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ISOMORPHISM THEOREMS FOR σ-IDEALS OF MICROSCOPIC SETS IN VARIOUS METRIC SPACES

Summary

In the paper we generalise the notion of a microscopic set onto an arbitrary metric space. We examine microscopic sets in the Cantor space 2^ω and give a series of isomorphism theorems concerning the families of microscopic sets in 2^ω, $[0, 1]$, $\mathbb{R}$ as well as their finite products. Moreover, we investigate some connections between cardinal invariants for the mentioned σ- ideals.

Keywords and phrases: microscopic sets, isomorphisms of ideals, small sets, σ-ideals, cardinal invariants

1. Introduction

The notion of a *microscopic set* was introduced in 2001 by Appell in [2] (cf. [1, 3]). Namely, a set $A \subseteq \mathbb{R}$ is microscopic if for every $\varepsilon > 0$ there exists a sequence of intervals $(I_n)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} I_n$ and $|I_n| \leq \varepsilon^n$ for $n \in \mathbb{N}$. The family of all microscopic sets on $\mathbb{R}$ is in fact a σ-ideal situated between the σ-ideal of measure zero sets and the σ-ideal of strong measure zero sets. The notion of microscopic sets has been recently studied by several authors; see for example [4, 5, 6, 7, 8, 9].

In [8] Karasińska and Wagner-Bojakowska have introduced two generalisations of microscopic sets in $\mathbb{R}^2$. They call a set $A \subseteq \mathbb{R}^2$ microscopic if for every $\varepsilon > 0$ there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of rectangles (with sides parallel to coordinate axis) such that $\lambda_2(A) \leq \varepsilon^n$ for each $n \in \mathbb{N}$, whereas the set is called *strongly* microscopic if the same condition holds true for squares with sides parallel to coordinate axis (where λ_2 denotes the Lebesgue measure on $\mathbb{R}^2$). In this paper we generalise the notion of a microscopic set onto an arbitrary metric space. Our generalisation is coherent with the latter approach. We then examine some properties of microscopic sets in metric spaces and then concentrate on microscopic sets in the Cantor space 2^ω. We construct several Borel isomoprhisms between the families of microscopic sets in 2^ω, $[0,1]$ and their finite products. We also prove that the σ-ideal of microscopic sets in $\mathbb{R}$ and in the Cantor space 2^ω equipped with the Baire metric are Borel-isomorphic. It is not uncommon that dealing with the Cantor space is easier that with the real line. We believe that our result will help in proving combinatorial properties of the σ-ideal of microscopic sets or will at least simplify the existing proofs. Combinatorial properties of the σ-ideal of microscopic sets on the real line have been recently studied by A. Kwela in [9], where he proved that the additivity of this σ-ideal is equal to ω_1.

2. Microscopic sets in metric spaces

In this section we examine a natural generalisation of the notion of a microscopic set onto an arbitrary metric space. We give a series of conditions which are equivalent to being microscopic. We also formulate some well-known facts about the family of microscopic sets in the case of a metric space, giving a reader a general overview on the subject.

Let (X,d) be a metric space. Let $x \in N$ and $r > 0$. We denote the open (closed) ball with the centre x and radius r by $B(x,r)$ ($\overline{B}(x,r)$). Let $A \subseteq X$. We denote the diameter of the set A by $diam(A)$.

We call a set $A \subseteq X$ *microscopic*, if for every, $\varepsilon > 0$ there exists a sequence $(I_n)_{n\in\mathbb{N}}$ of subsets of X, such that $A \subseteq \bigcup_{n\in\mathbb{N}} I_n$ and $\operatorname{diam}(I_n) \leq \varepsilon^n$ for every $n \in \mathbb{N}$. We denote the family of all microscopic sets in the space X by $\mathcal{M}ic_X$.

It turns out, that in an arbitrary metric space, one obtains the following convenient result, whose proof is an easy exercise. Before we state it, let us denote $c_0^+ := \{(a_k)_{k\in\mathbb{N}} \in c_0 : \forall_{k\in\mathbb{N}}\ a_k > 0\}$.

Proposition 2.1. *Let $A \subseteq X$. The following conditions are equivalent:*

(i) $A \in \mathcal{M}ic_X$;

(ii) $\forall_{\varepsilon>0}\ \exists_{\{x_n:n\in\mathbb{N}\}\subseteq X}\ A \subseteq \bigcup_{n\in\mathbb{N}} B(x_n,\varepsilon^n)$;

(iii) $\forall_{\varepsilon>0}\ \exists_{\{x_n:n\in\mathbb{N}\}\subseteq X}\ A \subseteq \bigcup_{n\in\mathbb{N}} \overline{B}(x_n,\varepsilon^n)$;

(iv) $\forall_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{\{x_n:n\in\mathbb{N}\}\subseteq X}\ A \subseteq \bigcup_{n\in\mathbb{N}} B(x_n,a_k^n)$;

(v) $\forall_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{(B_n)_{n\in\mathbb{N}}}\ \left(A \subseteq \bigcup_{n\in\mathbb{N}} B_n \wedge \forall_{n\in\mathbb{N}}\ \operatorname{diam}(B_n) \leq a_k^n\right)$, *where $(B_n)_{n\in\mathbb{N}}$ is a sequence of open balls in X;*

(vi) $\forall_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{(J_n)_{n\in\mathbb{N}}}\ \left(A \subseteq \bigcup_{n\in\mathbb{N}} J_n \wedge \forall_{n\in\mathbb{N}}\ \operatorname{diam}(J_n) \leq a_k^n\right)$, *where $(J_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X;*

(vii) $\exists_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{(J_n)_{n\in\mathbb{N}}}\ \left(A \subseteq \bigcup_{n\in\mathbb{N}} J_n \ \wedge \forall_{n\in\mathbb{N}}\ \operatorname{diam}(J_n) \le a_k^n\right)$,
where $(J_n)_{n\in\mathbb{N}}$ *is a sequence of subsets of* X*;*

(viii) $\exists_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{(B_n)_{n\in\mathbb{N}}}\ \left(A \subseteq \bigcup_{n\in\mathbb{N}} B_n \ \wedge \forall_{n\in\mathbb{N}}\ \operatorname{diam}(B_n) \le a_k^n\right)$,
where $(B_n)_{n\in\mathbb{N}}$ *is a sequence of open balls in* X*;*

(ix) $\exists_{(a_k)_{k\in\mathbb{N}}\in c_0^+}\ \forall_{k\in\mathbb{N}}\ \exists_{\{x_n:n\in\mathbb{N}\}\subseteq X}\ A \subseteq \bigcup_{n\in\mathbb{N}} B(x_n, a_k^n)$.

It is obvious that in the conditions *1.*, *2.* i *3.* of the above theorem one can equivalently take $\varepsilon \in (0,1)$.

