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# Estimation of Value-at-Risk using Weibull distribution – portfolio analysis on the precious metals market

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**Abstract.** In this paper, we present a modification of the Weibull distribution for the Value-at-Risk (VaR) estimation of investment portfolios on the precious metals market. The reason for using the Weibull distribution is the similarity of its shape to that of empirical distributions of metals returns. These distributions are unimodal, leptokurtic and have heavy tails. A portfolio analysis is carried out based on daily log-returns of four precious metals quoted on the London Metal Exchange: gold, silver, platinum and palladium. The estimates of VaR calculated using GARCH-type models with non-classical error distributions are compared with the empirical estimates. The preliminary analysis proves that using conditional models based on the modified Weibull distribution to forecast values of VaR is fully justified.

**Keywords:** risk analysis, Value-at-Risk, metals market, GARCH-type models, two-sided Weibull distribution

**JEL:** C32, C58, G11, G17

## 1. Introduction

The last decade saw a growing interest in other forms of investment than those offered by the capital market, which is mainly the effect of the uncertainty and unpredictability of the global economy trends. The crisis of 2008–2009 caused some investors to transfer their capital to other, alternative markets, in order to minimise the risk involved in their investment activity. One of these alternative markets is the metals market. The level of the volatility of metals returns depends on the moods observed on the market and is directly related to the uncertainty of the trends of many economic indicators and the occurrence of unpredictable random events that may affect these trends. Moreover, uncertainty produces risk that the future return will be below the expected level. Risk is therefore a random variable and its level is determined by measures defined for this variable.

## 2. Value-at-Risk

In the literature there are numerous studies on risk measurement, many of which concern Value-at-Risk (VaR). VaR has been proposed as a measure of risk by the RiskMetrics Group (the leading provider of risk management and corporate governance products and services to financial market participants). Daníelsson et al. (2013)

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examined certain properties of VaR, which showed that the VaR risk measure is subadditive in the respective tail region of the return distribution. The authors also observed that the VaR estimates calculated using the historical simulation method can lead to a violation of the subadditivity assumption. As a result, they suggested estimating VaR by means of the semi-parametric extreme value theory (EVT). Alexander and Sarabia (2012) proposed to estimate risk related to VaR and to adjust its estimates to the estimation error and model specification. Chinhamu et al. (2015) predicted the values of VaR using EVT and the generalised Pareto distribution (GPD). Other researchers analysed the quality of VaR forecasts using GARCH-type models, e.g. Chkili et al. (2014), who applied non-linear FIAPARCH models. Yu et al. (2018) measured values of VaR using GARCH-type models and EVT jointly with copula models. The results of the backtesting showed that the GARCH-EVT and copula models were able to increase the accuracy of VaR estimations. In contrast, Cheung and Yuen (2020) introduced an uncertainty model for the distribution of returns and examined the impact of this uncertainty on VaR through the worst-case scenario approach. The researchers proved that the selection of a loss model is essential when applying an uncertainty model.

Value-at-Risk is defined as a statistical measure which indicates (in an explicit manner) the amount of a potential loss of market value of a financial asset, for which the probability of reaching or exceeding this value within a specified time horizon is equal to the tolerance level determined by the decision-maker (Doman & Doman, 2009; Dowd, 1999; Trzpiot, 2004). Another definition of VaR sees it as a measure of the maximum loss that an individual can incur within a certain time horizon for an investment realised under normal market conditions, within a predefined tolerance level (Krawczyk, 2017). Assuming random variable $X$, the mathematical definition of VaR is as follows:

$$VaR_\alpha(X) = inf\{x|F_X(x) \geq \alpha\} = F_X^{-1}(\alpha), \quad (1)$$

where $F_X^{-1}(\alpha)$ is the quantile function of random variable $X$, and $\alpha$ is the level of the quantile of the probability distribution of this random variable. In particular, random variable $X$ may represent return $r_t$ of any financial asset at time $t$.

The advantage of defining VaR through the quantile function is the possibility to apply any probability distribution of a random variable to estimate its value. Thus, the selection of a suitable probability distribution is crucial. Empirical studies on financial data show that time series are characterised by a high level of volatility, clustering of variance, significant skewness and leptokurtosis, and the presence of outliers. These features explicitly exclude the possibility of estimating VaR through symmetrical distributions, such as normal or Student's $t$-distribution. Therefore, in empirical analyses, it is necessary to use probability distributions which take into consideration the above-mentioned characteristics.

