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Yield Curve Modelling with the Nelson-Siegel Method for Poland*

Modelowanie krzywej dochodowości dla Polski z wykorzystaniem metody Nelsona-Siegla

Abstract

Yield curve modelling is an essential task for the governance of the modern economy and in particular for financial market participants, and hence it is an extensively researched topic. This paper presents yield curve modelling using the Nelson-Siegel approach for Poland, which was recently recognised as a developed country. Yield curve studies available for Poland are quite scarce and were conducted when Poland was still classified as a developing country. Therefore, it is worthwhile to examine the yield curve construction after three decades of economic transition. This study offers a model which, with certain assumptions, derives zero-coupon yield curves from the market prices of Treasury bonds. The simplifying assumptions reduce model development time, while delivering yield curves of higher accuracy than those commercially available.

Keywords:
yield curve, bootstrapping, Nelson-Siegel model

JEL classification codes:
C52, C55, E43, G12

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Streszczenie

Modelowanie krzywej dochodowości należy do kluczowych zadań w zarządzaniu nowoczesną gospodarką, jest również szczególnie istotne dla uczestników rynku finansowego, dlatego pozostaje przedmiotem szeroko zakrojonych badań naukowych. W artykule zaprezentowano modelowanie struktury terminowej stóp procentowych z wykorzystaniem modelu Nelsona–Siegla dla Polski, która została uznana za kraj wysoko rozwinięty. Studia krzywych dochodowości dla Polski są stosunkowo rzadko dostępne i zostały wykonane, kiedy Polskę zaliczano do grona krajów rozwijających się. W związku z tym, pożądane jest przeprowadzenie badań dotyczących konstrukcji krzywej dochodowości po trzech dekadach transformacji gospodarczej. W prezentowanym badaniu zaproponowano model, który – przy pewnych założeniach – szacuje zero-kuponowe krzywe dochodowości na podstawie cen rynkowych obligacji skarbowych. Przyjęte założenia upraszczają konstrukcję modelu i skracają czas jego budowy, przy czym dokładność oszacowanych krzywych jest na poziomie wyższym niż w przypadku wykorzystania krzywych dostępnych w płatnych serwisach informacyjnych.

Słowa kluczowe:
krzywa dochodowości, bootstrapping, model Nelsona–Siegla

Kody klasyfikacji JEL:
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Introduction

Developing a model that would deliver an accurate yield curve is important for financial market participants and governments. It would make it possible to accurately price interest rate derivatives, estimate the credit risk of counterparties as well as capital and liquidity requirements for banks and other financial institutions. Unlike central banks in other developed economies, the National Bank of Poland does not publish an official daily yield curve, hence practitioners often rely on curves supplied by commercial financial information services such as Thomson Reuters Eikon or Bloomberg. Yield curve estimation is not a trivial task, yet for the Polish market it is particularly challenging because the Treasury bond market is not sufficiently mature. We argue, however, that it is possible, with some limitations, to construct a yield curve based on Treasury bond market prices using a pragmatic methodology that can be applied by practitioners. We present a relatively simple method which derives the yields from free public data and in the next step estimates a curve applying the Nelson-Siegel (NS) approach. Based on in-sample pricing, we confirm the accuracy of the obtained curves, which proves that they can be viewed as an alternative to commercially available ones. In certain applications, which rely on factors derived from the NS model (level, slope, curvature), this can provide better accuracy, while not being overly demanding in terms of development due to simplifying assumptions.

The article is organised as follows. In Section 2, we review the literature on yield curve modelling using the Nelson-Siegel model with a particular focus on the Polish economy. In Section 3, we describe data used for the modelling. Section 4 provides the methodology, Section 5 presents the results of the yield curve estimation and bond valuations. The next section concludes.

Literature review

The interpretable affine model proposed by **Nelson and Siegel** [**1987**] and its subsequent extensions have gained a lot of recognition in yield curve modelling. They are widely used in academia as well as central and commercial banking applications. Nelson and Siegel proposed a model based on expectation theory of the term structure of interest rates. It describes the yields dynamic with four parameters: level, slope, curvature, and speed of decay. They modelled instantaneous forward rates for the US market, which can be interpreted as expected future spot rates under the assumption that the forward term premium is negligible. The model was developed based on bid-ask discount quotes which were used for price derivation.

The Nelson-Siegel approach was extended by **Svensson** [**1994**], who introduced additional parameters allowing a better fit. Svensson modelled a yield curve for the Swedish economy; his study was based on market data. In turn, **Diebold and Li** [**2006**] provided a dynamic extension to the NS approach by modelling the level, slope and curvature parameters (beta factors) as autoregressive series using the market prices of US Treasury bonds. Curve modelling with the NS approach (and its extensions) is typically performed using market data, such as bond prices or yields. However, when the focus is on using beta factors from the NS model for yield forecasting or on analysing interactions between them and other variables, the yield derivation is often omitted in favour of using zero-coupon curves supplied by financial providers or central banks. The direct market data approach was applied by **Diebold, Rudebusch, Aruoba** [**2006**], who used market bid-ask quotes for US Treasury bonds. They put the NS model into space-state form and applied the Kalman filter to produce smooth estimates of the underlying beta factors. They also related the factors to the macroeconomy. **Hladíková and Radová** [**2012**] concluded that the NS model produces reasonable yield curves for the Czech Republic. Their curve estimation was based on bond market prices and yields to maturity. **Ibanez** [**2015**] applied the NS model to curve fitting presenting two approaches, the first one being a rigorous approach, from the academic point of view and the second one more pragmatic, focusing on the easiness of implementation. Both methodologies used US Treasury market data published by the Federal Reserve. **Annaert et al.** [**2013**] researched a multicollinearity problem among the NS model parameters and offered a conditional ridge regression procedure as a solution. They used short Euribor rates from the official website and euro swap curves with maturities between two and 10 years collected from Thomson DataStream and through a bootstrapping technique they constructed zero-coupon curves.

