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COEFFICIENTS INEQUALITIES OF k^{th} ROOT TRANSFORMATION FOR UNIVERSALLY PRESTARLIKE FUNCTIONS

Summary

In the present paper, we consider the class of universally prestarlike functions of complex order. The main result is the solution of the Fekete-Szegö problem for k^{th} root transformation of functions from the defined class.

Keywords and phrases: analytic functions, prestarlike functions, universally prestarlike functions, Fekete-Szegö inequality, quasi subordination

1. Introduction

Let $\mathcal{H}(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For domain Ω containing the origin $\mathcal{H}_0(\Omega)$ stands for the set of all functions $f \in \mathcal{H}(\Omega)$ with $f(0) = 1$. We also use the notation $\mathcal{H}_1(\Omega) = \{zf : f \in \mathcal{H}_0(\Omega)\}$. In the special case when Ω is the open disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| \leq 1\},$$

we use the abbreviation $\mathcal{H}, \mathcal{H}_0$ and $\mathcal{H}_1$ respectively for $\mathcal{H}(\Omega), \mathcal{H}_0(\Omega)$ and $\mathcal{H}_1(\Omega)$.

A function $f \in \mathcal{H}_1$ is called *starlike of order* α with $(\alpha < 1)$ satisfying the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (z \in \mathbb{U})$$

and the set of all such functions is denoted by $\mathcal{S}_\alpha$.

The Hadamard product (or convolution) of two functions $f, g \in \mathcal{H}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \qquad (z \in \mathbb{U}).$$

A function $f \in \mathcal{H}_1(\Omega)$ is called prestarlike of order α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in \mathcal{S}_\alpha.$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$.

The notion of prestarlike functions has been extended, from the unit disk to other disk or half planes containing the origin, by Ruscheweyh *et al.* [10]-[12].

Let Ω be one such disk or half plane. Then there are two unique parameters $\gamma \in \mathbb{C}\backslash\{0\}$ and $\rho \in [0,1]$ such that $\Omega = \Omega_{\gamma,\rho} := \omega_{\gamma,\rho}(\mathbb{U})$, where

$$\omega_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z} \qquad (z \in \mathbb{U}).$$

Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$.

Definition 1. [10]-[12] Let $\alpha < 1$ and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in \mathcal{H}_1(\Omega)$ is called *prestarlike of order α in Ω* if

$$f_{\gamma,\rho} = \frac{1}{\gamma}\left(f \circ \omega_{\gamma,\rho}\right) \in \mathcal{R}_\alpha.$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let

$$\mathfrak{F}(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz}, \quad a_k = \int_0^1 t^k d\mu(t),$$

where $\mu(t)$ is a probability measure on $[0,1]$. By T we denote the set of all such functions $\mathfrak{F}$ which are analytic in the slit domain $\Lambda = \mathbb{C}\backslash[1,\infty)$ (the slit being along the positive real axis).

Definition 2. [12] Let $\alpha \leq 1$. A function f is called *universally prestarlike of order α* if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. The set of all such functions is denoted by $\mathcal{R}_\alpha^u$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided that there is an analytic function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In 1970 Robertson [13] introduced the concept of quasi-subordination. An analytic function $f(z)$ is quasi-subordinate to an analytic function $g(z)$ in $\mathbb{U}$ if there exist functions $\varphi, w \in \mathcal{H}$ with $w(0) = 0$, such that $|\varphi(z)| \leq 1, |w(z)| < 1$ and

$$f(z) = \varphi(z) g[w(z)] \qquad (z \in \mathbb{U}).$$

Then we write $f(z) \prec_q g(z)$.

If $\varphi(z) = 1$, then the quasi-subordination reduces to the subordination. Also, if $w(z) = z$ then $f(z) = \varphi(z) g(z)$ and in this case we say that $f(z)$ is majorized by $g(z)$ and it is written as $f(z) << g(z)$. Hence it is obvious that quasi-subordination is the generalization of subordination as well as majorization.

Motivated by Ruscheweyh *et al.* [12] (see also [14]) we define the following class of functions.

Definition 3. Let $\alpha \leq 1, \gamma \neq 0, \phi \in \mathcal{H}_0$. We denote by $\mathcal{R}^u_{\alpha,\gamma}(\phi)$ the class of functions $f \in \mathcal{H}_1(\Lambda)$ such that $D^{2-2\alpha f(z) \neq 0}$ $(z \in \mathbb{C} \setminus \{0\})$ and

$$\frac{1}{\gamma}\left[\frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} - 1\right] \prec_q \phi(z) - 1, \tag{1}$$

where

$$(D^\beta f)(z) := \frac{z}{(1-z)^\beta} * f\,(z)\,.$$

In particular, by taking $\alpha = \frac{1}{2}$ we get the class $\mathcal{S}^*(\alpha, \phi) := \mathcal{R}^u_{\frac{1}{2}}(\phi)$ which consists of all analytic functions $f \in \mathcal{H}_1(\Lambda)$ satisfying

$$\frac{1}{\gamma}\left[\frac{D^2 f(z)}{D^1 f(z)} - 1\right] \prec_q \phi(z) - 1. \tag{2}$$

Moreover, if we put $\gamma = 1$, then we get the class $\mathcal{S}^*(\alpha, \phi) := \mathcal{R}^u_{\frac{1}{2}}(\phi)$ of functions $f \in \mathcal{H}_1(\Lambda)$ such that

$$\frac{D^2 f(z)}{D^1 f(z)} - 1 \prec_q \phi(z) - 1. \tag{3}$$

Throughout this paper, let

$$\varphi(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots \quad (z \in \mathbb{U})$$

and

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (z \in \mathbb{U}),$$

where $B_n \in \mathbb{R}$, $B_1 > 0$ and $|C_n| \leq 1$.