## 3. Two-sided Weibull distribution

In this study, we propose the Weibull distribution as the theoretical tool for estimating VaR. This distribution belongs to the family of extreme distributions; therefore, it considers the presence of outliers in time series, which results in a high level of asymmetry, kurtosis and heavy tails. Technically, random variable $X$ is described by the Weibull distribution if its density function takes the following form:

$$f(x;k,\lambda)=\begin{cases}\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{\left(\frac{x}{\lambda}\right)^{k}} & \text{if} \quad x\geq 0\\ 0 & \text{if} \quad x<0\end{cases}, \tag{2}$$

where $k>0$ is the shape parameter and $\lambda>0$ is the scale parameter. The density function given by formula (2) can also be defined as

$$f(x;k,b)=\begin{cases}bkx^{k-1}e^{-bx^{k}} & \text{if} \quad x\geq 0\\ 0 & \text{if} \quad x<0\end{cases}, \tag{3}$$

where $b=\lambda^{-k}$ is the scale parameter.

As mentioned above, the Weibull distribution is applied in EVT and therefore can be used to describe rare events which significantly affect the estimates of the tail risk measure for a relatively low level of the quantile. Formulas (2)–(3) demonstrate that the density function of the Weibull distribution is equal to zero for negative values of random variable $X$. Chen and Gerlach (2013) proposed a certain generalisation of the classical (one-sided) Weibull distribution over the entire set of real numbers by introducing a standardised two-sided Weibull distribution, for which the density function has the form of

$$f_{\mathrm{DW}}(x;k_1,\lambda_1,k_2,\lambda_2)=\begin{cases}b_p\left(\frac{-b_px}{\lambda_1}\right)^{k_1-1}\exp\left[-\left(\frac{-b_px}{\lambda_1}\right)^{k_1}\right] & \text{if} \quad x<0\\ b_p\left(\frac{b_px}{\lambda_2}\right)^{k_2-1}\exp\left[-\left(\frac{b_px}{\lambda_2}\right)^{k_2}\right] & \text{if} \quad x\geq 0\end{cases}, \tag{4}$$

where $k_1,k_2>0$ are shape parameters and $\lambda_1,\lambda_2>0$ are scale parameters. In addition,

$$b_p^2=\frac{\lambda_1^3}{k_1}\Gamma\left(1+\frac{2}{k_1}\right)+\frac{\lambda_2^3}{k_2}\Gamma\left(1+\frac{2}{k_2}\right)-\left[-\frac{\lambda_1^2}{k_1}\Gamma\left(1+\frac{1}{k_1}\right)+\frac{\lambda_2^2}{k_2}\Gamma\left(1+\frac{1}{k_2}\right)\right]^2 \tag{5}$$

and $\frac{\lambda_1}{k_1}+\frac{\lambda_2}{k_2}=1.$

The estimates of VaR using two-sided Weibull distribution can be obtained by using the quantile function:

$$VaR_\alpha = F^{-1}(\alpha;k_1,\lambda_1,k_2,\lambda_2) = \begin{cases} -\frac{\lambda_1}{b_p}\left[-\ln\left(\frac{k_1}{\lambda_1}\alpha\right)\right]^{\frac{1}{k_1}} & \text{if} \quad 0 \le \alpha < \frac{\lambda_1}{k_1} \\ \frac{\lambda_2}{b_p}\left[-\ln\left(\frac{k_2}{\lambda_2}(1-\alpha)\right)\right]^{\frac{1}{k_2}} & \text{if} \quad \frac{\lambda_1}{k_1} \le \alpha < 1 \end{cases}. \tag{6}$$

Considering quantile function for returns $r_t$, a one-day-ahead VaR forecast of $\alpha$-quantile is defined as

$$\alpha = P(r_{t+1} < VaR_\alpha|\mathrm{I}_t), \tag{7}$$

where $r_{t+1}$ is the return at time $t+1$, $\alpha$ is the level of the quantile, and $\mathrm{I}_t$ is the information set at time $t$. Consequently, resulting from the above, VaR is defined as the $\alpha$-quantile of a conditional distribution of $r_t$.