Geyer and Mader [1999] discussed the properties of the NS model and its Svensson extension in describing the interest rate structure in Western Europe, the United States and Japan. Their research based on bond prices obtained from Datastream concluded that the parsimonious NS approach outperformed Svensson's one in terms of parameter stability in time as it was less sensitive to outliers. Gilli, Große and Schumann [2010] examined the calibration of the NS model and its Svensson extension through the application of Differential Evolution – an optimisation algorithm to obtain the parameters. For their experiment they reused the data set collected by Diebold and Li [2006].

For the Polish market, Kliber [2009] used bond prices augmented with WIBOR rates for the comparison of yield construction using Nelson and Siegel [1987], Svensson [1994] and piece-wise polynomials. Sepielak and Borowski [2013] used bond prices for their yield curve estimation with the Svensson approach involving a number of limitations regarding the shape of the curve and non-negative rates. Marciniak [2006] conducted a comparative analysis to show that the Svensson extension of the NS model performed relatively well in comparison to the B-spline – Variables Roughness Penalty model in explaining the interest rate structure. He used data from the Polish bond market complemented with WIBOR rates for the short end of the curve.

The second approach, in which already estimated zero-coupon yields are used, was applied by Yu and Zivot [2010]. They used rates provided by Bloomberg for the comparison of the two-step estimation procedure proposed by Diebold and Li [2006] to the one-step Kalman filtering estimation method discussed by Diebold, Rudebusch, Aruoba [2006]. This study also examined the inclusion of the macroeconomic variables to improve the forecasting accuracy. Similarly, Rubaszek [2016] studied dynamic affine models with autoregressive, vector-autoregressive and Bayesian autoregressive processes, which include an analysis of the impact of macroeconomic variables on beta factors. He used yield curves published by the Federal Reserve. Sengupta [2010] confirmed that curves fitted with the NS model are better for actuarial valuation than curves published by the National Stock Exchange in India. The study was based on zero-coupon yields obtained from Bloomberg.

For the Polish market, Dziwok [2013] analysed which criterion should be used to fit the short end of the yield curve using the Svensson [1994] model. Her study indicated that the best results were achieved by minimising the squared difference between actual and theoretical prices weighted by the reciprocal of the duration. She used publicly available WIBOR rates. Kostyra and Rubaszek [2020] analysed interest rate forecasting by predicting beta factors through the application of the autoregression, vector-autoregression techniques as well as machine learning. They used swap rates provided by Thomson Reuters Eikon.

Our study presents a method that derives spot rates directly from Treasury bond prices, and then estimates zero-coupon yield curves using the NS approach. These curves are more accurate than those estimated by NS models where zero-coupon yields were collected from Thomson Reuters Eikon.

Data

Treasury bond market prices

For our model, we collect daily Polish Treasury bond prices[1] published by BondSpot[2], using the Python BeautifulSoup package to web-scrape the data directly from the BondSpot website, specifically the second fixing prices from the 2019:07–2021:06 period for 25 wholesale zero-coupon and fixed-coupon Treasury bonds (Table 1; all tables and figures are at the end of the article). On average, 19 bonds were used to construct a daily yield curve. In Poland, there are very few bonds with maturities longer than 10 years, therefore the curve estimation is limited for maturities up to 10 years, which is common practice in yield estimation.

1 Price is the bond value expressed in percentage of the nominal; the nominal value is PLN 1,000.

2 Treasury BondSpot Poland is a wholesale market dedicated to the trading of Polish Treasury bonds and Treasury bills. Bond prices can be found at www.bondspot.pl/fixing_obligacji.

We consider the following maturity buckets: 1Y, 2Y, 3Y, 4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y. For any given day, bonds with maturities longer than 10 years are ignored. BondSpot removes Treasury bonds with maturities shorter than three months from fixing. These bonds are typically excluded from yield curve estimation due to their low liquidity, as pointed out by **Nymand-Andersen** [**2018**].

Benchmark zero-coupon curves

We consider the zero-coupon curves based on Polish Treasury bonds provided by Thomson Reuters Eikon as the benchmark. We have gathered daily series zero-coupon curves from the same period and maturities as above. For each day we transform the series of yields (r_τ) for fixed maturities (τ) 1Y, 2Y,..., 10Y into continuously compounded yields with the following formula:

$$R_\tau = \ln\left(1+\frac{r_\tau}{100}\right)\cdot 100. \tag{1}$$

Methodology

Yields derivation

We follow the approach outlined by **Munk** [**2004**] and **Glova** [**2010**] to use matrix theory to find discount factors and associated zero-coupon spot rates. It is possible to derive *n* yields if we have *n* bonds with *n* different maturities (τ) and where cash flows (coupon and notional payments) generated by bonds span across all maturities, i.e., we need to have at least one notional payment for each maturity. The cash flows are mapped to the maturity buckets in the following way: payments occurring within one year are mapped to the 1Y maturity bucket, payments occurring between the first year and the end of the second year are mapped to the 2Y maturity bucket, etc. This will make it possible to construct a system of linear equations for each day, where the clean bond prices (*P*) equal the sum of total cash flow payments (*CF*) multiplied by discount factors (*d*), which can be expressed in the matrix notation as:

$$\boldsymbol{P}_n = \boldsymbol{CF}_{n,\tau}\cdot \boldsymbol{D}_\tau, \tag{2}$$

or:

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} = \begin{pmatrix} CF_{11} & 0 & \dots & 0 \\ CF_{21} & CF_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CF_{n1} & CF_{n2} & \dots & CF_{n\tau} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_\tau \end{pmatrix}. \tag{3}$$

Since this system of equations has *n* equations and *n* unknowns (d_τ), we will be able to find unique solutions. However, payment dates can differ, and we may have more than one payment date in a single maturity bucket. Specifically, for the Polish market coupons are paid out annually, typically in April, July or October. For illustration, suppose that on December 31 BondSpot published the prices of 11 zero-coupon and fixed-coupon Treasury bonds. Two bonds are maturing the following year, one in April (4-month maturity) and another in July (7-month maturity). The remaining nine bonds have payment dates in October and notional payments in each maturity bucket from 2Y to 10Y (in 22 months for the 2Y bucket, in 34 months for the 3Y bucket, etc.). Thus, in the 1Y maturity bucket, we have two maturing bonds at different dates (April and July of the following year). We offer two variants to deal with this situation in order to satisfy the square (10 by 10) matrix requirement. In the base model, we average the bond prices and coupons using the simple arithmetic mean. In our example, for the 1Y bucket, we would use the average price and coupon of these two bonds to populate P_1 and CF_{11} respectively. Alternatively, we only use bonds with the longest maturity in each bucket, disregarding those with shorter maturities. In the example, this means dropping the bond which matures in April and using the price and cash flow of the bond maturing in July to populate P_1 and CF_{11} respectively, hence no averaging is required.

Having found the discount factors by solving a system of linear equations for each day, we arrive at the daily spot yields for each maturity (y_τ) by using the following formula, where d_τ is the discount factor for the maturity bucket, and τ is the average maturity in this bucket:

$$y_\tau = -\ln\left(d_\tau\right) / \overline{\tau}. \tag{4}$$

In the case of multiple payment dates in a single maturity bucket, we need to find the average maturity for each bucket across all bonds, in other words, for each column in the square cash flow matrix. Continuing with the example, since there are two bonds maturing in the 1Y bucket (in April and July), we find their average maturity, which is 5.5 months and corresponds to element CF_{11}. Now we can find the mean maturity of all 10 cash flows in the first column of the matrix (1Y bucket), which is 9.6 months. Since in other maturity buckets all payments occur in October, the average maturities are equal to the payment periods, i.e., 22 months in the 2Y bucket, 34 months in the 3Y bucket, etc. In the alternative approach, since we dropped the bond maturing in April, the maturity corresponding to element CF_{11} is the maturity of the remaining bond, i.e., seven months, therefore the average maturity in the 1Y bucket is 9.7 months. All other bonds have payment dates in October, therefore the average maturities in the buckets from 2Y to 10Y are equal to the payment periods, i.e., 22 months in the 2Y bucket, 34 months in the 3Y bucket, etc.

Both approaches suffer from some simplifications employed to satisfy the square matrix requirement. The source of error in the base approach is the averaging of bond prices, coupons and maturities. In the alternative, the information loss is due to the exclusion of some of the bonds and the averaging of maturities. On the other hand, these simplifications reduce computational complexity, especially for the alternative model, and the code development time.

We have now estimated yields for each maturity of our sampled time period. However, to estimate the entire yield curve and to find the yields for the fixed maturities (1Y, 2Y,..., 10Y), we use the Nelson-Siegel interpolation approach.

Curve fitting with Nelson-Siegel (NS)

Nelson and Siegel [1987] proposed a relatively simple parametrisation of the term structure of interest rates. We need to find four parameters β_1, β_2, β_3 and λ to estimate yield Y_τ for a particular maturity τ. The β_1 parameter does not depend on the time to maturity, its weight is constant and set to 1, therefore it can be viewed as the long-term yield level. β_2 is weighted by a function of time to maturity. The limit of this function when τ approaches 0 equals 1, however as τ increases, the function exponentially decays to 0, hence β_2 impacts the short-term interest rate and can be interpreted as the slope due to exponential decay. β_3 is also weighted by a function of time to maturity, but the limit of this function when τ approaches 0 is 0, then as τ grows the function increases, reaches its maximum and declines to zero as τ approaches infinity. β_3 has the strongest impact on the medium-term interest rates; it can also be thought of as a curvature as it adds a hump to the yield curve. The λ parameter governs the speed of the exponential decay of β_2 and β_3; the higher the value of λ the faster the decay. It also determines the position of the hump. The parameters β_1, β_2, β_3 are estimated using the spot rate at different maturities derived from the market prices according to the process described in Section 4.1. For the λ parameter we define a search range between 0.2 and 1 following **Ibanez [2015]** to find the value which minimises the root mean squared error (RMSE). We also verify a special case with the value set at 0.7308, as per **Diebold and Li [2006]**[3]. We use the NS model in the functional form proposed by **Diebold and Li [2006]**.

