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The minimal-time growth problem and 'very strong' turnpike theorem

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Abstract. This paper refers to the author's previous work, in which the 'weak' turnpike theorem in the stationary Gale economy was proved. This theorem states that each optimal growth process $\{y^*(t)\}_{t=0}^{t_1^*}$ that leads the economy in the shortest possible time t_1^* from the (initial) state of y^0 to the set of target/postulated states Y^1 almost always runs in the neighbourhood of the production turnpike, where the economy remains in a specific dynamic equilibrium (peak growth equilibrium). This paper presents a proof of the 'very strong' turnpike theorem in the stationary Gale economy, which states that if the optimal process (the solution to the minimal-time growth problem) reaches a turnpike in a certain period of time $\check{t} < t_1^* - 1$, then it stays on it everywhere else, except for, at most, final period t_1^*. The obtained result confirms the well-known Samuelson hypothesis about the specific turnpike stability of optimal growth paths in multiproduct/multisectoral von Neumann-Leontief-Gale-type models, also in the case where the growth criterion is not the (normally assumed) utility of production but the time needed by the economy to achieve the postulated target level or volume of production.

Keywords: stationary Gale economy, von Neumann equilibrium, minimum-time growth problem, turnpike effect

JEL: C62, C67, O41, O49

1. Introduction

There are several turnpike theorems (production, capital, consumption turnpikes, etc.) in the literature proved in various multi-product/multi-sector input-output models of economic dynamics – see e.g. Babaei (2019), Babaei et al. (2020), Giorgi and Zuccotti (2016), Jensen (2012), Khan and Piazza (2011), Majumdar (2009), Makarov and Rubinov (1977), Nikaido (1968, chap. 4), Panek (2003, part 2, chap. 5–6; 2014, 2015, 2019), Takayama (1985, chap. 6–7). An extensive bibliography on the turnpike theory is presented in McKenzie (2005), Mitra and Nishimura (2009), Panek (2011), and others. The role of the growth criterion is most frequently embraced by the production utility generated in the economy either in the last period of defined horizon $T = \{0, 1, \dots, t-1\}$ or in all periods of the horizon. A paper by Panek (2021), however, presents a different approach – it proves a 'weak' turnpike theorem in the stationary Gale economy with n products and with a single production turnpike. In that paper, the time needed for an economy starting from a fixed initial state (production vector) of $y^0 = (y_1^0, \dots, y_n^0) > 0$ to reach the

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desirable target set of states (production vectors)[1] $Y^1 = \{y \in R^n_+ | y \geqq y^1\}, y^1 > y^0$ assumes the role of the growth criterion. According to this theorem, almost all optimal growth processes[2] — regardless of the distance of the target set of states Y^1 from initial vector y^0 – take place in an arbitrarily close (in the angular measure sense) neighbourhood of the production turnpike, where the economy develops at the maximum rate and achieves the highest technological and economic efficiency. It is a state of a specific dynamic equilibrium (peak equilibrium of growth) in the Gale economy.

This paper refers to the aforementioned article and contains a proof of a 'very strong' turnpike theorem. It states that if optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$ (the solution to the minimal-time growth problem) reaches the turnpike in a certain period of $\check{t} < t_1^* - 1$, it remains on it everywhere else, except for, at most, the last period of horizon $\{0, 1, \dots, t_1^*\}$. The potential precipitation of the economy from the turnpike in period t_1^* results from the necessity to reach the target set of states Y^1.

The paper further consists of Section 2, where a model of the stationary Gale economy is presented and selected properties of the production turnpike and the von Neumann equilibrium state are defined and discussed, Section 3, which presents the minimal-time growth issue and the conditions for the existence of a feasible stationary and optimal growth process, Section 4, which provides the formulation and proof of a 'very strong' turnpike theorem (Theorem 3), and Section 5, which features a certain specific version of the 'very strong' turnpike theorem in the stationary Gale economy with a single turnpike and a minimum-time growth criterion (Theorem 3'). The paper concludes with the author's indication of the possible directions for further development of the current research.