## 4. Empirical study

Metals are commodities used in many areas of human activity. These include heavy industry (military, construction and infrastructure), aerospace (spacecraft, orbital probes, telescopes) and the automotive industry (production of cars and car components). Metals are used in the production of household appliances, they are also used as alloys in various steel compounds, mainly to improve their quality and expand their physical properties. Metals are not only related to industry, but they are also used in the jewellery trade (mainly precious metals), medicine (including aesthetic), biotechnology and in gastronomy (gold and silver). From an investment point of view, metals, being commodities quoted on stock exchanges, can be the subject of financial investments (direct and indirect). The above refers primarily to precious metals, which are an alternative form of investing if compared to the classical capital market assets, such as stocks or bonds.

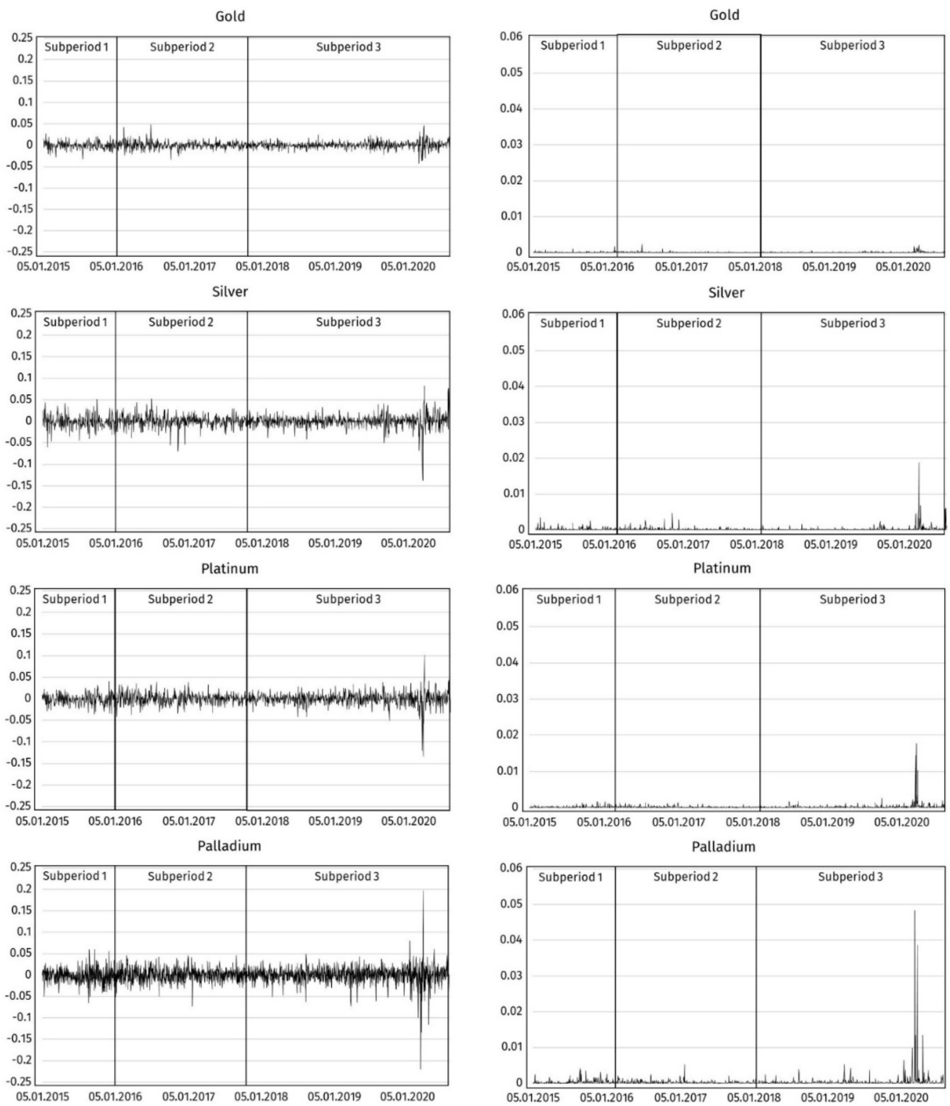
The metals market is not a popular area of interest among researchers, although the number of papers on risk analysis in this area has clearly increased in the recent years. However, research is mainly concerned with gold. Zijing and Zhang (2016) analysed the volatility and risk of precious metals returns using GARCH-type models with a random error described by the GED distribution, while Włodarczyk (2017) analysed the impact of asymmetry and long memory effects on forecasting conditional volatility and the risk of gold and silver using linear and non-linear GARCH models. Chen and Qu (2019) analysed the risk and volatility of precious metals returns using copula and dynamic conditional correlation (DCC) models. In turn, Krężołek (2020) has conducted extensive research on risk modelling of the base and precious metals markets. The author showed in his research, among other things, that fat-tailed distributions (including alpha-stable distributions) and ARMA-GARCH-type models should be used for risk modelling. Other methods were proposed by Wang et al. (2019), who predicted the volatility and risk of copper prices by comparing complex hybrid networks with traditional artificial neural network techniques. The results demonstrated that the proposed hybrid models were able to achieve a favourable predictive effect both in forecasting the levels of risk and volatility in copper prices.