[3] This is a scaled value from the original 0.0609, as we measure time to maturity in years, not months, we need to multiply by 12.

$$Y_\tau = \beta_1 + \beta_2 \left(\frac{1 - e^{-\tau\lambda}}{\tau\lambda} \right) + \beta_3 \left(\frac{1 - e^{-\tau\lambda}}{\tau\lambda} - e^{-\tau\lambda} \right). \tag{5}$$

Results

Zero-coupon yield curves

We produce daily zero-coupon yield curves for the 2019:07–2021:06 period using the NS model with two variants of zero-coupon yields derived from BondSpot prices and with zero-coupon curves from Thomson Reuters Eikon, as shown by Figure 1 and the descriptive statistics in Table 2. Figure 1 shows that the highest differences are for maturities of up to 1Y and to a lesser extent for maturities up to 2Y. However, the yields converge after a series of nominal interest rate decreases. The National Bank of Poland cut interest rates three times between March and May 2020, with the reference rate dropping from 1% to 0.1%. Table 2 shows that on average yields for maturities up to 7Y for the base and alternative models are mostly higher than the benchmark, with the alternative model having the highest values. For maturity 8Y the average yield is the same for all three models. For maturities longer than 8Y the benchmark model has the highest yields, with yields for the alternative model being the lowest. The standard deviations for all the models are similar in their respective maturities.

Beta factors

We provide a time series of beta factors for each model in Figure 4 and their descriptive statistics in Table 3. Figure 4 shows that the Level factors are very similar across the examined period for all the models. However, we can observe some differences for the slope and curvature factors especially at the beginning of the examined period. Table 3 shows that the average values for the level factor are very close, and the Thomson Reuters Eikon model is sandwiched between the base and the alternative, with the base model having the lowest average value and the alternative having the highest average. Standard deviations for the level factor are almost the same for all three models. The slope factor has a significantly lower average value for the Thomson Reuters Eikon model than the base and alternative, followed by the base model, with the alternative having the highest average value. The standard deviations for the slope factor are the same for the base and the alternative, with Thomson Reuters Eikon having the smallest value. The curvature factor has a significantly higher average value for the alternative model than the other two. It is followed by Thomson Reuters Eikon, with the base model having the lowest average value. The standard deviations for the curvature factor are practically the same for the base and the Thomson Reuters Eikon, with the alternative having a significantly higher standard deviation. Overall, the base model has closer beta factor statistics to the benchmark.

Root mean squared errors

In the literature, there are two approaches to verify the accuracy of yields. The verification can be done by pricing bonds with obtained rates and comparing the results to market prices. The second method relies on comparing the yields. We conduct the price verification because the pool of zero-coupon bonds in the Polish market is small. Therefore, it would be difficult, if not impossible, to do a meaningful yield comparison. We priced the Treasury bonds using curves produced by three models to generate clean theoretical prices. We present the root mean square error (RMSE) results[4] in Table 4 for different values of the λ parameter between 0.2 and 1. Also, for a selection of bonds from the sample, we provide graphs with the actual fixing prices and prices estimated with each model where λ was set to 0.7308 (Fig. 5).

4 The RMSE was computed for all the sampled bonds, including those which were not selected to derive zero – coupon yields.

For the base model, the smallest RMSE is achieved with λ set at 0.63, and for the alternative at 0.56, for which the RMSE is 0.2879 and 0.3036 respectively. These results are somewhat better compared to errors when λ is set at 0.7308 as per **Diebold and Li [2006]**, the RMSE equal to 0.2936 (base) and 0.3149 (alternative). The sensitivity analysis of the RMSE to parameter λ presented in Table 4 indicates that errors for our both models are rather insensitive to the parameter when its value is between 0.2 and 1. On the other hand, the NS benchmark model is much more sensitive to λ, with the lowest RMSE 0.3624 for λ set to 0.69. The minimal RMSE is distinctly lower than 0.4345, which is achieved when λ is fixed at 0.7308. Overall, there is an opportunity to decrease the RMSE for all three models by setting the λ value slightly below 0.7308, however the gains for the base and alternative models are negligible.

The best performing model, with the smallest RMSE, is the base model. It is constructed using averages, therefore the prices of all the bonds are reflected in the yield curves. Next in ranking is the alternative model, where we simply select bonds which are closest to the higher maturity bound of each maturity bucket. The alternative approach reduces development, as the sample always has 10 bonds with principal payments in each maturity bucket, which satisfies the square matrix requirement. The RMSE difference between the base model and alternative is in fact negligible. The RMSE is the highest when pricing the bonds with Thomson Reuters Eikon zero-coupon rates. The difference between this model and our approaches is small, yet distinct. Overall, the base model has consistently the smallest RMSE in the whole sample period. The benchmark model underperforms particularly during the period when the first COVID-19 lockdown was anticipated and finally introduced in Poland (March 2020), with the RMSE reaching an all-time high when the nominal interest rates were reduced by the central bank (Fig. 2). For all the models, the errors are the highest for maturities longer than eight years (Fig. 3).