Clearly countable sets are microscopic. Moreover, if one considers a discrete metric space, then the opposite implication is true as well. Suppose that A is a microscopic set in a discrete metric space. By Proposition 2.1 (viii), there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of open balls covering the set A such that $\operatorname{diam}(B_n) \le (\frac{1}{2})^n$ for $n \in \mathbb{N}$. Since for every $n \in \mathbb{N}$ the ball B_n is in fact a singleton, A must be countable.

It is a well known fact that the family of all microscopic sets on the real line (considered with Euclidean metric) forms a σ-ideal, i.e. a family closed under taking subsets and countable unions. It turns out that the same is true in case of an arbitrary metric space. Some other facts about microscopic sets are true in arbitrary metric spaces, namely:

Proposition 2.2. *Let* (X, d) *be a metric space. Then*

- (a) $\mathcal{M}ic_X$ *is a* σ*-ideal;*
- (b) *if* (Y, ρ) *is a metric space and* $f : X \to Y$ *is a Lipschitz mapping, then* $A \in \mathcal{M}ic_X$ *implies* $f(A) \in \mathcal{M}ic_Y$*;*
- (c) *if* d,ρ *are Lipschitz equivalent metrics on* X*, then* $\mathcal{M}ic_{(X,d)} = \mathcal{M}ic_{(X,\rho)}$*;*
- (d) *microscopic sets have Hausdorff dimension zero.*

The proof of (a) in the above theorem is analogous to the one in case of $\mathbb{R}$, which can be found in several papers concerning microscopic sets. Points (b), (c) and (d) are simple observations.

Recall that we call the ideals $\mathcal{A}$ and $\mathcal{B}$ on a nonempty set X orthogonal if there exist sets $A \in \mathcal{A}$, $B \in \mathcal{B}$ such that $A \cap B = \emptyset$ and $X = A \cup B$.

We can now formulate another generalisations of facts concerning families microscopic sets which are well-known in the case of the real line.

Fact 2.3. *In every separable and complete metric space, there exists a residual microscopic set. In other words, the* σ*-ideals* $\mathcal{M}ic_X$ *and* $\mathcal{M}_X$ *are orthogonal (where* $\mathcal{M}_X$ *denotes the* σ*-ideal of the sets of the first category).*

Proof. First observe that the σ-ideal $\mathcal{M}ic_X$ has a base consisting of G_δ-sets. Indeed, let $A \in \mathcal{M}ic_X$. From Proposition 2.1(ii) for every $k \in \mathbb{N}$ there exists a set $\{x_{k,n} : n \in \mathbb{N}\}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B(x_{k,n}, (\frac{1}{k})^n)$. Put $B := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} B(x_{k,n}, (\frac{1}{k})^n)$. Then obviously

B is a microscopic, G_δ superset of A. Now, let $D \subseteq X$ be dense and countable. Clearly D is microscopic and its G_δ microscopic superset is residual. $\square$

3. Microscopic sets in the Cantor space 2^ω

Let us denote the set of all infinite (finite) binary sequences by 2^ω $(2^{<\omega})$. For any $s \in 2^{<\omega}$, $s = (s_0, s_1, ...s_n)$ and $i \in \{0,1\}$ by $s\hat{\ }i$ we denote the concatenation of the sequence and the term i, i.e. $s\hat{\ }i = (s_0, s_1, ..., s_n, i)$ and by $|s|$ we denote the length of the sequence s (we assume $|\emptyset| = 0$). Moreover, for $t \in 2^\omega$, by $t|n$ we denote the restriction of t to n initial terms.

In this section we take into consideration the family of all microscopic sets in the Cantor space 2^ω, i.e. the set of all infinite binary sequences with the so-called Baire metric. The metric, which is in fact an ultrametric is defined as follows. For $x, y \in 2^\omega$, $x = (x(n))_{n\in\omega}$, $y = (y(n))_{n\in\omega}$ we put

$$\rho(x,y) := \begin{cases} 2^{-\min\{n\in\omega : x(n)\neq y(n)\}}, \text{ if } x \neq y; \\ 0, \text{ if } x = y. \end{cases}$$

Of course the space $(2^\omega, \rho)$ is compact and separable. The base of the topology consists of sets $\langle s\rangle = \{x \in 2^\omega : s \prec x\}$ for $s \in 2^{<\omega}$, where $s \prec x$ iff $x||s| = s$. Thus, the set $\langle s\rangle$ consists of all infinite extensions of the finite sequence s.

Let us define a mapping between the Cantor space and the unit interval $[0,1]$ which will enable us to identify microscopic sets in 2^ω with microscopic subsets of the unit interval (with Euclidean metric). Consider a mapping $g : 2^\omega \longrightarrow [0,1]$ given by:

$$g(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}, \text{ for } x = (x(n))_{n\in\omega} \in 2^\omega.$$

Observe, that if $x \in 2^\omega$, then $g(x)$ is a number in $[0,1]$, whose binary expansion is the sequence x. Since there exist numbers which have two binary expansions (the so called *dyadic fractions*), g is not injective. Thus, let us consider the set $E := \{x \in 2^\omega : \exists_{n\in\omega} \forall_{m\geq n}\ x(m) = 1\} \setminus \{(1,1,1,...)\}$ and put $h := g\restriction_{2^\omega\setminus E}$. Then the following is true.

Proposition 3.1. *The mapping $h : 2^\omega \setminus E \to [0,1]$ is a Lipschitz bijection.*

Proof. Since $2^\omega \setminus E$ consists precisely of the sequence $(1,1,1,...)$ and binary sequences which have infinitely many zeros, then h is an injection. Let $z \in [0,1]$. If z is a dyadic fraction, i.e. $z = \frac{m}{2^k}$ for some $m, k \in \omega$ such that $m \leq 2^k$, then z has two binary expansions and exactly one of them belongs to $2^\omega \setminus E$. Otherwise, z has only one binary expansion and it does not belong to E. Hence h is a bijection. Let $x, y \in 2^\omega \setminus E$ and assume $x \neq y$. Put $n_0 := \min\{n \in \omega : x(n) \neq y(n)\}$. Thus $\rho(x,y) = \frac{1}{2^{n_0}}$. We have:

$$|h(x)-h(y)| = \Big|\sum_{k=0}^{\infty}\frac{x(k)}{2^{k+1}} - \sum_{k=0}^{\infty}\frac{y(k)}{2^{k+1}}\Big| = \Big|\sum_{k=n_0+1}^{\infty}\frac{x(k)}{2^{k+1}} - \sum_{k=n_0+1}^{\infty}\frac{y(k)}{2^{k+1}}\Big| =$$

$$= \Big|\sum_{k=n_0+1}^{\infty}\frac{x(k)-y(k)}{2^{k+1}}\Big| \le \sum_{k=n_0+1}^{\infty}\frac{1}{2^{k+1}} = \frac{1}{2^{n_0+1}} = \frac{1}{2}\rho(x,y).$$

Hence h is Lipschitz (with constant $\frac{1}{2}$). $\square$

The following technical lemmas will be useful in the proof of the main theorem in this section.