2. Technological and economic production efficiency. Von Neumann equilibrium[3]

In the economy we have $n < +\infty$ consumed and/or produced commodities. We consider a model with discrete time $t = 0, 1, \dots$. By $x = (x_1, \dots, x_n)$ we denote the input vector that is used in the economy in a specific unit of time, e.g. for a year (we also call it a production factors vector), and by $y = (y_1, \dots, y_n)$ the output vector that is produced in a unit of time (also called a production vector). If the technology at the disposal of the economy allows the achievement of production y from inputs x, then the pair (x, y) is said to create (describe) a technologically feasible

[1] If $a, b \in R^n$, then $a \geqq b$ means that $\forall i (a_i \geq b_i)$. The notation $a \geq b$ means that $a \geqq b$ and $a \neq b$. We define notation $a \leqq b$ similarly.

[2] Leading in the shortest time from initial state y^0 to set Y^1.

[3] The notation used further in this paper refers to Panek (2021).

production process.[4] Non-empty set $Z \subset R_+^{2n}$ of all technologically feasible production processes is called the Gale production space (or the technological set) if the following conditions are met:

(G1) $\forall (x^1, y^1) \in Z \; \forall (x^2, y^2) \in Z \; \forall \, \lambda_1, \lambda_2 \geq 0 \; (\lambda_1(x^1, y^1) + \lambda_2(x^2, y^2) \in Z)$
(inputs/outputs proportionality condition and the additivity of production processes),

(G2) $\forall (x, y) \in Z \; (x = 0 \Rightarrow y = 0)$
('no cornucopia' condition),

(G3) $\forall (x, y) \in Z \; (x' \geqq x \Longrightarrow (x', y) \in Z) \; \& \; \forall (x, y) \in Z 0 \leqq y' \leqq y \Longrightarrow (x, y') \in Z$
(possibility of wasting the inputs/outputs),

(G4) production space Z is a closed subset of R_+^{2n}.

The Gale production set is a closed cone in R_+^{2n} with a vertex at 0. If $(x, y) \in Z$ and $x = 0$, then, according to (G2), also $y = 0$. We are only interested in processes $(x, y) \in Z \backslash \{0\}$. The number

$$\alpha(x, y) = \max\{\alpha | \; \alpha x \leqq y\}$$

is called the index of the technological efficiency of process $(x, y) \in Z \backslash \{0\}$. It follows from the definition that function $\alpha(\cdot)$ is non-negative and positively homogeneous of degree 0 on $Z \backslash \{0\}$.

□ Theorem 1. If conditions (G1)–(G4) are satisfied, then a solution to the problem exists:

$$\max_{(x,y) \in Z \backslash \{0\}} \alpha(x, y) = \alpha(\bar{x}, \bar{y}) \geq 0.$$

For proof, see Panek (2022, th.1), Takayama (1985, th. 6.A.1). ■

The number α_M is called the optimal indicator of the technological production efficiency. Process $(\bar{x}, \bar{y}) \in Z \backslash \{0\}$ is called the optimal production process. In the stationary Gale economy it is determined with the accuracy of a multiplication

[4] Due to the technology that the economy has at its disposal.

by a positive constant (with a structure accuracy); if $\alpha(\bar{x},\bar{y}) = \alpha_M$, then $\forall\, \lambda > 0(\alpha(\lambda\bar{x},\lambda\bar{y}) = \alpha_M)$.

We are interested in an economy where in optimal production process $(\bar{x},\bar{y})$ all commodities are produced and the production of the commodities exceeds (on all coordinates) the inputs.

This is ensured by the following condition:

(G5) $\exists(\bar{x},\bar{y}) \in Z\backslash\{0\}(\alpha(\bar{x},\bar{y}) = \alpha_M > 1 \,\&\, \bar{y} > 0)$.