Discussion and conclusions

This paper offers a model to estimate zero-coupon yield curves using the Nelson-Siegel approach based on the daily market prices of Treasury bonds published by BondSpot. The motivation was to offer a simple model to derive zero-coupon yields through a bootstrapping technique using matrices and then apply the NS approach with a fixed λ parameter to obtain the yield curve. The simplicity is achieved by using all the fixed coupon bonds available and applying arithmetic means to build a cash flow matrix and bond price vector and by fixing the λ parameter at 0.7308 in the NS method. The price paid for the simplification is some level of inaccuracy in the bond valuations. However, the model proved to deliver more accurate curves than zero-coupon yield curves obtained with the same method but based on zero-coupon yields provided by Thomson Reuters Eikon. The study indicates that when the yield curve is estimated with the NS model, it may pay off to derive the yields directly from market data, as opposed to using commercially available zero-coupon yields. The presented approach would be particularly useful for pricing or studies which rely on the use of beta factors, for example to examine their correlation with other variables or to use them for yield forecasting. However, the potential benefits come at the expense of the time spent to develop the code to scrape the bond prices from the BondSpot website and to solve systems of equations to obtain daily yields. Commercially available curves are also provided for longer maturities of up to 20 years.

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Table 1. Polish Treasury bonds

No	ISIN	Coupon (%)	Maturity Date
1	PL0000102646	5.75	23 Sep 2022
2	PL0000105391	5.75	25 Apr 2029
3	PL0000105441	5.50	25 Oct 2019
4	PL0000106126	5.25	25 Oct 2020
5	PL0000106670	5.75	25 Oct 2021
6	PL0000107264	4.00	25 Oct 2023
7	PL0000107611	2.75	25 Apr 2028
8	PL0000108197	3.25	25 Jul 2025
9	PL0000108510	1.50	25 Apr 2020
10	PL0000108866	2.50	25 Jul 2026
11	PL0000108916	2.00	25 Apr 2021
12	PL0000109153	1.75	25 Jul 2021
13	PL0000109427	2.50	25 Jul 2027
14	PL0000109492	2.25	25 Apr 2022
15	PL0000110151	2.50	25 Jan 2023
16	PL0000110375	0.00	25 Jul 2020
17	PL0000111191	2.50	25 Apr 2024
18	PL0000111274	0.00	25 May 2021
19	PL0000111498	2.75	25 Oct 2029
20	PL0000111720	2.25	25 Oct 2024
21	PL0000112165	0.00	25 Jul 2022
22	PL0000112728	0.75	25 Apr 2025
23	PL0000112736	1.25	25 Oct 2030
24	PL0000112900	0.00	25 Apr 2023
25	PL0000113460	0.25	25 Oct 2026

Source: https://www.gov.pl/web/finanse/bony-i-obligacje-hurtowe1.

Table 2. Zero-coupon yields – descriptive statistics

Maturity	Mean	St. Dev.	Min.	Max.	ACF	ADF
Base						
1Y	0.50	0.65	–0.17	1.64	0.99	–18.05***
2Y	0.60	0.66	–0.13	1.62	1.00	–18.63***
3Y	0.79	0.61	0.03	1.78	1.00	–18.90***
4Y	0.99	0.56	0.24	1.92	1.00	–18.97***
5Y	1.16	0.50	0.45	2.05	1.00	–18.68***
6Y	1.30	0.46	0.63	2.14	0.99	–18.23***
7Y	1.42	0.43	0.77	2.22	0.99	–17.77***
8Y	1.51	0.40	0.89	2.28	0.99	–17.37***
9Y	1.59	0.38	0.99	2.36	0.99	–17.05***
10Y	1.65	0.36	1.06	2.38	0.98	–16.80***
Alternative						
1Y	0.55	0.67	–0.13	1.81	0.99	–18.60***
2Y	0.66	0.70	–0.14	1.78	1.00	–18.71***
3Y	0.88	0.65	0.01	1.87	1.00	–18.76***
4Y	1.03	0.59	0.23	1.98	0.99	–18.82***
5Y	1.18	0.52	0.44	2.08	0.99	–18.65***
6Y	1.32	0.47	0.63	2.16	0.99	–18.31***

Maturity	Mean	St. Dev.	Min.	Max.	ACF	ADF
7Y	1.43	0.42	0.78	2.23	0.99	–17.95***
8Y	1.51	0.39	0.91	2.28	0.99	–17.61***
9Y	1.58	0.36	1.01	2.32	0.99	–17.33***
10Y	1.64	0.34	1.09	2.36	0.98	–17.10***
Thomson Reuters Eikon						
1Y	0.46	0.57	–0.21	1.38	0.99	–17.54***
2Y	0.64	0.64	–0.04	1.60	1.00	–16.33***
3Y	0.80	0.63	0.04	1.76	1.00	–14.64***
4Y	0.97	0.58	0.19	1.91	1.00	–14.07***
5Y	1.13	0.52	0.37	2.03	0.99	–13.92***
6Y	1.28	0.47	0.56	2.13	0.99	–13.92***
7Y	1.40	0.43	0.74	2.20	0.99	–14.02***
8Y	1.51	0.39	0.90	2.26	0.99	–14.18***
9Y	1.61	0.36	1.04	2.32	0.99	–14.36***
10Y	1.70	0.34	1.17	2.38	0.99	–14.51***

Notes: ACF and ADF refer to the values of the autocorrelation coefficient for the first lag and the Augmented Dickey Fuller test for the first difference. Asterisks ***, ** and * denote the rejection of the null that series is a non-stationary series at 1%, 5% and 10% significance level respectively.

