DOI: 10.26485/0459-6854/2017/67.2/9 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017 Vol. LXVII

Recherches sur les déformations no. 2

pp. 93–99

Ikkei Hotta and Andrzej Michalski

SOME REMARKS ON SCHUR'S THEOREM FOR BOUNDED ANALYTIC FUNCTIONS

Summary

In the present paper we are concerned with some estimates for coefficients of analytic functions $\omega(z) = \sum_{n=0}^{\infty} c_n z^n$ defined on the unit disc satisfying $|\omega(z)| \leq 1$. Our results are based on commonly known Schur's theorem [J. Reine Angew. Math. **147** (1917), 205-232].

Keywords and phrases: bounded analytic function, coefficient estimate, Schur's theorem

1. Introduction

Let Δ stand for the open unit disc in the complex plane $\mathbb{C}$ and denote by B the class of all analytic functions $\omega : \Delta \to \overline{\Delta}$. It is well-known that each function $\omega \in B$ is uniquely determined by coefficients of its power series expansion, i.e.

$$\omega(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta, \tag{1.1}$$

for some $c_n \in \mathbb{C}$, $n \in \mathbb{Z}_0$. Here as well as in the whole paper $\mathbb{Z}_p := \{n \in \mathbb{Z} : p \leq n\}$ and $\mathbb{Z}_{p,q} := \{n \in \mathbb{Z} : p \leq n \leq q\}$ for any fixed $p, q \in \mathbb{Z}$, where $\mathbb{Z}$ stands for the set of all integer numbers.

Properties of the class B are widely used in complex analysis, therefore it is thoroughly investigated. There are whole books devoted to bounded analytic functions (e.g. [2]). In particular, such functions play an important role in the theory of planar harmonic mappings (see e.g. [1]). In fact, our researches of this topic (see [4],[5],[3]) gave us motivation to study properties of bounded analytic functions and led us to several remarks, which are the main results of this paper.

Since we focus on the coefficients of a function $\omega \in B$, we start with recalling one of the most interesting results in this area, commonly known as Schur's theorem [7] (see also [6]).

Theorem 1.1. *For a function ω analytic in Δ with the power series expansion* (1.1), *the following conditions are equivalent:*

1) $\omega \in B$;

2) for all $N \in \mathbb{Z}_0$ and for all $\lambda_0, \ldots, \lambda_N \in \mathbb{C}$ we have

$$\sum_{k=0}^{N} \left| \sum_{n=k}^{N} c_{n-k} \lambda_n \right|^2 \leq \sum_{k=0}^{N} |\lambda_k|^2.$$

Note, that if we fix $N \in \mathbb{Z}_0$ and apply Theorem 1.1 with $\lambda_i = 0$ for $i \in \mathbb{Z}_{0,N-1}$ and $\lambda_N = 1$ we immediately obtain the following classical result (usually derived from the Parseval-Gutzmer formula) known as Gutzmer's inequality:

$$\sum_{k=0}^{N} |c_n|^2 \leq 1. \tag{1.2}$$

2. First remark

Our first aim is to improve the inequality (1.2) by more careful and tricky use of Theorem 1.1.

Theorem 2.1. *If $\omega \in B$ then*

$$\sum_{k=0}^{N} |c_k|^2 \leq 1 - |c_0|^2 \left(1 - \sum_{k=0}^{N-1} |c_k|^2 \right) \tag{2.1}$$

for any fixed $N \in \mathbb{Z}_1$.

Proof. For a function $\omega \in B$ and any fixed $N \in \mathbb{Z}_1$ the inequality (2.1) trivially holds in the case when $|c_0| = 1$. Otherwise, by applying Theorem 1.1 with $\lambda_0 = x$, $\lambda_i = 0$ for $i = \mathbb{Z}_{1,N-1}$ and $\lambda_N = e^{i\theta}$, where $x > 0$ and $\theta \in \mathbb{R}$, we have

$$|xc_0 + c_N e^{i\theta}|^2 + \sum_{k=0}^{N-1} |c_k e^{i\theta}|^2 \leq x^2 + 1.$$

Choosing θ so that $|xc_0 + c_N e^{i\theta}| = |xc_0| + |c_N e^{i\theta}|$, the above inequality takes the form

$$(x|c_0| + |c_N e^{i\theta}|)^2 + \sum_{k=0}^{N-1} |c_k e^{i\theta}|^2 \leq x^2 + 1,$$

which can be rewritten as

$$-(1-|c_0|^2)\left(x-\frac{|c_0||c_N|}{1-|c_0|^2}\right)^2+\frac{|c_0|^2|c_N|^2}{1-|c_0|^2}+\sum_{k=0}^{N}|c_k|^2-1\leq 0$$

for all $x>0$, since $|c_0|<1$. This implies

$$\frac{|c_0|^2|c_N|^2}{1-|c_0|^2}+\sum_{k=0}^{N}|c_k|^2-1\leq 0, \tag{2.2}$$

which after suitable rearrangement completes the proof. □

Remark 2.2. Observe that one can rewrite (2.1) as

$$|c_N|^2\leq(1-|c_0|^2)\left(1-\sum_{k=0}^{N-1}|c_k|^2\right).$$

It immediately follows the known inequality

$$|c_N|\leq 1-|c_0|^2. \tag{2.3}$$

for all $N\in\mathbb{Z}_1$.