In this study we use daily log-returns of four precious metals: gold, silver, platinum and palladium for the construction of investment portfolios. The data come from the London Metal Exchange (LME) from the period of January 2015–July 2020, which has further been divided into three sub-periods:

- sub-period 1 (2015): portfolio construction;
- sub-period 2 (2016–2017): model estimation;
- sub-period 3 (2018–2020): forecasting of VaR.

The main goal of this research is to estimate the Value-at-Risk of investment portfolios using selected models of conditional volatility (GARCH-type models) with error terms described by the following non-classical probability models: Student's $t$-distribution, skewed Student's $t$-distribution, GED, skewed GED and the two-sided Weibull distribution. Four investment portfolios have been constructed, for which the values of VaR (for the quantile of 0.01 and 0.05) have been estimated, according to the proposed theoretical model. The quality of the forecasts has been assessed using the test of exceedance proposed by Kupiec (1995) and the independence test introduced by Christoffersen (1998). Figure 1 presents returns and squares of returns of all the studied precious metals, while Table 1 presents the descriptive statistics of returns by sub-periods.

**Figure 1.** Log-returns (left) and squares of log-returns (right) of selected precious metals between January 2015 and July 2020




Source: author's work based on data from LME.

The first and second sub-period saw a comparatively stable level of variance, while in the third sub-period a relatively significant clustering of volatility was observed (early 2020), which resulted from the socio-economic condition in the worldwide economy caused by the COVID-19 pandemic. Moreover, the data show that gold returns, compared to other metals, did not react strongly to the information from the market during the pandemic period. This results from the fact that gold is perceived as a 'safe haven' in times of increasing uncertainty in the global economy (Salisu et al., 2021). However, some studies indicate that during the pandemic, for some assets, gold lost its 'safe haven' property (Cheema et al., 2020).

**Table 1.** Descriptive statistics of log-returns for three sub-periods

| Statistics | Gold | Silver | Platinum | Palladium |
|---|---|---|---|---|
| **Sub-period 1** | | | | |
| Mean | −0.00044 | −0.00051 | −0.00115 | −0.00134 |
| Standard deviation | 0.00858 | 0.01488 | 0.01228 | 0.01862 |
| Coefficient of variation in % | −1966.40 | −2922.26 | −1066.04 | −1392.01 |
| Skewness | 0.00002 | −0.06384 | −0.07000 | 0.01858 |
| Kurtosis | 1.16474 | 1.94523 | 0.41096 | 1.11559 |
| Minimum | −0.03280 | −0.05967 | −0.03641 | −0.06474 |
| Maximum | 0.02712 | 0.05112 | 0.04046 | 0.05992 |
| **Sub-period 2** | | | | |
| Mean | 0.00040 | 0.00039 | 0.00008 | 0.00124 |
| Standard deviation | 0.00819 | 0.01309 | 0.01200 | 0.01573 |
| Coefficient of variation in % | 2053.50 | 3324.77 | 14880.94 | 1271.04 |
| Skewness | 0.46418 | −0.32880 | 0.08377 | −0.23870 |
| Kurtosis | 3.86023 | 2.96838 | 0.74131 | 1.02741 |
| Minimum | −0.03300 | −0.06882 | −0.04139 | −0.07233 |
| Maximum | 0.04867 | 0.05258 | 0.03814 | 0.04602 |
| **Sub-period 3** | | | | |
| Mean | 0.00061 | 0.00049 | −0.00004 | 0.00101 |
| Standard deviation | 0.00821 | 0.01548 | 0.01568 | 0.02275 |
| Coefficient of variation in % | 1354.91 | 3157.36 | −42793.61 | 2242.77 |
| Skewness | −0.06169 | −0.76277 | −1.29900 | −0.99925 |
| Kurtosis | 4.68469 | 14.09777 | 15.30709 | 24.14889 |
| Minimum | −0.04196 | −0.13719 | −0.13300 | −0.21994 |
| Maximum | 0.04605 | 0.08243 | 0.10163 | 0.19665 |

Source: author's calculations based on data from LME.