Source: Own calculations.

Table 3. Beta factors – descriptive statistics

Beta factors	Mean	St. Dev.	Min.	Max.	ACF	ADF
Base						
β_1 (Level)	2.25	0.28	1.43	3.08	0.96	–15.05***
β_2 (Slope)	–1.58	0.57	–2.82	–0.12	0.98	–15.64***
β_3 (Curvature)	–2.79	1.46	–5.01	0.67	0.99	–13.32***
Alternative						
β_1 (Level)	2.19	0.26	1.26	3.03	0.95	–15.45***
β_2 (Slope)	–1.50	0.57	–2.82	–0.14	0.98	–16.61***
β_3 (Curvature)	–2.50	1.75	–4.89	1.27	0.99	–14.60***
Thomson Reuters Eikon						
β_1 (Level)	2.37	0.29	1.88	3.23	0.97	–14.62***
β_2 (Slope)	–1.96	0.41	–2.98	–1.04	0.97	–17.45***
β_3 (Curvature)	–2.74	1.45	–5.06	0.26	0.98	–18.65***

Notes: ACF and ADF refer to the values of the autocorrelation coefficient for the first lag and the Augmented Dickey Fuller tests for the first difference. Asterisks ***, ** and * denote the rejection of the null that the series is a non-stationary series at 1%, 5% and 10% significance level respectively.

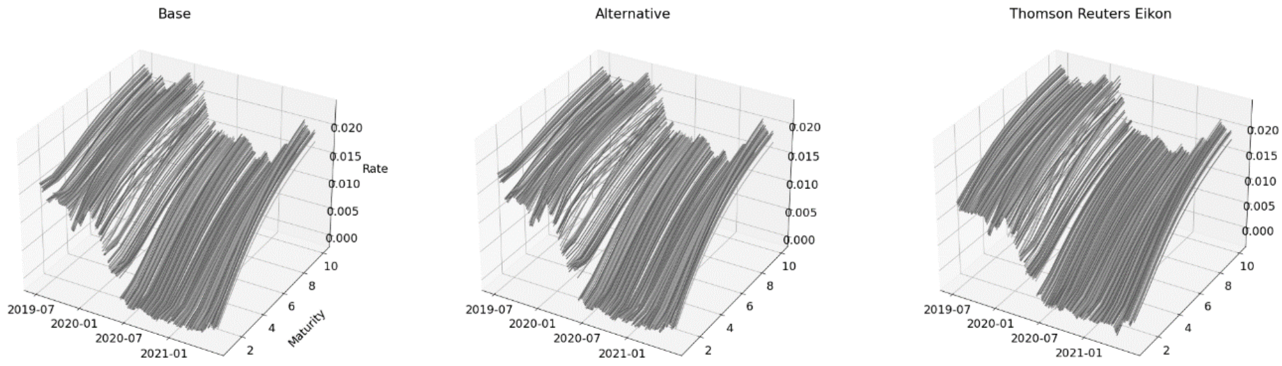
Source: Own calculations.

Table 4. Root mean squared errors (RMSE) sensitivity to λ

λ	0.2	0.4	0.6	0.7308	0.8	1.0
Base	0.3383	0.3104	0.2883	0.2936	0.3035	0.3517
Alternative	0.3320	0.3126	0.3041	0.3149	0.3259	0.3712
Thomson Reuters Eikon	4.5285	2.2624	0.6597	0.4345	0.6885	1.4239

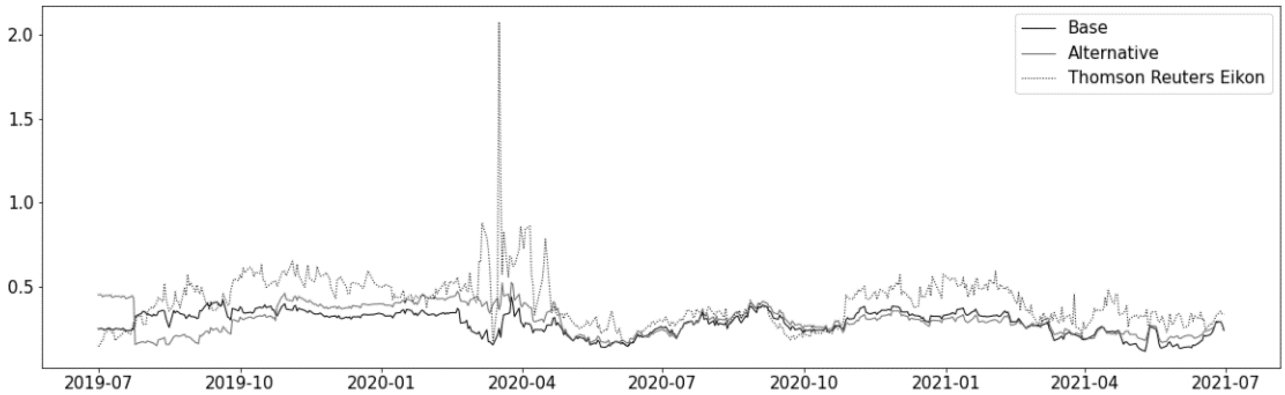
Source: Own calculations.