Lemma 3.2. *Let I be a nonempty open subinterval of $(0,1)$. Then*

$$\log_2 \operatorname{diam}(h^{-1}(I)) = -\min\{n\in\mathbb{N} : \exists_{k\in\{0,1,\dots,2^n-1\}}\ \frac{k}{2^n}\in I\}.$$

Proof. Let an interval $\emptyset \neq I \subseteq [0,1]$ be open in $\mathbb{R}$. For any distinct $a,b\in I$ let us denote: $(a,b\rangle := (\min\{a,b\},\max\{a,b\}]$ $a,b\in I$. For arbitrary $a<b$ there exist: $n\in\mathbb{N}$ and $k\in\{0,1,\dots,2^n-1\}$ such that $a<\frac{k}{2^n}<b$. Hence:

$$\begin{aligned}
\log_2 \operatorname{diam}(h^{-1}(I)) &= \log_2 \sup\{\rho(x,y) : x,y\in h^{-1}(I)\}\\
&= \log_2 \sup\{2^{-\min\{n\in\omega : x(n)\neq y(n)\}} : x,y\in h^{-1}(I),\ x\neq y\}\\
&= \log_2(\inf\{2^{\min\{n\in\omega : x(n)\neq y(n)\}} : x,y\in h^{-1}(I),\ x\neq y\})^{-1}\\
&= -\min\{\min\{n\in\omega : x(n)\neq y(n)\} : x,y\in h^{-1}(I),\ x\neq y\}\}\\
&\overset{(\star)}{=} -\min\{\min\{n\in\omega : \exists_{k\in\{0,1,\dots,2^{n+1}-1\}}\ \frac{k}{2^{n+1}}\in(h(x),h(y)\rangle\} : x,y\in h^{-1}(I), x\neq y\}\\
&= -\min\{n\in\omega : \exists_{a,b\in I}\ \exists_{k\in\{0,1,\dots,2^{n+1}-1\}}\ a<\frac{k}{2^{n+1}}\le b\}\\
&= -\min\{n\in\omega : \exists_{k\in\{0,1,\dots,2^{n+1}-1\}}\ \frac{k}{2^{n+1}}\in I\}\\
&= -\min\{n\in\mathbb{N} : \exists_{k\in\{0,1,\dots,2^{n}-1\}}\ \frac{k}{2^{n}}\in I\}.
\end{aligned}$$

We will only prove $(\star)$. Let $x,y\in h^{-1}(I)$, $x\neq y$. Observe, that $x,y\neq(1,1,1,\dots)$. Indeed, if $I\subseteq[0,1]$ is open in $\mathbb{R}$, then $1\notin I$, so $(1,1,1,\dots)\notin h^{-1}(I)$. It is sufficient to show that

$$\min\{n\in\omega : x(n)\neq y(n)\} = \min\Big\{n\in\omega : \exists_{k\in\{0,1,\dots,2^{n+1}-1\}}\ \frac{k}{2^{n+1}}\in(h(x),h(y)\rangle\Big\}.$$

" $\geq$ " Put $n_0 := \min\{n\in\omega : x(n)\neq y(n)\}$. Without the loss of generality let us assume that $x(n_0)=0$ and $y(n_0)=1$. Firstly, observe that

$$h(x) = \sum_{n=0}^{n_0-1}\frac{x(n)}{2^{n+1}} + 0 + \sum_{n=n_0+1}^{\infty}\frac{x(n)}{2^{n+1}}$$

and

$$h(y)=\sum_{n=0}^{n_0-1}\frac{y(n)}{2^{n+1}}+\frac{1}{2^{n_0+1}}+\sum_{n=n_0+1}^{\infty}\frac{y(n)}{2^{n+1}}.$$

Then $h(y)>h(x)$, because

$$h(y)-h(x)=\frac{1}{2^{n_0+1}}+\sum_{n=n_0+1}^{\infty}\frac{y(n)-x(n)}{2^{n+1}}\overset{(\diamond)}{>}\frac{1}{2^{n_0+1}}+\sum_{n=n_0+1}^{\infty}\frac{-1}{2^{n+1}}=0,$$

where $(\diamond)$ results from the fact that $\sum\limits_{n=n_0+1}^{\infty}\frac{x(n)}{2^{n+1}}<\sum\limits_{n=n_0+1}^{\infty}\frac{1}{2^{n+1}}=\frac{1}{2^{n_0+1}}$, because $x\in 2^\omega\setminus E$ and $x\neq(1,1,1...)$. Moreover:

$$\sum_{n=0}^{n_0-1}\frac{y(n)}{2^{n+1}}+\frac{1}{2^{n_0+1}}=\frac{\left(\sum\limits_{n=0}^{n_0-1}y(n)2^{n_0-n}\right)+1}{2^{n_0+1}}$$

(when $n_0=0$, we put $\sum\limits_{n=0}^{n_0-1}\frac{y(n)}{2^{n+1}}=0$). Put $k:=\left(\sum\limits_{n=0}^{n_0-1}y(n)2^{n_0-n}\right)+1\in\mathbb{Z}$. Then $1\leq k\leq\sum\limits_{n=0}^{n_0}2^{n_0-n}=\sum\limits_{n=0}^{n_0}2^n=2^{n_0+1}-1$. It is obvious that $h(y)\geq\frac{k}{2^{n_0+1}}$ and $h(x)<\frac{k}{2^{n_0+1}}$, because $\sum\limits_{n=n_0+1}^{\infty}\frac{x(n)}{2^{n+1}}<\frac{1}{2^{n_0+1}}$. Hence $\frac{k}{2^{n_0+1}}\in(h(x),h(y)]=(h(x),h(y)\rangle$, so $n_0\geq\min\{n\in\omega:\exists_{k\in\{0,1,\ldots,2^{n+1}-1\}}\ \frac{k}{2^{n+1}}\in(h(x),h(y)\rangle\}$.