In the first sub-period, compared to the others, the returns of all precious metals had a negative average value. Regardless of the sub-period, a high level of volatility is observed. Additionally, the empirical distributions of returns are skewed and leptokurtic in all the sub-periods (especially in the third one). Based on the data from the first sub-period, investment portfolios of three components have been constructed in such a way that each portfolio contains a different combination of components:

- $P_1$ – gold, silver, platinum;
- $P_2$ – gold, silver, palladium;

- $P_3$ – gold, platinum, palladium;
- $P_4$ – silver, platinum, palladium.

Optimal portfolios have been determined with the assumption that there is no possibility of short selling, whereas the optimisation criterion involves the minimisation of the portfolio's risk (measured by variance). Weights of metals in optimal portfolios are presented in Table 2, whereas the expected return and risk for equally weighted and optimal portfolios are presented in Table 3 and Figure 2.

**Table 2.** Weights of components in optimal portfolios

| Metal | $P_{1opt.}$ | $P_{2opt.}$ | $P_{3opt.}$ | $P_{4opt.}$ |
|---|---|---|---|---|
| | in % | | | |
| Gold | 100.00 | 96.72 | 96.72 | . |
| Silver | 0.00 | 0.00 | . | 20.79 |
| Platinum | 0.00 | . | 0.00 | 73.09 |
| Palladium | . | 3.28 | 3.28 | 6.13 |

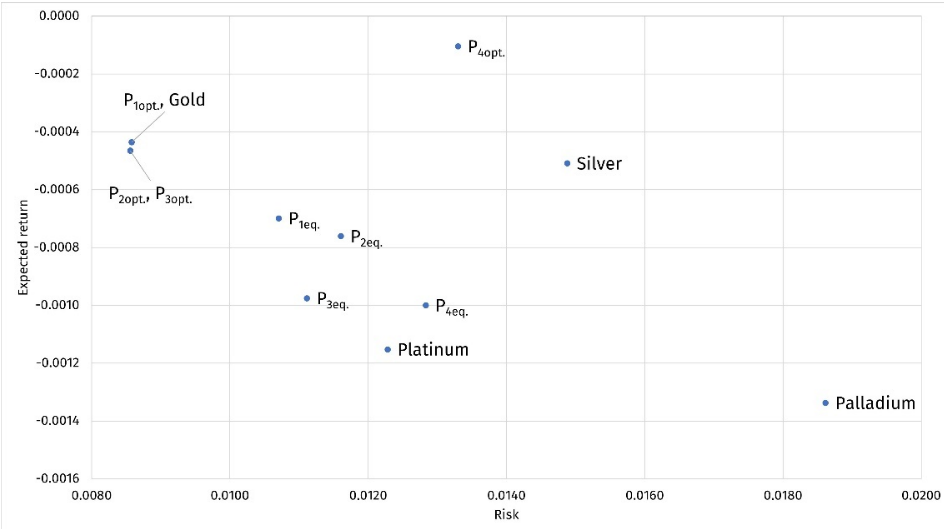
Source: author's calculations based on data from LME.

**Table 3.** The risk and expected return for equally weighted and optimal portfolios in sub-period 1

| Portfolio | Risk | Expected return |
|---|---|---|
| **Equally weighted** | | |
| $P_{1eq.}$ | 0.01071 | –0.00070 |
| $P_{2eq.}$ | 0.01161 | –0.00076 |
| $P_{3eq.}$ | 0.01112 | –0.00098 |
| $P_{4eq.}$ | 0.01283 | –0.00100 |
| **Optimal** | | |
| $P_{1opt.}$ | 0.00858 | –0.00044 |
| $P_{2opt.}$ | 0.00856 | –0.00047 |
| $P_{3opt.}$ | 0.00856 | –0.00047 |
| $P_{4opt.}$ | 0.00010 | –0.00010 |
| **Individual assets** | | |
| Gold | 0.00858 | –0.00044 |
| Silver | 0.01488 | –0.00051 |
| Platinum | 0.01228 | –0.00115 |
| Palladium | 0.01862 | –0.00134 |

Source: author's calculations based on data from LME.