Figure 1. Zero-coupon curves



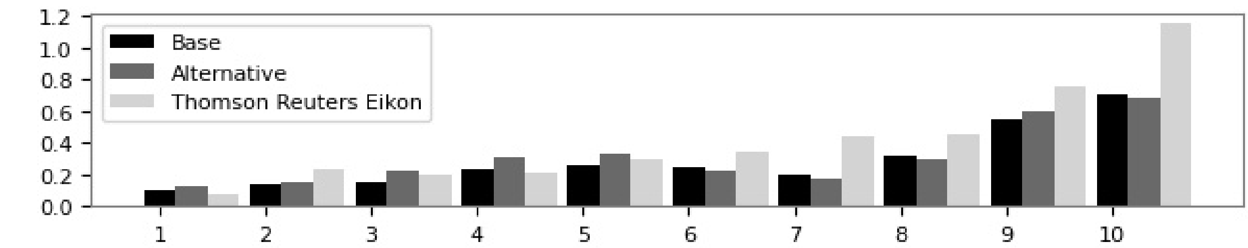
Source: Own calculations.

Figure 2. Daily RMSE



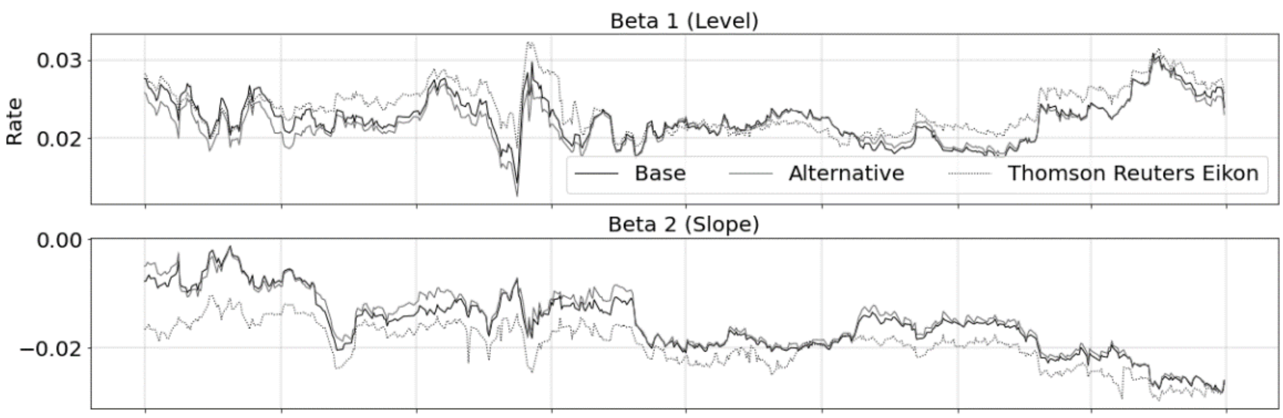
Source: Own calculations.

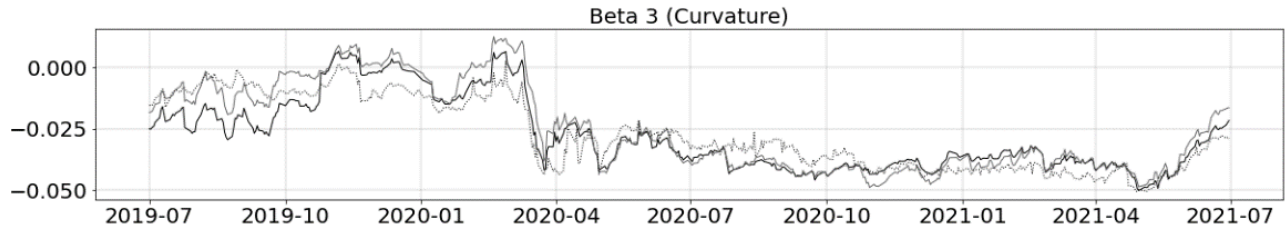
Figure 3. RMSE by maturity



Source: Own calculations.