By repeated use of Theorem 2.1 we can deduce another interesting improvement of Gutzmer's inequality (1.2).

Corollary 2.3. *If $\omega\in B$ then*

$$\sum_{k=1}^{N}|c_k|^2\leq(1-|c_0|^2)(1-|c_0|^{2N}).$$

for any fixed $N\in\mathbb{Z}_0$.

Proof. The proof follows by induction from Theorem 2.1. □

Remark 2.4. The inequality given in Corollary 2.3 immediately implies the following estimate

$$|c_N|^2\leq(1-|c_0|^2)(1-|c_0|^{2N})$$

for all $N\in\mathbb{Z}_1$, which is an improvement of inequality (2.3).

3. Second remark

Using Theorem 1.1 in another way leads to the following result.

Theorem 3.1. *If $\omega\in B$ then*

$$\left|\sum_{k=0}^{N-1}\gamma_{N-k}c_k\right|\leq|c_0||\gamma_1|+\sqrt{(N-1)\left(\sum_{k=0}^{N-1}|\gamma_{N-k}-\gamma_{N-k-1}|^2-|c_0|^2|\gamma_1|^2\right)}$$

for a fixed $N\in\mathbb{N}$, $\gamma_0:=0$ and any numbers $\gamma_k\in\mathbb{C}$, $k\in\mathbb{Z}_{1,N}$.

Proof. Setting $\lambda_k := \gamma_{N-k} - \gamma_{N-k-1}$ for $k \in \mathbb{Z}_{0,N-1}$ we have

$$\sum_{k=0}^{N-1} \gamma_{N-k} c_k = \sum_{k=0}^{N-1} \left(c_k \sum_{j=k}^{N-1} \lambda_j \right) = \sum_{k=0}^{N-1} \left(\sum_{j=k}^{N-1} c_{j-k} \lambda_j \right)$$
$$= c_0 \lambda_{N-1} + \sum_{k=0}^{N-2} \left(\sum_{j=k}^{N-1} c_{j-k} \lambda_j \right).$$

Hence, by applying the triangle inequality we get

$$\left| \sum_{k=0}^{N-1} \gamma_{N-k} c_k \right| \leq |c_0 \lambda_{N-1}| + \sum_{k=0}^{N-2} \left| \sum_{j=k}^{N-1} c_{j-k} \lambda_j \right|$$
$$= |c_0 \lambda_{N-1}| + \sqrt{\left(\sum_{k=0}^{N-2} \left| \sum_{j=k}^{N-1} c_{j-k} \lambda_j \right| \right)^2}.$$

Using the Cauchy-Schwarz inequality we have

$$\sqrt{\left(\sum_{k=0}^{N-2} \left| \sum_{j=k}^{N-1} c_{j-k} \lambda_j \right| \right)^2} \leq \sqrt{(N-1) \sum_{k=0}^{N-2} \left| \sum_{j=k}^{N-1} c_{j-k} \lambda_j \right|^2}$$
$$= \sqrt{(N-1) \left(\sum_{k=0}^{N-1} \left| \sum_{j=k}^{N-1} c_{j-k} \lambda_j \right|^2 - |c_0 \lambda_{N-1}|^2 \right)}.$$

This followed by application of Theorem 1.1 (Schur's theorem) yields

$$\left| \sum_{k=0}^{N-1} \gamma_{N-k} c_k \right| \leq |c_0 \lambda_{N-1}| + \sqrt{(N-1) \left(\sum_{k=0}^{N-1} |\lambda_k|^2 - |c_0 \lambda_{N-1}|^2 \right)},$$

which, in view of the definition of λ_k, $k \in \mathbb{Z}_{0,N-1}$, completes the proof. □

Corollary 3.2. *If* $\omega \in B$ *and* $\omega(0) = 0$ *then*

$$\left| \sum_{k=1}^{N-1} (N-k) c_k \right| \leq N - 1$$

for a fixed $N \in \mathbb{N}$.

Proof. By applying the inequality given in Theorem 3.1 with $\gamma_0 := 0$, $\gamma_k := k$ for $k \in \mathbb{Z}_{1,N-1}$ and $\gamma_N := \gamma_{N-1}$ together with the identity $c_0 = 0$ we obtain

$$\left|\sum_{k=1}^{N-1}(N-k)c_k\right| = \left|\sum_{k=0}^{N-1}(N-k)c_k\right| \le \sqrt{(N-1)\sum_{k=0}^{N-1}\left|\gamma_{N-k}-\gamma_{N-k-1}\right|^2}$$
$$= \sqrt{(N-1)\sum_{k=1}^{N-1}|\gamma_{N-k}-\gamma_{N-k-1}|^2} = N-1,$$

which is the desired result. □

Theorem 3.3. *If $\omega \in B$ and $\omega(0) = 0$ then*

$$\left|\sum_{j=1}^{n-1}(n-j)a_{n-j}c_j\right| \le (n-1)a_{n-1}$$

for a fixed $n \in \mathbb{N}$ and any numbers $a_j \in \mathbb{R}$, $j \in \mathbb{Z}_{1,n-1}$, such that $a_2 \ge a_1 \ge 0$ and $ja_j + (j-2)a_{j-2} \ge 2(j-1)a_{j-1}$, $j \in \mathbb{Z}_{3,n-1}$.