An economy that satisfies condition (a) is called regular, and an economy which meets condition (b) is called productive. If condition (G5) is met, then due to (G3):

$$\exists(\bar{x},\bar{y}) \in Z\backslash\{0\}(\bar{y} = \alpha_M\bar{x} > 0).$$

Everywhere else, when we talk about optimal process $(\bar{x},\bar{y})$, we mean the production process that meets the above-mentioned condition. We say that vector

$$\bar{s} = \frac{\bar{y}}{\|\bar{y}\|}$$

represents the production structure in optimal process $(\bar{x},\bar{y})$[5]. Ray

$$N = \{\lambda\bar{s}|\lambda > 0\} \subset R_+^n$$

is called the production turnpike (the von Neumann ray) in the stationary Gale economy.

By $p = (p_1, \dots, p_n) \geq 0$ we denote the commodity price vector in the Gale economy. Let $(x,y) \in Z\backslash\{0\}$. Then $\langle p,x\rangle = \sum_{i=1}^n p_i x_i$ is the inputs value and $\langle p,y\rangle = \sum_{i=1}^n p_i y_i$ the production value in process (x,y). The number

$$\beta(x,y,p) = \frac{\langle p,y\rangle}{\langle p,x\rangle}$$

$(\langle p,x\rangle \neq 0)$ is called the index of the economic efficiency of process (x,y). Let $(\bar{x},\bar{y}) \in Z\backslash\{0\}$ be the optimal production process in the Gale economy. Then

$$\alpha_M\bar{x} = \bar{y} > 0. \tag{1}$$

[5] If $a \in R_+^n\backslash\{0\}$, then $\|a\| = \sum_{i=1}^n a_i$ and $\frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \dots, \frac{a_n}{\|a\|}\right)$.

□ Theorem 2. If conditions (G1)–(G5) are satisfied, then such a price vector $\bar{p} \geq 0$ exists that

$$\forall (x, y) \in Z(\langle \bar{p}, y \rangle \leq \alpha_M \langle \bar{p}, x \rangle). \tag{2}$$

For proof, see e.g. Panek (2003; chap. 5, th. 5.4). ■

Since in optimal process $(\bar{x}, \bar{y})$ the production vector is positive and the price vector is at least semi-positive, then

$$\langle \bar{p}, \bar{y} \rangle > 0. \tag{3}$$

From (1)–(3) it follows that

$$\beta(\bar{x}, \bar{y}, \bar{p}) = \frac{\langle \bar{p}, \bar{y} \rangle}{\langle \bar{p}, \bar{x} \rangle} = \max_{(x,y) \in Z \setminus \{0\}} \beta(x, y, \bar{p}) = \alpha_M.$$

We say that the triple $\{\alpha_M, (\bar{x}, \bar{y}), \bar{p}\}$ represents the (optimal) von Neumann equilibrium state in the stationary Gale economy. Price vector $\bar{p}$ is called the von Neumann price vector. In the equilibrium state, the technological production efficiency matches its economic efficiency (at the maximum possible level of α_M that can be achieved by the economy).

In the von Neumann equilibrium state production process $(\bar{x}, \bar{y})$ and price vector $\bar{p}$ are determined with a structure accuracy (multiplication by a positive constant).

To ensure the uniqueness of turnpike N, we assume that the economy satisfies the following condition:

(G6) $\forall (x, y) \in Z \setminus \{0\} (x \notin N \Rightarrow \beta(x, y, \bar{p}) < \alpha_M)$.

Condition $x \notin N = \{\lambda \bar{s} | \lambda > 0\}$ holds if and only if $\frac{x}{\|x\|} \neq \bar{s}$. Therefore, if in a certain production process the inputs structure differs from the turnpike structure, then according to (G6), the economic efficiency of such a process is lower than optimal.[6]

□ Lemma 1. If conditions (G1)–(G6) are satisfied, then

[6] If conditions (G1)–(G6) are satisfied, then not only input vector $\bar{x}$ and output vector $\bar{y}$, but also the von Neumann equilibrium price vector $\bar{p}$ is positive.

