherical triangle $\Delta(g_1, g_2, g_3)_S$ is not contained in the spherical triangle $\Delta(g_4, g_5, g_6)_S$.*

In the case (c) the points g, g' are not antipodal. A polyhedron with the set of normal vectors equal to G is a truncated prism (see FIGURE 13c), i.e. a prism with nonparallel bases. The shortest of lateral faces must be a parallelogram. In the case (c′) the points g, g' are antipodal. A corresponding polyhedron is a prism (see FIGURE 13c′), i.e. a direct sum of a convex polygon and a line segment. In the case (d) the subset $G \cap l$ is contained in some semicircle. A corresponding polyhedron is a wedge (see FIGURE 13d), i.e. a truncated polyhedron with nonparallel bases which have a common edge. Moreover, no segment parallel to this edge and contained in the polyhedron is longer than the edge. In the case (e) the set G is contained in three semicircles. A corresponding polyhedron is a skew cube (see FIGURE 13e), i.e. a polyhedron with six quadrilateral faces, exactly three of them being trapezoids.

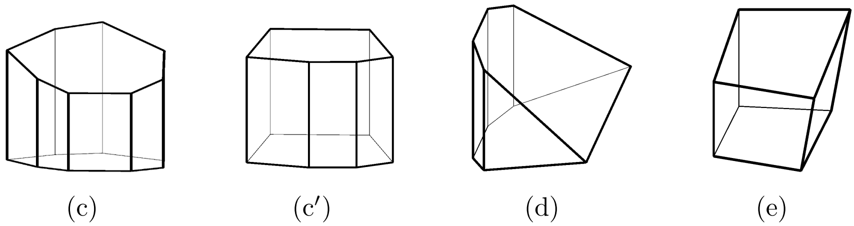


Fig. 13: Four types of polyhedra from Theorem 4.3.

Proof. By Theorem 2.1 the family $\mathcal{B}_G(\mathbb{R}^3)$ is closed with respect to Minkowski addition if and only if G is a set described in Theorem 4.2 and the intersection of a family of half-spaces determined by all normal vectors from G is bounded. Any polyhedron with normal vectors from G is bounded if and only if G is contained in no closed hemisphere. Then the cases from FIGURE 12a and 12b are not suitable. Moreover, in the cases (c), (c′) and (d) from Theorem 4.2 the set G is contained in no closed hemisphere if and only if the set $(G \cap l) \cup ([g, g']_S \cap l)$ is contained in no semicircle. In the case (e) the set G is contained in no closed hemisphere if and only if the spherical triangle $\Delta(g_1, g_2, g_3)_S$ is not contained in the spherical triangle $\Delta(g_4, g_5, g_6)_S$. □

5. Conclusions

Monocrystals can be modeled with the help of convex polyhedra in three dimensional space and the growth of monocrystals can be described using Minkowski addition and subtraction (detailed description in [5], [11] and [13]). Since Minkowski addition may lead to crystals with new faces and new normal vectors that do not belong to a fixed set of normal vectors, Reinhold and Briesen [13] limit their considerations to a family of homothetic copies of summands of a given polyhedron. In general it means that the class of polyhedra which they study is almost never a family $\mathcal{B}_G(\mathbb{R}^3)$. Having a set G of normal vector such that the family $\mathcal{B}_G(\mathbb{R}^3)$ is closed with respect to Minkowski addition we avoid similar limitations. In Theorem 4.3 we found all such sets G.

From [5] we know that a uniformly growing crystal is a function of time variable t and can be expressed as $A(t) = \frac{t-t_1}{t_2-t_1} A(t_2) \dot{-} \frac{t-t_2}{t_2-t_1} A(t_1)$, $t \geqslant t_2$ where $A(t_1)$ and $A(t_2)$ are polyhedra representing crystal at times t_1 and t_2. Since $A \dot{-} B = \bigcap_{b \in B} (A - b)$ a growing crystal belonging to $\mathcal{B}_G(\mathbb{R}^3)$ at time t_2 stays within the family of $\mathcal{B}_G(\mathbb{R}^3)$ afterward. In fact any change of a crystal with constant speed of growth (or recession) of all its faces can be described by $A(t) = A(0) + tB \dot{-} tC$ where the pair (B, C) is inclusion minimal [3, 4]. However, the sets B and C do not have to belong $\mathcal{B}_G(\mathbb{R}^3)$.

We believe it would be profitable to find out whether minimal pairs exist within $\mathcal{B}_G(\mathbb{R}^3)$.

We discovered that appropriate sets G (see Theorem 4.3) coincide with sets of normal vectors of monotypic polytopes (see (37) and (38) in [9]). A monotypic polytope is such a convex polyhedron that the intersection of two translates of this polyhedron is a homothetic copy of summand of it. As we see the only centrally symmetric monotypic polytopes are prisms with centrally symmetric bases (see (36) in [9]) and the only appropriate sets G such that elements of $\mathcal{B}_G(\mathbb{R}^3)$ are centrally symmetric, are represented by FIGURE (c′). There exist crystals that assume a shape of a prism. In particular arfvedsonite $Na_3Fe_4^{2+}Fe^{3+}(Si_8O_{22})(OH)_2$, aquamarine $Be_3Al_2(SiO_3)_6$, tourmalines $XY_3Z_6(T_6O_{18})(BO_3)_3V_3W$ (see [14]) and synthetic emeralds grown by Richard Nacken (see [15]). In crystal of spodumene was found included tetrahedral crystal – possibly pyrochlore (see FIGURE 41 in [14]). We noticed that some optic crystals (e.g. synthetic quartz) are prisms. Moreover, we can see hexagonal prisms in heterogeneous ice nucleation (see [8]).

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KLASY WIELOŚCIANÓW WYPUKŁYCH ZAMKNIĘTE ZE WZGLĘDU NA DODAWANIE MINKOWSKIEGO

S t r e s z c z e n i e

Powyższy artykuł dotyczy rozważań na temat klasy wielościanów wypukłych, których zbiór wektorów normalnych do ścian zawiera się w zadanym, skończonym zbiorze wektorów G. Suma Minkowskiego $A + B$ dwóch wielościanów może posiadać wektor normalny do ściany, który nie jest wektorem normalnym żadnej ze ścian wielościanów A czy B, stąd powyższa klasa może nie być zamknięta ze względu na dodawanie Minkowskiego. Głównym rezultatem artykułu jest podanie warunków koniecznych i dostatecznych aby klasa wielościanów wypukłych o zadanym zbiorze wektorów normalnych do ścian była zamknięta ze względu na dodawanie Minkowskiego.

Słowa kluczowe: stożek abstrakcyjny wielościanów wypukłych, suma Minkowskiego, wzrost kryształu, wielościan wypukły