We also refer to [4], [8], [9].

2. Coefficient bounds for the function class $\mathcal{R}^u_{\alpha,\gamma}(\phi)$

To prove our main result, we need the following lemma.

Lemma 1. [5] *If* $w \in \mathcal{H}$, *with* $w(0) = 0$, $|w(z)| \leq 1$ *and*

$$w(z) = w_1 z + w_2 z^2 + ... \quad (z \in \mathbb{U}) \tag{4}$$

then

$$\left|w_2 - tw_1^2\right| \le \max\{1, |t|\},$$

for any complex number t. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

The k^{th} root transform of a function $f \in \mathcal{H}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}) \tag{5}$$

is defined by

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{6}$$

Now we determine the Fekete-Szegö inequality $|b_{2k+1} - \mu b_{k+1}^2|$ for $f \in \mathcal{R}_{\alpha,\gamma}^u(\phi)$; cf. [5]-[3], [6], [7].

Theorem 2. *Let $f \in \mathcal{R}_{\alpha,\gamma}^u(\phi)$ be of the form (5) and let F be defined by (6). Then*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \frac{|\gamma|}{k(3-2\alpha)} \cdot$$
$$\cdot \left[B_1 + \max\left\{B_1, \left|(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right| |\gamma| B_1^2 + |B_2|\right\}\right]. \tag{7}$$

Proof. Let $f \in \mathcal{R}_{\alpha,\gamma}^u(\phi)$. Then there exist two analytic functions $w, \varphi \in \mathcal{H}$, with $w(0) = 0$, $|w(z)| \le 1$ and $|\varphi(z)| \le 1$ such that

$$\frac{1}{\gamma}\left[\frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} - 1\right] = \varphi(z)[\phi(w(z)) - 1]. \tag{8}$$

Thus we have

$$\varphi(z)[\phi(w(z)) - 1] = B_1 C_0 w_1 z + \left[B_1 C_1 w_1 + C_0\left\{B_1 w_2 + B_2 w_1^2\right\}\right] z^2 + \cdots \tag{9}$$

and

$$\frac{1}{\gamma}\left[\frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} - 1\right] = \frac{1}{\gamma}\left[A_1 z + A_2 z^2 + A_3 z^3 + \cdots\right], \tag{10}$$

where

$$A_1 = [\mathfrak{C}'(\alpha,2) - \mathfrak{C}(\alpha,2)]a_2, \tag{11}$$
$$A_2 = [\mathfrak{C}'(\alpha,3) - \mathfrak{C}(\alpha,3)]a_3 + [\mathfrak{C}(\alpha,2)a_2]^2 - [\mathfrak{C}(\alpha,2)\mathfrak{C}'(\alpha,2)]{a_2}^2 \tag{12}$$

and

$$\mathfrak{C}(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}, \quad \mathfrak{C}'(\alpha, n) = \frac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!}$$

$$b_n = \int_0^1 t^n d\mu(t), \quad n = 2, 3, 4, ...,$$

and $\mu(t)$ is a probability measure on [0,1]. Equating the coefficients of z and z^2 respectively and simplifying we have

$$a_2 = \gamma B_1 C_0 w_1, \tag{13}$$
$$a_3 = \frac{\gamma}{(3-2\alpha)}\left[B_1C_1w_1 + C_0\left\{B_1w_2 + [(2-2\alpha)\gamma B_1^2C_0 + B_2]w_1^2\right\}\right]. \tag{14}$$

For a function f given by (5), a simple computation shows that

$$\left[f(z^k)\right]^{\frac{1}{k}} = z + \frac{1}{k}a_2 z^{k+1} + \left[\frac{1}{k}a_3 - \frac{1}{2}\left(\frac{k-1}{k^2}\right)a_2^2\right]z^{2k+1} + \cdots \tag{15}$$

Equating the coefficients of z^{k+1} and z^{2k+1} in view of (6) and (15), we get

$$b_{k+1} = \frac{1}{k}a_2, \tag{16}$$
$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\left(\frac{k-1}{k^2}\right)a_2^2. \tag{17}$$

Now, substituting the equations (13) and (14) in (16) and (17) we get

$$b_{k+1} = \frac{\gamma B_1 C_0 w_1}{k}, \tag{18}$$

and

$$b_{2k+1} = \frac{\gamma}{k(3-2\alpha)}\left[B_1C_1w_1 + B_1C_0w_2 + C_0\left\{\left[(2-2\alpha) - \frac{1}{2}\left(\frac{k-1}{k}\right)(3-2\alpha)\right]\gamma B_1^2C_0 + B_2\right\}w_1^2\right]. \tag{19}$$