"$\leq$" Let $n_0=\min\{n\in\omega:\exists_{k\in\{0,1,\ldots,2^{n+1}-1\}}\ \frac{k}{2^{n+1}}\in(h(x),h(y)\rangle\}$. Suppose that $\min\{n\in\omega:x(n)\neq y(n)\}>n_0$, that is for $n\leq n_0$ we have $x(n)=y(n)$. Hence:

$$h(x)=\sum_{n=0}^{n_0}\frac{x(n)}{2^{n+1}}+\sum_{n=n_0+1}^{\infty}\frac{x(n)}{2^{n+1}}$$

once

$$h(y)=\sum_{n=0}^{n_0}\frac{y(n)}{2^{n+1}}+\sum_{n=n_0+1}^{\infty}\frac{y(n)}{2^{n+1}}.$$

Analogously as in the first part: $\sum\limits_{n=0}^{n_0}\frac{x(n)}{2^{n+1}}=\frac{k_0}{2^{n_0+1}}$ for some $k_0\in\{0,...,2^{n_0+1}-1\}$. Observe that $h(x),h(y)\geq\frac{k_0}{2^{n_0+1}}$. If $k_0=2^{n_0+1}-1$, then we immediately reach a contradiction with the definition of n_0. If $k_0<2^{n_0+1}-1$, then let us consider the number $\frac{k_0+1}{2^{n_0+1}}$. Since $x,y\in 2^\omega\setminus E$ and $x,y\neq(1,1,1,...)$, then binary expansions of x,y contain infinitely many zeros and therefore we have that $h(x),h(y)<\frac{k_0}{2^{n_0+1}}+\sum\limits_{n=n_0+1}^{\infty}\frac{1}{2^{n+1}}=\frac{k_0}{2^{n_0+1}}+\frac{\frac{1}{2^{n_0+2}}}{\frac{1}{2}}=\frac{k_0}{2^{n_0+1}}+\frac{1}{2^{n_0+1}}=\frac{k_0+1}{2^{n_0+1}}$, so $\frac{k_0+1}{2^{n_0+1}}>h(x),h(y)\rangle$, which again contradicts the definition of n_0. □

Lemma 3.3. *Let $n\in\mathbb{N}$. Let $I\subseteq[0,1]$ be an interval, open in $\mathbb{R}$ of length not greater* *than* $\frac{1}{2^n}$. *If*

$$\exists_{m\le n}\ \exists_{k\in\{0,\ldots,2^m-1\}}\ \frac{k}{2^m}\in I,$$

then

$$\forall_{l\le n}\ \forall_{r\in\{0,\ldots,2^l-1\}}\ \Big(\frac{r}{2^l}\in I\Rightarrow \frac{r}{2^l}=\frac{k}{2^m}\Big).$$

In other words, I contains at most one element with binary expansion of length not greater than n.

Proof. Assume that there are $m\le n$ and $k\in\{0,...,2^m-1\}$ such that $\frac{k}{2^m}\in I$. Let $l\le n$ and $r\in\{0,1,...,2^l-1\}$ be such that $\frac{r}{2^l}\in I$. Suppose that $\frac{r}{2^l}\neq\frac{k}{2^m}$. Then

$$\frac{r2^{n-l}}{2^n}\neq\frac{k2^{n-m}}{2^n}\iff r2^{n-l}\neq k2^{n-m}\iff |r2^{n-l}-k2^{n-m}|\ge 1.$$

Hence we obtain that

$$\Big|\frac{r}{2^l}-\frac{k}{2^m}\Big|=\frac{|r2^{n-l}-k2^{n-m}|}{2^n}\ge\frac{1}{2^n}.$$

Since $\frac{r}{2^l},\frac{k}{2^m} inI$ and I is open of length not greater than $\frac{1}{2^n}$, we reach a contradiction. □

Theorem 3.4. *$A\subseteq 2^\omega\setminus E$ is microscopic if and only if $h(A)\subseteq[0,1]$ is microscopic.*

Proof. $\Rightarrow$ By Proposition 2.2 (b), an image of a microscopic set through a Lipschitz mapping is also microscopic, so by Proposition 3.1, the proof of this implication is done.

$\Leftarrow$ Let $A\subseteq[0,1]$ be microscopic. Without the loss of generality assume that $0,1\notin A$. Put $a_n=\frac{1}{2^{3n}}$, for $n\in\mathbb{N}$. Let $k\in\mathbb{N}$. By Proposition 2.1 (viii) there is a sequence $(I_n)_{n\in\mathbb{N}}$ of open intervals covering the set A such that $\operatorname{diam}(I_n)\le\frac{1}{2^{3kn}}$. Fix $n\in\mathbb{N}$ and assume that $I_n=(a,b)$ for some $0\le a<b\le 1$. By Lemma 3.2, $\log_2\operatorname{diam}(h^{-1}(I_n))=-\min\{m\in\mathbb{N}:\exists_{p\in\{0,1,\ldots,2^m-1\}}\ \frac{p}{2^m}\in I_n\}$. Let us consider two cases:

1. $\min\{m\in\mathbb{N}:\exists_{p\in\{0,1,\ldots,2^m-1\}}\ \frac{p}{2^m}\in I_n\}\ge 3kn$. Then:
$$\log_2\operatorname{diam}(h^{-1}(I_n))=-\min\{m\in\mathbb{N}:\exists_{p\in\{0,1,\ldots,2^m-1\}}\ \frac{p}{2^m}\in I_n\}\le -3kn.$$
In this case put $J_{3n-2}:=\emptyset$, $J_{3n-1}:=\emptyset$, $J_{3n}:=h^{-1}(I_n)$. Observe that $\operatorname{diam}(J_{3n-2})=\operatorname{diam}(J_{3n-1})=0<\frac{1}{2^{(3n-1)k}}<\frac{1}{2^{(3n-2)k}}$ and $\operatorname{diam}(J_{3n})\le\frac{1}{2^{3kn}}$.
2. $N:=\min\{m\in\mathbb{N}:\exists_{p\in\{0,1,\ldots,2^m-1\}}\ \frac{p}{2^m}\in I_n\}<3kn$. Then there is $p\in\{0,1,...,2^N-1\}$ such that $\frac{p}{2^N}\in I_n$. By Lemma 3.3: $\forall_{l\le 3kn}\ \forall_{r\in\{0,\ldots,2^l-1\}}\ \Big(\frac{r}{2^l}\in(a,b)\Rightarrow\frac{r}{2^l}=\frac{p}{2^N}\Big)$, so $\frac{p}{2^N}=\frac{2^{3kn-N}p}{2^{3kn}}$ is the only number of the form $\frac{q}{2^{3kn}}$ in (a,b). Consider the intervals $(a,\frac{p}{2^N})$ and $(\frac{p}{2^N},b)$. Observe that no number of the form $\frac{q}{2^{3kn}}$ belongs to $(a,\frac{p}{2^N})$. By Lemma 3.2: $\operatorname{diam}\big(h^{-1}\big((a,\frac{p}{2^N})\big)\big)\le\frac{1}{2^{3kn}}$. Analogously $\operatorname{diam}\big(h^{-1}\big((\frac{p}{2^N},b)\big)\big)\le\frac{1}{2^{3kn}}$. Note that $h^{-1}(I_n)=J_{3n-2}\cup J_{3n-1}\cup J_{3n}$, where $J_{3n-2}:=h^{-1}\big(\{\frac{p}{2^N}\}\big)$, $J_{3n-1}:=h^{-1}\big((a,\frac{p}{2^N})\big)$, $J_{3n}:=$

$h^{-1}\left(\left(\frac{p}{2^N}, b\right)\right)$. Moreover $\operatorname{diam}(J_{3n-2}) = 0 < \frac{1}{2^{(3n-2)k}}$, $\operatorname{diam}(J_{3n-1}) \leq \frac{1}{2^{3kn}} < \frac{1}{2^{(3n-1)k}}$ and $\operatorname{diam}(J_{3n}) \leq \frac{1}{2^{3kn}}$.