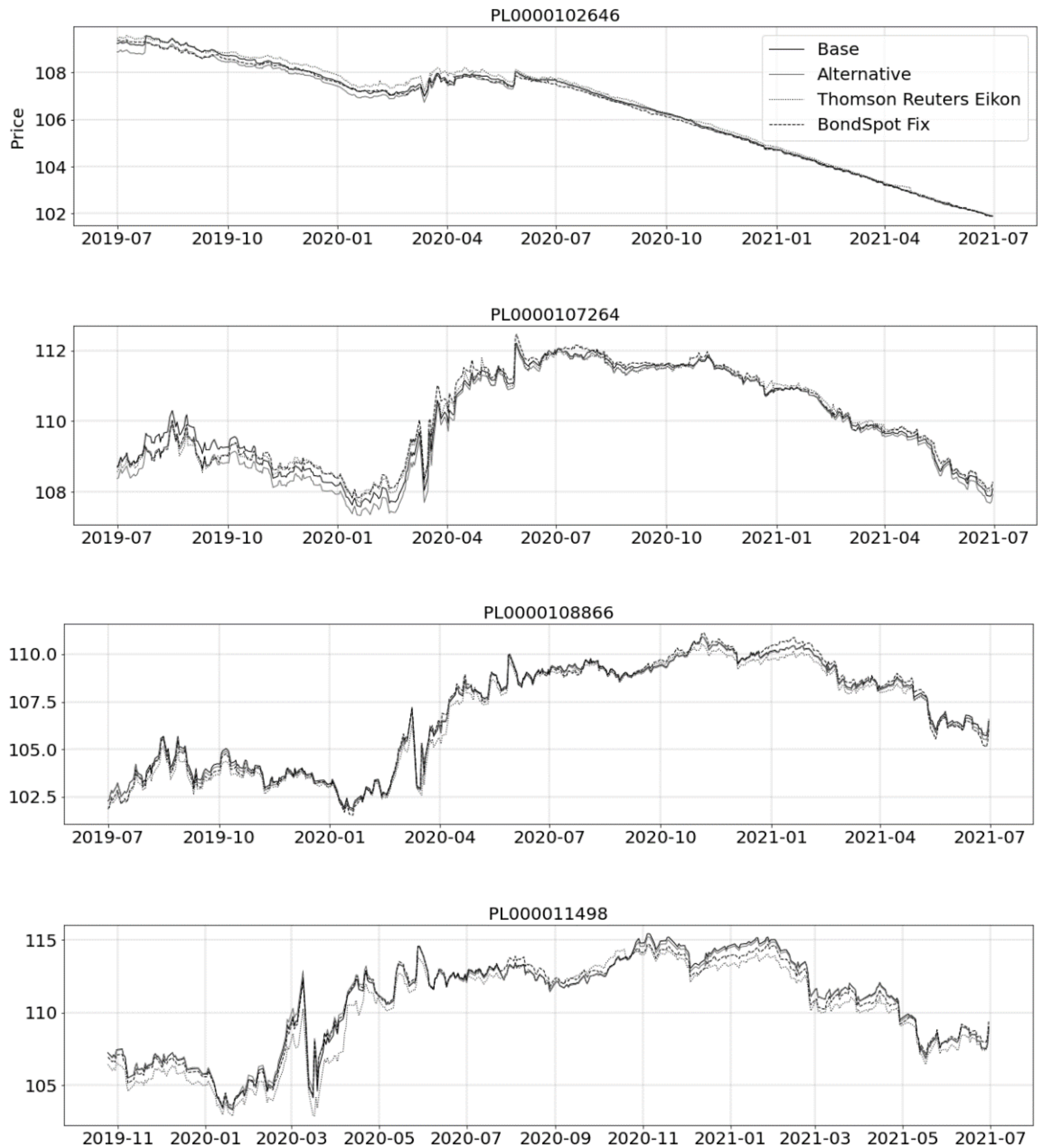
Figure 4. Beta factors

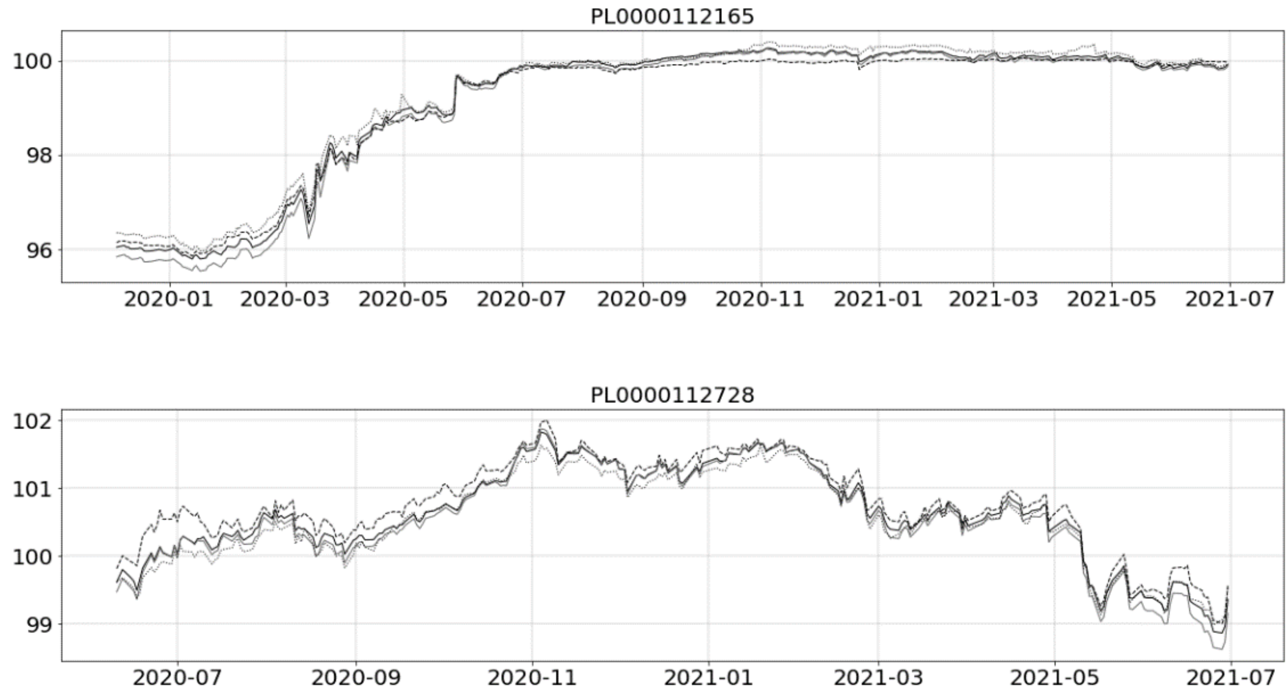




Source: Own calculations.

Figure 5. Theoretical and actual prices for a selection of Treasury bonds





Source: Own calculations.