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon \in (0, \alpha_M)\ \forall\ (x, y) \in$$
$$\in Z \left(\left\| \frac{x}{\|x\|} - \bar{s} \right\| \geq \varepsilon \Longrightarrow \beta(x, y, \bar{p}) = \frac{\langle \bar{p}, y \rangle}{\langle \bar{p}, x \rangle} \leq \alpha_M - \delta_\varepsilon \right).$$

For proof, see: Radner (1961), Takayama (1985; chap. 7), Panek (2003; chap. 5, lemma 5. 2). ■

3. Dynamics. Feasible, stationary and optimal growth processes

We assume that time t is discrete, $t = 0, 1, \dots$. We denote the input vector (or the production factors vector) that is used in the economy in period t by $x(t) = \big(x_1(t), \dots, x_n(t)\big)$, and the output vector (or the production vector) that is produced in period t by $y(t) = \big(y_1(t), \dots, y_n(t)\big)$. From the assumption that $\big(x(t), y(t)\big) \in Z\backslash\{0\}$ it follows that it is possible to produce production vector $y(t)$ from input vector $x(t)$ in period t. The economy is closed in the sense that inputs $x(t+1)$ (that are incurred in the next period) come from production $y(t)$ (produced in the previous period), i.e.

$$x(t+1) \leqq y(t), \quad t = 0, 1, \dots .$$

Hence, according to (G3), it follows that

$$\big(y(t), y(t+1)\big) \in Z\backslash\{0\}, \quad t = 0, 1, \dots . \tag{4}$$

Let y^0 be a given (positive) initial production vector (produced in period $t = 0$),

$$y(0) = y^0 > 0 \tag{5}$$

and

$$Y^1 = \{y \in R^n_+ | y \geqq y^1\}, \tag{6}$$

be a fixed target set of the desired states (production vectors); $y^1 > y^0$.

Each production vector sequence $\{y(t)\}_{t=0}^{t_1}$ satisfying conditions (4), (5) and the following condition:

$$y(t_1) \in Y^1 \tag{7}$$

is called (y^0, Y^1, t_1) – the feasible growth process.

Feasible process $\{y^*(t)\}_{t=0}^{t_1^*}$ is called (y^0, Y^1, t_1^*) − optimal, if it is a solution to the following minimal-time growth problem:

$$\begin{gathered}\min t_1 \\ \text{subject to (4), (5), (7),}\end{gathered} \tag{8}$$

in which vector y^0 and set Y^1 are fixed.

In problem (8), production vectors $y(1), \dots, y(t_1)$ and time t_1 are the decision variables. This problem has a solution if assumptions (G1)–(G6) are met (Panek, 2021).

If conditions (G1)–(G6) are satisfied, and particularly $y^0 = \bar{y} \in N$, then a growth process exists (satisfying conditions (4), (5)) of the following form:

$$\bar{y}(t) = \alpha_M^t \bar{y}, \quad t = 0, 1, \dots, \tag{9}$$

which is called the stationary growth process with the α_M rate.[7] Since in such a process the following condition is satisfied:

$$\forall t \left(\bar{s}(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|} = \frac{\alpha_M^t \bar{y}}{\|\alpha_M^t \bar{y}\|} = \frac{\bar{y}}{\|\bar{y}\|} = \bar{s} \right),$$

we therefore say that it is characterised by a constant (turnpike) production structure. Each stationary growth process lies on turnpike N. The production of all the commodities in such a process increases at a maximum rate of $\alpha_M > 1$ achievable by the economy. This fact still plays an important role in the proof of the 'very strong' turnpike theorem in the next section.

4. 'Very strong' turnpike effect

Let us introduce the following (angular) distance measure of production vector $y(t)$ from turnpike $N = \{\lambda \bar{s} | \lambda > 0\}$:

$$d(y(t), N) = \left\| \frac{y(t)}{\|y(t)\|} - \bar{s} \right\|.$$

[7] The stationary growth process exists if and only if condition $(\bar{y}, \alpha_M \bar{y}) \in Z \backslash \{0\}$ is satisfied. This condition is fulfilled in our model.