Proceeding analogously with all the intervals from the sequence $(I_n)_{n\in\mathbb{N}}$, we construct a sequence $(J_n)_{n\in\mathbb{N}}$ of subsets of $2^\omega \setminus E$ with the property $\operatorname{diam}(J_n) \leq \frac{1}{2^{kn}}$. By Proposition 2.1 (vi) $h^{-1}(A)$ is microscopic in $2^\omega \setminus E$. □

4. Isomorphisms of ideals

Theorem 3.4 enables us to compare microscopic sets on the unit interval and in the Cantor space 2^ω. To be more precise, in this section we will give attention to the existence of an isomorphism between ideals $\mathcal{M}ic_{[0,1]}$ and $\mathcal{M}ic_{2^\omega}$. Let us first recall the definition of an isomoprhism of ideals.

Definition 4.1. Let X, Y be nonempty sets, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y, respectively. We call the ideals $\mathcal{I}$ and $\mathcal{J}$ isomorphic, if there exists a bijection $F : X \to Y$ such that

$$\forall_{A\subseteq X} \ \Big(A \in \mathcal{I} \iff F(A) \in \mathcal{J}\Big).$$

We denote this fact by $\mathcal{I} \cong \mathcal{J}$ and call the bijection F an isomorphism of the ideals $\mathcal{I}$ and $\mathcal{J}$. If F is a Borel isomorphism, then we say that I and J are Borel isomorphic.

Thanks to Theorem 3.4, we can prove the following result.

Theorem 4.2. *The ideals $\mathcal{M}ic_{[0,1]}$ and $\mathcal{M}ic_{2^\omega}$ are Borel isomorphic.*

Proof. Put $P := \{\frac{1}{2^n} : n \in \mathbb{N}\} \subseteq [0,1]$. Observe that $E \cup h^{-1}(P)$ (where E is the same as in Proposition 3.1) is countable. Fix a bijection $b : E \cup h^{-1}(P) \to P$. Define $F : 2^\omega \to [0,1]$ as follows: for $\alpha \in 2^\omega$, put:

$$F(\alpha) := \begin{cases} b(\alpha), \text{ where } \alpha \in E \cup h^{-1}(P), \\ h(\alpha), \text{ where } \alpha \in 2^\omega \setminus (E \cup h^{-1}(P)). \end{cases}$$

Clearly, F is a Borel isomorphism. We will now show that it is the desired isomorphism of the ideals. Let $A \in \mathcal{M}ic_{2^\omega}$. $F(A) = h(A\setminus(E\cup h^{-1}(P)))\cup b(A\cap(E\cup h^{-1}(P)))$, so it is a sum of two microscopic sets, because $h(A \setminus (E \cup h^{-1}(P)))$ is microscopic by Theorem 3.4 and $b(A \cap (E \cup h^{-1}(P)))$ is countable. Now, let $A \subseteq 2^\omega$ and $F(A)$ be microscopic on $[0,1]$. Note that $A = F^{-1}(F(A)) = h^{-1}\Big(h\big(A \setminus \big(E \cup h^{-1}(P)\big)\big)\Big) \cup b^{-1}\Big(b\big(A \cap \big(E \cup h^{-1}(P)\big)\big)\Big)$. Again using Theorem 3.4 one can show that A is microscopic as well. □

We will now present further theorems concerning ideal isomorphisms. Note that all product spaces will be considered with the maximum metric.

Theorem 4.3. *The ideals $\mathcal{M}ic_{2^\omega\times 2^\omega}$ and $\mathcal{M}ic_{2^\omega}$ are Borel isomoprhic.*

Proof. Let $H : 2^\omega \times 2^\omega \to 2^\omega$ be defined as follows:

$$H(x, y) := (x(0), y(0), x(1), y(1), x(2), y(2), ...).$$

H is of course a homeomorphism. By Proposition 2.1 (vii) A is microscopic in $2^\omega \times 2^\omega$ with the maximum metric, if

$$\forall_{n\in\mathbb{N}} \ \exists_{\substack{(\langle s_k\rangle\times\langle t_k\rangle)_{k\in\mathbb{N}}\\ s_k,t_k\in 2^{kn}}} \ A \subseteq \bigcup_{k\in\mathbb{N}} \langle s_k\rangle \times \langle t_k\rangle. \tag{$\bowtie$}$$

Let $(x, y) \in A$, $n \in \mathbb{N}$ and $(\langle s_k\rangle \times \langle t_k\rangle)_{k\in\mathbb{N}}$ be as in ($\bowtie$). Then there is $k \in \mathbb{N}$ such that $x \in \langle s_k\rangle$ and $y \in \langle t_k\rangle$, namely $x(i) = s_k(i)$ for $i \in \{0, 1, ..., kn - 1\}$ and $y(i) = t_k(i)$ for $i \in \{0, 1, ..., kn - 1\}$. Hence $H(x, y) = (x(0), y(0), x(1), y(1), ...) \in \langle s_k(0), t_k(0), s_k(1), t_k(1), ..., s_k(kn - 1), t_k(kn - 1)\rangle = H(\langle s_k\rangle \times \langle t_k\rangle)$. Because $\mathrm{diam}(\langle s_k(0), t_k(0), s_k(1), t_k(1), ..., s_k(kn - 1), t_k(kn - 1)\rangle) = \frac{1}{2^{2kn}}$ and $H(A) \subseteq \bigcup_{k\in\mathbb{N}} H((\langle s_k\rangle \times \langle t_k\rangle))$, so for $n \in \mathbb{N}$, the set $H(A)$ can be covered with a countable union of the sets $H(\langle s_k\rangle \times \langle t_k\rangle)$ with diameters $(\frac{1}{4^n})^1$, $(\frac{1}{4^n})^2$, $(\frac{1}{4^n})^3$, ... , respectively. So by Proposition 2.1 (vii) $H(A)$ is microscopic in 2^ω.