Proof.

Set $\lambda_1 := a_1$ and

$$\lambda_j := ja_j - \sum_{k=2}^{j} k\lambda_{j+1-k} \tag{3.1}$$

for $j \in \mathbb{Z}_{2,n-1}$. We firstly remark that standard calculations show the following equations which are used in the later part of the proof:

$$(n-j)a_{n-j} = \sum_{k=1}^{n-j} k\lambda_{n-j+1-k}, \quad (n-j) \in \mathbb{Z}_2, \tag{3.2}$$

$$\lambda_n = na_n - 2(n-1)a_{n-1} + (n-2)a_{n-2}, \quad n \in \mathbb{Z}_3, \tag{3.3}$$

and

$$\sum_{j=1}^{n}\lambda_j = na_n - (n-1)a_{n-1}, \quad n \in \mathbb{Z}_2. \tag{3.4}$$

Now, by (3.2) we have

$$\sum_{j=1}^{n-1}(n-j)a_{n-j}c_j = \sum_{j=1}^{n-1}\left(\lambda_j\sum_{k=1}^{n-j}(n-j+1-k)c_k\right).$$

Observe, that by the assumptions and (3.3), $\lambda_j \ge 0$ for each $j \in \mathbb{Z}_{1,n-1}$. Hence using the triangle inequality we obtain

$$\left|\sum_{j=1}^{n-1}(n-j)a_{n-j}c_j\right| \le \sum_{j=1}^{n-1}\left(\lambda_j\left|\sum_{k=1}^{n-j}(n-j+1-k)c_k\right|\right).$$

By applying the inequality given in Corollary 3.2 with $N := n - j + 1$, $j \in \mathbb{Z}_{1,n-1}$ we get

$$\left|\sum_{k=1}^{n-j}(n-j+1-k)c_k\right| \le n-j,$$

which yields

$$\left|\sum_{j=1}^{n-1}(n-j)a_{n-j}c_j\right| \le \sum_{j=1}^{n-1}(n-j)\lambda_j.$$

Together with the equality

$$\sum_{j=1}^{n-1}(n-j)\lambda_j = (n-1)a_{n-1}$$

which is proven by using (3.4), we complete the proof. □

Remark 3.4. Observe, that the inequalities given in Corollary 3.2 and Theorem 3.3 are sharp and the equality is attained e.g. for the identity mapping.

Acknowledgement

The first author was supported by JSPS KAKENHI Grant Numbers 26800053.

References

[1] P. L. Duren, *Harmonic mappings in the plane*, Cambridge Univ. Press, Cambridge 2004.

[2] J. B. Garnett, *Bounded Analytic Functions*, Acad. Press, New York 1981.

[3] I. Hotta, A. Michalski, *Locally one-to-one harmonic functions with starlike analytic part*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **64**, no. 2 (2014), 19–27.

[4] D. Klimek, A. Michalski, *Univalent anti-analytic perturbation of the identity in the unit disc*, Sci. Bull. Chełm. **1** (2006), 67–78.

[5] D. Klimek, A. Michalski, *Univalent anti-analytic perturbation of convex conformal mapping in the unit disc*, Ann. Univ. Maria Curie-Skłodowska Sect. A **61** (2007), 39–49.

[6] R. A. Kortram, *A simple proof for Schur's theorem*, Proc. Amer. Math. Soc. **129**, no. 11 (2001), 3211–3212.

[7] J. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, J. Reine Angew. Math. **147** (1917), 205–232.

Department of Applied Science
Yamaguchi University
2-16-1 Tokiwadai Ube 755-8611
Japan
E-mail: ihotta@yamaguchi-u.ac.jp

Department of Complex Analysis
The John Paul II Catholic University of Lublin
Konstantynów 1H, PL-20-950 Lublin
Poland
E-mail: amichal@kul.lublin.pl

Presented by Zbiniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on March 23, 2017.

PEWNE UWAGI ZWIĄZANE Z TWIERDZENIEM SCHURA DLA OGRANICZONYCH FUNKCJI ANALITYCZNYCH

S t r e s z c z e n i e

W niniejszej pracy zajmujemy się oszacowaniami współczynników funkcji analitycznych $\omega(z) = \sum_{n=0}^{\infty} c_n z^n$ określonych w kole jednostkowym, które spełniają warunek $|\omega(z)| \leq 1$. Prezentowane wyniki uzyskaliśmy w oparciu o powszechnie znane twierdzenie Schura [J. Reine Angew. Math. **147** (1917), 205-232].

Słowa kluczowe: ograniczone funkcje analityczne, oszacowania współczynników, twierdzenie Schura