In a paper by Panek (2021, Th. 4), we proved the 'weak' turnpike theorem which states that if conditions (G1)–(G6) apply, and (*) such a number $M < +\infty$ exists that regardless of the distance between target states set Y^1 and initial state y^0, each vector $y^1 > y^0$ determining the shape of this set (see (6)) satisfies condition $\frac{\max_i y_i^1}{\min_i y_i^1} \leq M$, then – regardless of the distance between target states set Y^1 from initial state y^0 – the production structure in each optimal growth process (y^0, Y^1, t_1^*), i.e. the solution to problem (8), almost always[8] differs slightly, in an arbitrary way, from the turnpike production structure on which the economy develops at its maximum rate, achieving the highest possible technological and economic efficiency. According to condition (*), y^1 is any production vector (greater than initial vector y^0) in which with $\|y^1\| \to +\infty$, the distance (range) between the values of its coordinates does not increase 'too rapidly' (i.e. no faster than linearly).[9]

We will now trace the trajectory of optimal growth process $\{y^*(t)\}_{t=0}^{t_1^*}$, which in a certain time period of $\check{t} < t_1^* - 1$ reaches turnpike N, when condition (*) is replaced with the following condition:[10]

(G7) vector $y^1 > y^0$, on which the set of target states Y^1 depends, satisfies

$$\frac{\max_i \frac{y_i^1}{\bar{s}_i}}{\min_i \frac{y_i^1}{\bar{s}_i}} \leq \alpha_M.$$

For the proof of Theorem 3, the following lemma will be necessary.

□ Lemma 2. Let us assume that (y^0, Y^1, t_1^*) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1^*}$ and the solution to problem (8), in a certain period $\check{t} < t_1^* - 1$, reaches the turnpike:

$$y^*(\check{t}) \in N.$$

If conditions (G1)–(G7) are satisfied, then there exists such a (y^0, Y^1, t_1) – a feasible process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

[8] Except for a number of time periods, independent of Y^1 or t_1^*.

[9] For example, if we have sequence of problems (8) with the sets of target states $Y^{1,i} = \{y \in R_+^n | y \geqq y^{1,i} > y^0\}$, $\|y^{1,i}\| \underset{i}{\to} +\infty$, then condition (G7) excludes the situation in which for a certain k-th coordinate, $y_k^{1,i} \underset{i}{\to} \bar{y}_k^1 < +\infty$, and for a certain (different) j-th coordinate, $y_j^{1,i} \underset{i}{\to} +\infty$.

[10] By vector y^1 we may also understand any production vector greater than initial vector y^0, in which with $\|y^1\| \to +\infty$ the distance between the values of its coordinates, relativised with respect to the coordinates of turnpike production structure vector $\bar{s}$, increases no faster than linearly with coefficient $M = \alpha_M$.

$$\tilde{y}(t) = \begin{cases} y^*(t), t = 0, 1, \dots, \check{t}, \\ \alpha_M^{t-\check{t}} y^*(\check{t}), t = \check{t}+1, \dots, t_1 \end{cases}, \tag{10}$$

$t_1 \geq t_1^*$, that

$$\tilde{y}(t_1) \geqq y^1 \text{ and } \tilde{y}(t_1 - 1) \leqq y^1.$$

Proof.[11] If assumptions (G1)–(G7) are satisfied, then there exists (y^0, Y^1, t_1) – a feasible growth process (10), in which $\tilde{y}(t_1) \geqq y^1$, and t_1 is the smallest natural number which satisfies the condition

$$t_1 \geq \tau_1, \tag{11}$$

where $\tau_1 = \check{t} + \frac{\ln A_1}{\ln \alpha_M}$, $A_1 = \max_i \frac{y_i^1}{y_i^*(\check{t})} = \sigma^{-1} \max_i \frac{y_i^1}{\bar{s}_i} > 1$, $\sigma = \|y^*(\check{t})\| > 0, \bar{s} = \frac{y^*(\check{t})}{\|y^*(\check{t})\|}$.