Let $A \in \mathcal{M}ic_{2^\omega}$. By Proposition 2.1 (iv)

$$\forall_{n\in\mathbb{N}} \ \exists_{\substack{(s_k)_{k\in\mathbb{N}}\\ s_k\in 2^{2kn}}} \ A \subseteq \bigcup_{k\in\mathbb{N}} \langle s_k\rangle.$$

Observe that $H^{-1}(\langle s_k\rangle) = \langle s_k(0), s_k(2), ..., s_k(2kn - 2)\rangle \times \langle s_k(1), s_k(3), ..., s_k(2kn - 1)\rangle$. Hence $\mathrm{diam}(H^{-1}(\langle s_k\rangle)) = \frac{1}{2^{kn}}$. Moreover $H^{-1}(A) \subseteq \bigcup_{k\in\mathbb{N}} H^{-1}(\langle s_k\rangle)$, which proves that $H^{-1}(A) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. □

Theorem 4.4. *The ideals $\mathcal{M}ic_{[0,1]\times[0,1]}$ and $\mathcal{M}ic_{2^\omega\times 2^\omega}$ are Borel isomorphic.*

Proof. Let $D : 2^\omega \times 2^\omega \to [0, 1] \times [0, 1]$ be given by $D(x, y) = (F(x), F(y))$ (where F is the same as in the proof of Theorem 4.2). Then D is a Borel isomorphism. Put $Q := E \cup h^{-1}(P)$. Let $A \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. Note that $A \subseteq \tilde{A} \cup B$, where $\tilde{A} := A \setminus ((Q \times \pi_2(A)) \cup (\pi_1(A) \times Q))$, $B := ((Q \times \pi_2(A)) \cup (\pi_1(A) \times Q))$ and π_i denotes the projection on the i-th coordinate. Observe that $D(A) \subseteq D(\tilde{A}) \cup D(B)$ and $D(B) = F(Q) \times F(\pi_2(A)) \cup F(\pi_1(A)) \times F(Q)$. Let $F(Q) = \{q_1, q_2, ...\}$. Since $A \in \mathcal{M}ic_{2^\omega\times 2^\omega}$ and projections are Lipschitz mappings, then $\pi_1(A), \pi_2(A) \in \mathcal{M}ic_{2^\omega}$. Hence $F(\pi_i(A)) \in \mathcal{M}ic_{[0,1]}$ for $i = 1, 2$. It is an easy observation that $\{q_n\} \times F(\pi_2(A)) \in \mathcal{M}ic_{[0,1]\times[0,1]}$ for $n \in \mathbb{N}$, so $F(Q) \times F(\pi_2(A)) \in \mathcal{M}ic_{[0,1]\times[0,1]}$ because Q is countable. Similarly, $F(\pi_1(A)) \times F(Q) \in \mathcal{M}ic_{[0,1]\times[0,1]}$, so $D(B) \in \mathcal{M}ic_{[0,1]\times[0,1]}$.

We will show that $D(\tilde{A}) \in \mathcal{M}ic_{[0,1]\times[0,1]}$. Let $\varepsilon > 0$. $A \in \mathcal{M}ic_{2^\omega\times 2^\omega}$, so by Proposition 2.1 there is a sequence $(\langle s_n\rangle \times \langle t_n\rangle)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} \langle s_n\rangle \times \langle t_n\rangle$ i $\mathrm{diam}(\langle s_n\rangle), \mathrm{diam}(\langle t_n\rangle) \leq \varepsilon^n$ for $n \in \mathbb{N}$. Observe that $(D((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}))_{n\in\mathbb{N}}$ covers the set $D(\tilde{A})$ and for $n \in \mathbb{N}$:

$$D((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}) \subseteq D(\pi_1((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}) \times \pi_2((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}))$$

$$
\begin{aligned}
&= F(\pi_1(((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A})) \times F(\pi_2(((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}))\\
&= h(\pi_1(((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A})) \times h(\pi_2(((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A}))\\
&\subseteq h(\pi_1(\langle s_n\rangle \times \langle t_n\rangle)) \times h(\pi_2(\langle s_n\rangle \times \langle t_n\rangle))\\
&\subseteq h(\langle s_n\rangle) \times h(\langle t_n\rangle).
\end{aligned}
$$

At the same time, $\mathrm{diam}(h(\langle s_n\rangle)) = \mathrm{diam}(\langle s_n\rangle) \le \varepsilon^n$ and $\mathrm{diam}(h(\langle t_n\rangle)) = \mathrm{diam}(\langle t_n\rangle) \le \varepsilon^n$. Finally, we obtain that $\mathrm{diam}(D(((\langle s_n\rangle \times \langle t_n\rangle) \cap \tilde{A})) \le \varepsilon^n$, which finishes the first part of the proof.

Now, let $A \in \mathcal{M}ic_{[0,1]\times[0,1]}$. We will show that $D^{-1}(A) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. For $x, y \in [0,1]$, $D^{-1}(x,y) = (F^{-1}(x), F^{-1}(y))$. Similarly to the first part of the proof, $A \subseteq \tilde{A} \cup B$, where $\tilde{A} = A \setminus ((F(Q) \times \pi_2(A)) \cup (\pi_1(A) \times F(Q)))$ and $B = (F(Q) \times \pi_2(A)) \cup (\pi_1(A) \times F(Q))$. It is sufficient to show that $D^{-1}(\tilde{A}), D^{-1}(B) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. Note that $D^{-1}(F(Q) \times \pi_2(A)) = F^{-1}(F(Q)) \times F^{-1}(\pi_2(A)) = Q \times F^{-1}(\pi_2(A))$, so $Q \times F^{-1}(\pi_2(A)) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. Analogously, $F^{-1}(\pi_1(A)) \times Q \in \mathcal{M}ic_{2^\omega\times 2^\omega}$, so $D^{-1}(B) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. Consider the set $D^{-1}(\tilde{A}) \in \mathcal{M}ic_{2^\omega\times 2^\omega}$. Put $a_k = \frac{1}{2^{9k}}$ for $k \in \mathbb{N}$ and fix $k \in \mathbb{N}$. By Proposition 2.1, there is a sequence $(I_n \times J_n)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} I_n \times J_n$ and $\mathrm{diam}(I_n), \mathrm{diam}(J_n) \le \frac{1}{2^{9kn}}$ for any $n \in \mathbb{N}$. Observe that $(D^{-1}((I_n \times J_n) \cap \tilde{A}))_{n\in\mathbb{N}}$ is a covering of the set $D^{-1}(\tilde{A})$ and for $n \in \mathbb{N}$ we obtain that:

$$
\begin{aligned}
D^{-1}((I_n \times J_n) \cap \tilde{A}) &\subseteq D^{-1}(\pi_1((I_n \times J_n) \cap \tilde{A}) \times \pi_2((I_n \times J_n) \cap \tilde{A}))\\
&= F^{-1}(\pi_1((I_n \times J_n) \cap \tilde{A}) \times F^{-1}(\pi_2((I_n \times J_n) \cap \tilde{A})\\
&= h^{-1}(\pi_1((I_n \times J_n) \cap \tilde{A}) \times h^{-1}(\pi_2((I_n \times J_n) \cap \tilde{A})\\
&\subseteq h^{-1}(\pi_1(I_n \times J_n)) \times h^{-1}(\pi_2(I_n \times J_n))\\
&\subseteq h^{-1}(I_n) \times h^{-1}(J_n).
\end{aligned}
$$