**Figure 2.** The risk and expected return for equally weighted and optimal portfolios, and for individual assets




Source: author's work based on data from LME.

As a result of optimisation, the level of risk decreased for all portfolios and, in addition, the expected loss was reduced. Optimal portfolios $P_{2opt.}$ and $P_{3opt.}$ have the same characteristics because the optimisation resulted in the same components for these two portfolios (in the further part of the analysis, these two portfolios are denoted as one, namely $P_{2.3opt.}$). Moreover, individual investments show a higher level of risk than optimal portfolios. Gold remains the only exception, for which both a low level of risk and a relatively low level of the expected loss are observed.

In the next step of the analysis, involving data from the second sub-period, the parameters of conditional volatility models for optimal portfolio returns have been estimated at GARCH(1,1) and APARCH(1,1) for different error distributions. The conditional variance equations for the GARCH (Bollerslev, 1986) and APARCH models (Ding et al., 1993) are denoted by the following formulas:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{8}$$

$$\sigma_t^\delta = \alpha_0 + \sum_{i=1}^{q} \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^\delta + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^\delta, \tag{9}$$

where $\alpha_0 \geq 0$, $\alpha_i \geq 0$ for $i > 0$, $\beta_j \geq 0$, $\sum_{i=1}^{max(p,q)} (\alpha_i + \beta_i) < 1$, $\varepsilon_t \sim N(0,1)$ and $\varepsilon_t$ is iid. Based on the characteristics of the time series of metals returns

(Krężołek, 2020), the following error distributions for conditional models are proposed:

- Student's $t$-distribution:

$$f_{\text{st.}}(\varepsilon_t, \sigma_t^2; \theta) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sigma_t \Gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)}}\left(1 + \frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{\frac{v+1}{2}}, \tag{10}$$

where $v$ is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter $k$;

- Skewed Student's $t$-distribution:

$$f_{\text{sst.}}(x, v) = \frac{2}{\zeta + \frac{1}{\zeta}}\{g(\zeta(ax+b); v)I_{x<-\frac{b}{a}} + g\left(\frac{ax+b}{\zeta}; v\right)I_{x\geq -\frac{b}{a}}, \tag{11}$$

where $a = \frac{\Gamma\left(\frac{v-1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}\left(\zeta - \frac{1}{\zeta}\right)$, $b^2 = \left(\zeta^2 + \frac{1}{\zeta^2} - 1\right) - a^2$; $\zeta$ is the skewness parameter, and $g(\cdot)$ is the density function of a standard Student's $t$-distribution with $v$ degrees of freedom;

- GED distribution:

$$f_{\text{GED}}(\varepsilon_t, \sigma_t^2; \theta) = 2^{-\frac{v+1}{v}} \frac{v}{\sigma_t\sqrt{\frac{\Gamma(v^{-1})}{\Gamma(3v^{-1})}2^{-\frac{2}{v}}\Gamma(v^{-1})}} exp\left\{-\frac{1}{2}\left|\frac{\varepsilon_t}{\sigma_t\sqrt{\frac{\Gamma(v^{-1})}{\Gamma(3v^{-1})}2^{-\frac{2}{v}}}}\right|^{v}\right\}, \tag{12}$$

where $v$ is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter $k$;

- Skewed GED distribution:

$$f_{\text{sGED}}(x) = \frac{k^{1-\frac{1}{k}}}{2\sigma}\Gamma\left(\frac{1}{k}\right)^{-1}\exp\left(-\frac{1}{k}\frac{|u|^k}{(1+\text{sgn}(u)\zeta)^k\sigma^k}\right), \tag{13}$$

where $u = x - m$ ($m$ – the mode of random variable $X$), $\sigma$ is the scale parameter, $\zeta$ is the skewness parameter, $k$ is the kurtosis parameter, $\text{sgn}(\cdot)$ is the sign function, $v$ is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter $k$.