In this process, $\tilde{y}(\check{t}) = y^*(\check{t})$ and $\forall t \in \{\check{t}, \check{t}+1, \dots, t_1\}$ $(\tilde{y}(t) \in N)$. Let us denote by l_{y^1} the smallest number (not necessarily natural) for which the following inequality holds:

$$\alpha_M^{\tau_1 - \check{t} - l_{y^1}} y^*(\check{t}) \leqq y^1. \tag{12}$$

Such a number exists (since $y^1 > 0$ and $\alpha_M > 1$) and

$$l_{y^1} = \tau_1 - \check{t} - \frac{\ln A_2}{\ln \alpha_M} \geq 0,$$

where $A_2 = \min_i \frac{y_i^1}{y_i^*(\check{t})} = \sigma^{-1} \min_i \frac{y_i^1}{\bar{s}_i} > 0$ $(A_2 \leq A_1)$. From (G7) it follows that

$$\frac{A_1}{A_2} \leq \alpha_M, \text{ or } \frac{\ln A_2}{\ln \alpha_M} \geq \frac{\ln A_1}{\ln \alpha_M} - 1,$$

and hence, according to (11), we come to the conclusion that

$$l_{y^1} = \tau_1 - \check{t} - \frac{\ln A_2}{\ln \alpha_M} = \check{t} + \frac{\ln A_1}{\ln \alpha_M} - \check{t} - \frac{\ln A_2}{\ln \alpha_M} \leq 1.$$

[11] The proof is partially based on the proof of Lemma 3 from the paper by Panek (2021). The sequence elements (10) starting from $t = \check{t}$ belong to stationary growth process (9) with the α_M rate and initial production vector $\bar{y}(0) = \bar{y} = \alpha_M^{-\check{t}} y^*(\check{t}) \in N$.

Thus, each target production vector y^1 that satisfies condition (G7) corresponds to such a non-negative number $l_{y^1} \leq 1$ that condition (12) is satisfied. Particularly,

$$\tilde{y}(t_1 - 1) = \alpha_M^{t_1 - \check{t} - 1} y^*(\check{t}) \leqq \alpha_M^{\tau_1 - \check{t} - l_{y^1}} y^*(\check{t}) \leqq y^1,$$

and this concludes the proof. ■

In Theorem 3 we prove that if (y^0, Y^1, t_1^*) – optimal growth process in a certain time period $\check{t}$ meets condition $\check{t} < t_1^*$, then everywhere else, except possibly in the last period t_1^*, it remains on the turnpike.

□ Theorem 3. Suppose the following applies:

- conditions (G1)–(G7) are satisfied;
- (y^0, Y^1, t_1^*) – optimal growth process in a certain time period $\check{t} < t_1^* - 1$ reaches the turnpike:
  $y^*(\check{t}) \in N$,
- the solution to problem (8) is unique,
  then

$$\forall t \in \{\check{t} + 1, \dots, t_1^* - 1\}(y^*(t) \in N). \tag{13}$$

Proof. Let us consider any such (y^0, Y^1, t_1^*) – optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$ where $t_1^* > \check{t} + 1$. From (2) and from the definition of the optimal growth process we have

$$\langle \bar{p}, y^*(t+1) \rangle \leq \alpha_M \langle \bar{p}, y^*(t) \rangle, t = 0, 1, \dots, t_1^* - 1,$$

and hence, in particular,

$$\langle \bar{p}, y^*(t_1^*) \rangle \leq \alpha_M^{t_1^* - \check{t}} \langle \bar{p}, y^*(\check{t}) \rangle. \tag{14}$$

Let us assume that $y^*(\tau) \notin N$ for a certain $\tau \in \{\check{t} + 1, \dots, t_1^* - 1\}$. Then

$$\exists \varepsilon > 0 \left(d(y^*(\tau), N) = \left\| \frac{y^*(\tau)}{\|y^*(\tau)\|} - \bar{s} \right\| = \varepsilon > 0 \right).$$

According to Lemma 1, such a $\delta_\varepsilon \in (0, \alpha_M)$ number exists that

$$\langle \bar{p}, y^*(\tau+1) \rangle \leq (\alpha_M - \delta_\varepsilon) \langle \bar{p}, y^*(\tau) \rangle. \tag{15}$$

From (14), (15) we obtain the upper limit of the value of the outputs produced in period t_1^*

$$\langle \bar{p}, y^*(t_1^*)\rangle \leq \alpha_M^{t_1^*-\check{t}-1}(\alpha_M - \delta_\varepsilon)\langle \bar{p}, y^*(\check{t})\rangle. \tag{16}$$