Fix $n \in \mathbb{N}$. Proceeding as in the proof of Theorem 3.4, we can show that there exist sets $C_n^1, C_n^2, C_n^3, D_n^1, D_n^2, D_n^3 \subseteq 2^\omega$ such that $h^{-1}(I_n) = C_n^1 \cup C_n^2 \cup C_n^3$, $h^{-1}(J_n) = D_n^1 \cup D_n^2 \cup D_n^3$ and $\mathrm{diam}(C_n^i), \mathrm{diam}(D_n^j) \le \frac{1}{2^{9kn}}$ for $i, j \in \{1,2,3\}$. Of course $D^{-1}(\tilde{A}) \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i,j\in\{1,2,3\}} C_n^i \times D_n^j$. Fix a numeration $(L_n)_{n\in\mathbb{N}}$ of the family $\{C_n^i \times D_n^j : i,j \in \{1,2,3\}, n \in \mathbb{N}\}$ such that for $n \in \mathbb{N}$, $L_{9n-8} = C_n^1 \times D_n^1, L_{9n-7} = C_n^1 \times D_n^2, ..., L_{9n-1} = C_n^3 \times D_n^2, L_{9n} = C_n^3 \times D_n^3$. Observe that for $n \in \mathbb{N}$ and $l \in \{0,1,...,8\}$, $\mathrm{diam}(L_{9n-l}) \le \frac{1}{2^{9nk}} \le \frac{1}{2^{k(9n-l)}}$, so the sequence $(L_n)_{n\in\mathbb{N}}$ is the desired covering of the set $D^{-1}(\tilde{A})$. □

From the above theorems we immediately obtain the following. The σ-ideals of microscopic sets in the spaces $[0,1]$, 2^ω, $2^\omega \times 2^\omega$ i $[0,1] \times [0,1]$, are Borel isomorphic. In symbols:

$$\mathcal{M}ic_{[0,1]} \cong \mathcal{M}ic_{2^\omega} \cong \mathcal{M}ic_{2^\omega\times 2^\omega} \cong \mathcal{M}ic_{[0,1]\times[0,1]}.$$

From the above results it is an easy consequence that $\mathcal{M}ic_{\mathbb{R}} \cong \mathcal{M}ic_{2^\omega}$ and further, all ideals of microscopic sets on finite-dimensional Euclidean spaces, i.e. the ideals $\mathcal{M}ic_{\mathbb{R}^n}$, are pairwise isomorphic. This is particularly interesting, because it implies that the cardinal invariants for the ideals of microscopic sets in the above mentioned spaces are equal.

5. Cardinal invariants of microscopic σ-ideals

For a σ-ideal $\mathcal{I}$ of subsets of X containing all singletons we define the following cardinal invariants:

$$\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\},$$
$$\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\},$$
$$\operatorname{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \text{ and } Y \notin \mathcal{I}\},$$
$$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall_{B\in\mathcal{I}}\ \exists_{A\in\mathcal{A}}\ \ B \subseteq A\}.$$

As we have mentioned in the Introduction the σ-ideal $\mathcal{M}ic_{\mathbb{R}^2}$ coincides with the σ-ideal of strongly microscopic sets in the sense of Karasińska and Wagner-Bojakowska. By $\mathcal{M}ic^*_{\mathbb{R}^2}$ let us denote the σ-ideal of microscopic sets on the plane in sense of Karasińska and Wagner-Bojakowska. Clearly, the product $A \times B$ of two microscopic sets $A, B \in \mathcal{M}ic_{\mathbb{R}}$ is in $\mathcal{M}ic^*_{\mathbb{R}^2}$. On the other hand, the product of two microscopic sets on the real line is not necessary microscopic (strongly microscopic in sense of Karasińska and Wagner-Bojakowska) on the plane. Moreover, Karasińska and Wagner-Bojakowska proved in [8] a Fubini type result: product $A \times B \in \mathcal{M}ic^*_{\mathbb{R}^2}$ if and only if $A \in \mathcal{M}ic_{\mathbb{R}}$ or $B \in \mathcal{M}ic_{\mathbb{R}}$. They also proved that $A \in \mathcal{M}ic^*_{\mathbb{R}^2}$ if and only if there exists $U \in \mathcal{M}ic_{\mathbb{R}}$ such that $A \subseteq (U \times \mathbb{R}) \cup (\mathbb{R} \times U)$.

Proposition 5.1. *(i)* $\operatorname{add}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{add}(\mathcal{M}ic^*_{\mathbb{R}^2}) = \operatorname{add}(\mathcal{M}ic_{\mathbb{R}^2}) = \omega_1$*;*

(ii) $\operatorname{cov}(\mathcal{N}) \leq \operatorname{cov}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{cov}(\mathcal{M}ic^*_{\mathbb{R}^2}) = \operatorname{cov}(\mathcal{M}ic_{\mathbb{R}^2}) \leq \operatorname{non}(\mathcal{M})$*, where* $\mathcal{N}$ *stands for the σ-ideal of null subsets of the real line;*

(iii) $\operatorname{non}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{non}(\mathcal{M}ic^*_{\mathbb{R}^2}) = \operatorname{non}(\mathcal{M}ic_{\mathbb{R}^2}) \geq \operatorname{cov}(\mathcal{M})$*;*

(iv) $\operatorname{cof}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{cof}(\mathcal{M}ic^*_{\mathbb{R}^2}) = \operatorname{cof}(\mathcal{M}ic_{\mathbb{R}^2}) \geq \max\{\operatorname{cov}(\mathcal{M}), \operatorname{cov}(\mathcal{N})\}$*.*

Proof. If two ideals are isomorphic, then their respective cardinal invariants coincide. Therefore, by the results from the previous Section we have $\operatorname{add}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{add}(\mathcal{M}ic_{\mathbb{R}^2})$, $\operatorname{cov}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{cov}(\mathcal{M}ic_{\mathbb{R}^2})$, $\operatorname{non}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{non}(\mathcal{M}ic_{\mathbb{R}^2})$ and $\operatorname{cof}(\mathcal{M}ic_{\mathbb{R}}) = \operatorname{cof}(\mathcal{M}ic_{\mathbb{R}^2})$.

(i) Kwela proved in [9] that $\operatorname{add}(\mathcal{M}ic_{\mathbb{R}}) = \omega_1$. Let $\{A_\alpha : \alpha < \omega_1\}$ be a family of microscopic sets on the real line such that $A_\alpha \subseteq A_\beta$ provided $\alpha < \beta$ and $\bigcup_{\alpha<\omega_1} A_\alpha \notin \mathcal{M}ic_{\mathbb{R}}$. Then $A_\alpha \times A_\alpha \in \mathcal{M}ic^*_{\mathbb{R}^2}$ and $\bigcup_{\alpha<\omega_1} (A_\alpha \times A_\alpha) = (\bigcup_{\alpha<\omega_1} A_\alpha) \times (\bigcup_{\alpha<\omega_1} A_\alpha) \notin \mathcal{M}ic^*_{\mathbb{R}^2}$. Therefore $\operatorname{add}(\mathcal{M}ic^*_{\mathbb{R}^2}) = \omega_1$.