In addition, the standard two-sided Weibull distribution has also been applied, with the density function given by formula (4). The final model of the conditional volatility for a given optimal portfolio has been selected using the Akaike Information Criterion (AIC). The values of the AIC criterion are presented in Table 4.

**Table 4.** AIC information criterion for GARCH and APARCH models for optimal portfolios

| Model | $P_{1opt.}$ | $P_{2.3opt.}$ | $P_{4opt.}$ |
|---|---|---|---|
| $GARCH_{st.}(1,1)$ | –5452.99[a] | –5457.55 | –4823.82 |
| $GARCH_{sst.}(1,1)$ | –5451.01 | –5457.98 | –4823.37 |
| $GARCH_{GED}(1,1)$ | –5449.08 | –5455.38 | –4824.32 |
| $GARCH_{sGED}(1,1)$ | –5447.11 | –5453.39 | –4825.39 |
| $GARCH_{dW}(1,1)$ | –5447.98 | –5458.43[a] | –4829.65[a] |
| $APARCH_{st.}(1,1)$ | –5455.49[b] | –5460.55[b] | –4820.54 |
| $APARCH_{sst.}(1,1)$ | –5453.62 | –5458.63 | –4820.06 |
| $APARCH_{GED}(1,1)$ | –5450.34 | –5455.45 | –4820.72 |
| $APARCH_{sGED}(1,1)$ | –5448.39 | –5453.51 | –4821.83 |
| $APARCH_{dW}(1,1)$ | –5453.86 | –5419.51 | –4829.76[b] |

a The lowest value of AIC for GARCH models. b The lowest value of AIC for APARCH models.
Source: author's calculations based on data from LME.

The GARCH and APARCH models with error terms described by Student's $t$-distribution were selected for the first portfolio $P_{1opt.}$. For portfolio $P_{2.3opt.}$, the most convenient GARCH model is the one with an error term described by the two-sided Weibull distribution and the APARCH model with an error term described by Student's $t$-distribution. The GARCH and APARCH models with error terms described by two-sided Weibull distribution were selected for the last portfolio $P_{4opt.}$.

In the last phase of the study, one-day-ahead VaR forecasts are calculated for the data from the third sub-period. For this purpose, models of conditional volatility selected on the basis of the AIC criterion have been used. The verification of the number of exceedances has been carried out on the average VaR forecasts from the third sub-period for all optimal portfolios using the Kupiec ($LR_{POF}$) and Christoffersen ($LR_{IND}$) tests. All results are presented in Table 5 ($VaR_{0.01}$) and 6 ($VaR_{0.05}$).

**Table 5.** Average one-day-ahead $VaR_{0.01}$ forecasts within the third sub-period (Kupiec test and Independence test)

| Volatility model | $VaR_{0.01}$ | % of failure | Kupiec test | | Independence test | |
|---|---|---|---|---|---|---|
| | | | $LR_{POF}$ | $p$-value | $LR_{IND}$ | $p$-value |
| | | | **$P_{1opt.}$** | | | |
| Empirical | −0.02968 | 0.00978 | 0.00202 | 0.96419 | 0.82214 | 0.36346 |
| $GARCH_{st.}(1,1)$ | −0.03196 | 0.00733 | 0.32334 | 0.56961 | 1.15428 | 0.28265 |
| $APARCH_{st.}(1,1)$ | −0.03041 | 0.00978 | 0.00202 | 0.96419 | 1.73445 | 0.18784 |
| | | | **$P_{2.3opt.}$** | | | |
| Empirical | −0.02817 | 0.00978 | 0.00202 | 0.96419 | 0.82214 | 0.36346 |
| $GARCH_{dW}(1,1)$ | −0.02834 | 0.00978 | 0.00202 | 0.96419 | 0.82214 | 0.36346 |
| $APARCH_{st.}(1,1)$ | −0.02761 | 0.01222 | 0.19098 | 0.66211 | 2.51775 | 0.11257 |
| | | | **$P_{4opt.}$** | | | |
| Empirical | −0.04378 | 0.00978 | 0.00202 | 0.96419 | 0.82214 | 0.36346 |
| $GARCH_{dW}(1,1)$ | −0.04715 | 0.00978 | 0.00202 | 0.96419 | 1.15428 | 0.28265 |
| $APARCH_{dW}(1,1)$ | −0.04337 | 0.01222 | 0.19098 | 0.66211 | 2.51775 | 0.11257 |

Source: author's calculations based on data from LME.