Next, for any complex number μ

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{\gamma B_1}{k(3-2\alpha)}\Bigg[C_1w_1 + \left(w_2 - \left\{-\left[(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right]\gamma B_1C_0 - \left(\frac{B_2}{B_1}\right)\right\}{w_1}^2\right)C_0\Bigg]. \tag{20}$$

Using the inequalities $|w_n| \le 1$ and $|C_n| \le 1$, we have

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{|\gamma|\, B_1}{k(3-2\alpha)}\Bigg[1 + \left|w_2 - \left\{-\left[(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right]\gamma B_1C_0\left(\frac{B_2}{B_1}\right)\right\}{w_1}^2\right|\Bigg] =$$
$$= \frac{|\gamma|\, B_1}{k(3-2\alpha)}\left[1 + \left|w_2 - t{w_1}^2\right|\right], \tag{21}$$

where

$$t = -\left[(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right]\gamma B_1C_0 - \left(\frac{B_2}{B_1}\right).$$

By applying the Lemma 1 we obtain

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \frac{|\gamma|\, B_1}{k(3-2\alpha)} \cdot$$
$$\cdot \left[1 + \max\left\{1, \left|-\left[(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right]\gamma B_1 C_0 - \left(\frac{B_2}{B_1}\right)\right|\right\}\right].$$

Since,

$$\left|-\left[(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right]\gamma B_1 C_0 - \left(\frac{B_2}{B_1}\right)\right| \le$$
$$\le \left|(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right| |\gamma| B_1 + \left|\frac{B_2}{B_1}\right|,$$

we have

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{|\gamma|\, B_1}{k(3-2\alpha)} \cdot$$
$$\cdot \left[1 + \max\left\{1, \left|(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right| |\gamma| B_1 + \left|\frac{B_2}{B_1}\right|\right\}\right].$$

For $\mu = 0$, we get

$$|b_3| \le \frac{\gamma}{k(3-2\alpha)}\left[B_1 + \max\left\{B_1, \left|(2-2\alpha) - \left(\frac{k-1}{2k}\right)(3-2\alpha)\right| |\gamma| B_1^2 + |B_2|\right\}\right].$$

which completes the proof. □

In particular, from Theorem 2, we obtain the following two corollaries.

Corollary 3. *Let $f \in \mathcal{R}^u_{\frac{1}{2},\gamma}(\phi)$ be of the form* (5) *and let F be defined by* (6)*. Then*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \frac{|\gamma|}{2k}\left[B_1 + \max\left\{B_1, \left|1 - 2\left(\frac{k-1}{2k} + \mu\right)\right| |\gamma| B_1^2 + |B_2|\right\}\right].$$

Corollary 4. *Let $f \in \mathcal{R}^u_{\alpha,1}(\phi)$ given by* (5)*. Then*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le$$
$$\le \frac{1}{k(3-2\alpha)}\left[B_1 + \max\left\{B_1, \left|(2-2\alpha) - \left(\frac{k-1}{2k} + \mu\right)(3-2\alpha)\right| B_1^2 + |B_2|\right\}\right].$$

By taking $k = 1$ in Theorem 2 and Corollary (4), we state the following two results.

Corollary 5. *Let $f \in \mathcal{R}^u_{\alpha,\gamma}(\phi)$ be of the form* (5)*. Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{\gamma}{(3-2\alpha)}\left[B_1, \max\left\{B_1, |(2-2\alpha) - \mu(3-2\alpha)|\, |\gamma| B_1^2 + |B_2|\right\}\right].$$

Corollary 6. *Let $f \in \mathcal{R}^u_{\alpha,1}(\phi)$ be of the form* (5) *and let F be defined by* (6). *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{(3-2\alpha)} \left[B_1 + \max\left\{B_1, \left|(2-2\alpha) - \mu(3-2\alpha)\right| B_1^2 + |B_2|\right\}\right].$$

Remark 7. Putting $\gamma = 1$ in Corollary (3) we get the result obtained by Gurusamy et. al [3]. By various choices of the function ϕ and suitably choosing the values of B_1 and B_2, we state some interesting results analogous to Theorem 2 and the Corollaries 3, 4, 5 and 6. In particular, we can consieder the function

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1),$$

with $B_1 = (A-B), B_2 = -B(A-B)$.

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NIERÓWNOŚCI WSPÓŁCZYNNIKOWE DLA *K*-SYMETRYCZNYCH UNIWERSALNYCH FUNKCJI PREGWIAŹDZISTYCH

S t r e s z c z e n i e
W pracy zdefiniowana została klasa uniwersalnych funkcji pregwiaździstych o rzędzie zespolonym. Głównym rezultatem jest rozwiązanie problemu Fekete-Szegö dla k-symetrycznych funkcji z rozważanej klasy.

Słowa kluczowe: funkcje analityczne, uniwersalne funkcje pregwiaździste, nierówność Fekete-Szegö, quazi podporządkowanie