(ii) Since microscopic sets are null, then $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{M}ic_{\mathbb{R}})$. Since the ideals $\mathcal{M}ic_{\mathbb{R}}$ and $\mathcal{M}$ are orthogonal, then by the Rothberger Theorem we get the inequality $\mathrm{cov}(\mathcal{M}ic_{\mathbb{R}}) \leq \mathrm{non}(\mathcal{M})$. Let $\mathrm{cov}(\mathcal{M}ic_{\mathbb{R}}) = \kappa$ and let $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{M}ic_{\mathbb{R}}$ be such that $\bigcup_{\alpha<\kappa} A_\alpha = \mathbb{R}$. Then $A_\alpha \times \mathbb{R} \in \mathcal{M}ic^*_{\mathbb{R}^2}$ and $\bigcup_{\alpha<\kappa} (A_\alpha \times \mathbb{R}) = \mathbb{R}^2$. Thus $\mathrm{cov}(\mathcal{M}ic_{\mathbb{R}}) \geq \mathrm{cov}(\mathcal{M}ic^*_{\mathbb{R}^2})$. To prove the second inequality let $\kappa = \mathrm{cov}(\mathcal{M}ic^*_{\mathbb{R}^2})$ and let $\{B_\alpha : \alpha < \kappa\} \subseteq \mathcal{M}ic^*_{\mathbb{R}^2}$ be a cover of the plane. By the result of Karasińska and Wagner-Bojakowska, for every $\alpha < \kappa$ there exists $U_\alpha \in \mathcal{M}ic_{\mathbb{R}}$ such that $B_\alpha \subseteq (U_\alpha \times \mathbb{R}) \cup (\mathbb{R} \times U_\alpha)$. Then of course $\bigcup_{\alpha<\kappa} ((U_\alpha \times \mathbb{R}) \cup (\mathbb{R} \times U_\alpha)) = \mathbb{R}^2$. Suppose to the contrary that $\kappa < \mathrm{cov}(\mathcal{M}ic_{\mathbb{R}})$. Then $\bigcup_{\alpha<\kappa} U_\alpha \neq \mathbb{R}$. Thus there is $x \in \mathbb{R} \setminus \bigcup_{\alpha<\kappa} U_\alpha$ and $(x,x) \notin \bigcup_{\alpha<\kappa} ((U_\alpha \times \mathbb{R}) \cup (\mathbb{R} \times U_\alpha))$, contradiction.

(iii) Using the Rothberger Theorem again, we obtain $\mathrm{non}(\mathcal{M}ic_{\mathbb{R}}) \geq \mathrm{cov}(\mathcal{M})$. The inequality $\mathrm{non}(\mathcal{M}ic_{\mathbb{R}}) \geq \mathrm{non}(\mathcal{M}ic^*_{\mathbb{R}^2})$ can be deduced from the following: $A \notin \mathcal{M}ic_{\mathbb{R}} \implies A \times A \notin \mathcal{M}ic^*_{\mathbb{R}^2}$. The reverse inequality follows from: $B \notin \mathcal{M}ic^*_{\mathbb{R}^2} \implies \pi_1(B) \notin \mathcal{M}ic_{\mathbb{R}}$.

(iv) By (ii) and (iii) we obtain $\mathrm{cof}(\mathcal{M}ic_{\mathbb{R}}) \geq \max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\}$. Assume that $\mathcal{A}$ is a base for $\mathcal{M}ic_{\mathbb{R}}$. Then $\{(A \times \mathbb{R}) \cup (\mathbb{R} \times A) : A \in \mathcal{A}\}$ is a base for $\mathcal{M}ic^*_{\mathbb{R}^2}$. Thus $\mathrm{cof}(\mathcal{M}ic_{\mathbb{R}}) \geq \mathrm{cof}(\mathcal{M}ic^*_{\mathbb{R}^2})$. Let $\{B_\alpha : \alpha < \kappa\}$ be a base for $\mathcal{M}ic^*_{\mathbb{R}^2}$. Again, for every $\alpha < \kappa$ there exists $U_\alpha \in \mathcal{M}ic_{\mathbb{R}}$ such that $B_\alpha \subseteq (U_\alpha \times \mathbb{R}) \cup (\mathbb{R} \times U_\alpha)$. Let $\mathcal{A} = \{U_\alpha : \alpha < \kappa\}$. Let $A \in \mathcal{M}ic_{\mathbb{R}}$. Find $\alpha < \kappa$ with $A \times A \subseteq B_\alpha \subseteq (U_\alpha \times \mathbb{R}) \cup (\mathbb{R} \times U_\alpha)$. Then $A \subseteq U_\alpha$. Therefore $\mathrm{cof}(\mathcal{M}ic_{\mathbb{R}}) \leq \mathrm{cof}(\mathcal{M}ic^*_{\mathbb{R}^2})$. □

By Proposition 5.1, under Martin's Axiom, $\mathrm{cov}(\mathcal{M}ic_{\mathbb{R}}) = \mathrm{non}(\mathcal{M}ic_{\mathbb{R}}) = \mathrm{cof}(\mathcal{M}ic_{\mathbb{R}}) = \mathfrak{c}$. Moreover, consistently $\mathrm{cov}(\mathcal{M}ic_{\mathbb{R}}) < \mathfrak{c}$. There is an open question whether the invariants $\mathrm{non}(\mathcal{M}ic_{\mathbb{R}})$ and $\mathrm{cof}(\mathcal{M}ic_{\mathbb{R}})$ are equal to $\mathfrak{c}$ in ZFC.

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IZOMORFIZMY σ-IDEAŁÓW ZBIORÓW MIKROSKOPIJNYCH W PRZESTRZENIACH METRYCZNYCH

S t r e s z c z e n i e

W pracy uogólnione zostaje pojęcie zbioru mikroskopijnego na dowolną przestrzeń metryczną. Szczególny nacisk położony jest na zbiory mikroskopijne w przestrzeni Cantora 2^ω. Podana zostaje seria twierdzeń dotyczących izomorfizmów pomiędzy σ-ideałami zbiorów mikroskopijnych w przestrzeniach 2^ω, $[0,1]$, $\mathbb{R}$, jak i w ich skończonych produktach. Badane są także zależności pomiędzy niezmiennikami kardynalnymi powyższych σ-ideałów.

Słowa kluczowe: zbiory mikroskopijne, izomorfizm ideałów, zbiór mały, σ-ideał, niezmienniki kardynalne