**Table 6.** Average one-day-ahead $VaR_{0.05}$ forecasts within the third sub-period (Kupiec test and Independence test)

| Volatility model | $VaR_{0.05}$ | % of failure | Kupiec test | | Independence test | |
|---|---|---|---|---|---|---|
| | | | $LR_{POF}$ | $p$-value | $LR_{IND}$ | $p$-value |
| | | | **$P_{1opt.}$** | | | |
| Empirical | −0.01474 | 0.04890 | 0.01050 | 0.91840 | 0.71170 | 0.39888 |
| $GARCH_{st.}(1,1)$ | −0.01538 | 0.04156 | 0.64838 | 0.42069 | 1.78352 | 0.18172 |
| $APARCH_{st.}(1,1)$ | −0.01489 | 0.04645 | 0.11074 | 0.73931 | 1.17532 | 0.27831 |
| | | | **$P_{2.3opt.}$** | | | |
| Empirical | −0.01335 | 0.04890 | 0.01050 | 0.91840 | 0.71170 | 0.39888 |
| $GARCH_{dW}(1,1)$ | −0.01494 | 0.04156 | 0.64838 | 0.42069 | 1.78352 | 0.18172 |
| $APARCH_{st.}(1,1)$ | −0.01403 | 0.04890 | 0.01050 | 0.91840 | 0.92672 | 0.33572 |
| | | | **$P_{4opt.}$** | | | |
| Empirical | −0.02177 | 0.04890 | 0.01050 | 0.91840 | 0.71170 | 0.39888 |
| $GARCH_{dW}(1,1)$ | −0.02245 | 0.04645 | 0.11074 | 0.73931 | 0.01589 | 0.89969 |
| $APARCH_{dW}(1,1)$ | −0.02293 | 0.04645 | 0.11074 | 0.73931 | 0.01589 | 0.89969 |

Source: author's calculations based on data from LME.

The empirical forecasts of VaR for optimal portfolios differ depending on the model and the quantile level. VaR forecasts estimated using GARCH models, regardless of the assumed probability distribution for the error, were overestimated, while forecasts estimated using APARCH models were usually underestimated. Using the convergence criterion as the minimum value of the root mean square error (RMSE), the APARCH models allowed the estimation of the forecasts of VaR at a level relatively close to the empirical estimates. Referring to the results obtained in the context of the probability distribution for the error term, the models estimated by using two-sided Weibull distribution provided correct and accurate predictions of VaR. This was also confirmed by the results of the Kupiec and Christoffersen tests.

## 5. Conclusions

In this study we proposed the application of the two-sided Weibull distribution to forecast the values of VaR for investment portfolios on the precious metals market. A selection of conditional volatility models was used. The choice to apply the Weibull distribution resulted from the observed properties of precious metals' returns, including high-level volatility, clustering of variance, asymmetry and kurtosis, as well as the existence of outliers, which significantly affect the values of probability measured in the tail of the distribution. GARCH and APARCH models with non-classical error distributions were selected to describe the conditional volatility. The analysis was carried out for daily log-returns of four precious metals quoted on the LME between January 2015 and July 2020. This period was divided into three sub-periods, namely the construction of portfolios, model estimation and the forecasting of VaR. VaR was estimated at the quantile level of 0.01 and 0.05 for portfolio returns.

The results of the analysis show that the optimisation of portfolios on the precious metals market led to a simultaneous reduction in the level of risk and in the value of expected loss. The application of the AIC information criterion allowed the selection of conditional volatility models for each of the portfolios; these models had error terms described by Student's $t$-distribution and two-sided Weibull distributions. In the last phase of the research, one-day-ahead VaR forecasts were calculated on the basis of selected models. It was observed that, regardless of the error distribution, GARCH models overestimated and APARCH models underestimated the empirical values of VaR. The study also proved that the VaR estimates were accurate due to the use of models with an error term described by the two-sided Weibull distribution, which was confirmed by the Kupiec and Christoffersen tests. In conclusion, the two-sided Weibull distribution is an appropriate theoretical tool to determine forecasts of Value-at-Risk for investment portfolios on the precious metals market.

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