On the other hand, according to Lemma 2, there exists such a $(y^0, Y^1, t_1)-$ feasible process following (9) that satisfies the condition

$$\tilde{y}(t_1 - 1) = \alpha_M^{t_1-\check{t}-1} y^*(\check{t}) \leqq y^1.$$

Then

$$\langle \bar{p}, y^*(t_1^*)\rangle \geq \langle \bar{p}, y^1\rangle \geq \langle \bar{p}, \tilde{y}(t_1 - 1)\rangle = \alpha_M^{t_1-\check{t}-1}\langle \bar{p}, y^*(\check{t})\rangle. \tag{17}$$

From (16), (17) (according to $t_1 \geq t_1^*$), we obtain the following inequality:

$$\alpha_M - \delta_\varepsilon \geq \alpha_M^{t_1-t_1^*}. \tag{18}$$

If $t_1 = t_1^*$, then process $\{\tilde{y}(t)\}_{t=0}^{t_1}$ is (y^0, Y^1, t_1^*) $-$ optimal.[12] For $t_1 = t_1^* + 1$, we have $\alpha_M - \delta_\varepsilon \geq \alpha_M$, therefore $\delta_\varepsilon \leq 0$, in contradiction to our assumption. If $t_1 - t_1^* = k \geq 2$, then from (18) we get $\alpha_M - \delta_\varepsilon > \alpha_M^k$, which contradicts condition $\alpha_M > 1$. The obtained contradictions close the proof. ■

5. Final remarks

The necessity to leave the turnpike by optimal process $\{y^*(t)\}_{t=0}^{t_1^*}$, i.e. the solution to minimal-time growth problem (8), in final period t_1^* results simply from the postulate that the economy should reach the set of target states Y^1. In a particular case, when target production vector y^1 that determines the form of target state set (6) is located on the turnpike, $y^1 \in N$, condition (13) holds also for $t = t_1^*$.

The following version of Theorem 3 without the postulate of uniqueness of the solution also remains true:

□ Theorem 3'. Suppose the following applies:

- conditions (G1)–(G7) are satisfied;

[12] We deal with a similar situation in the paper by Panek (2021; Th. 3). When $t_1 = t_1^*$, then $\delta_\varepsilon < \alpha_M - 1$, which cannot be excluded, because $\alpha_M > 1$. Therefore, one of the assumptions of this theorem is the condition of the solution unique.

- a certain (y^0, Y^1, t_1^*) – optimal growth process in period $\check{t} < t_1^* - 1$ reaches the turnpike,

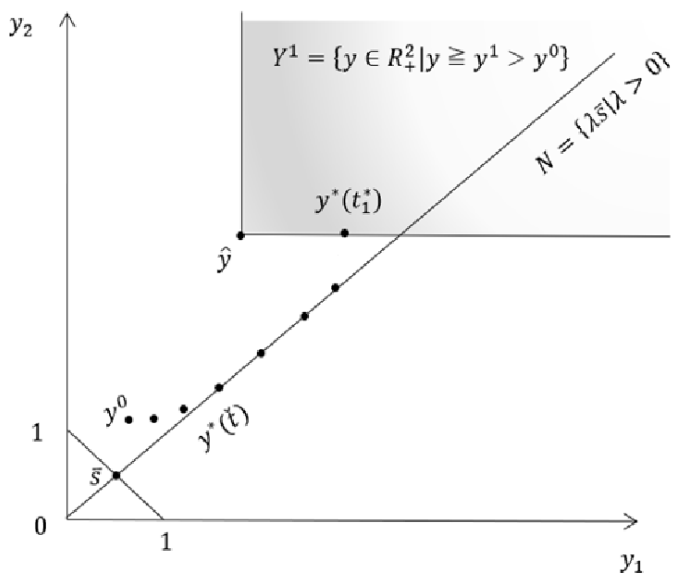
then there also exists such a (y^0, Y^1, t_1^*) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1^*}$ that

$$\forall t \in \{\check{t} + 1, \dots, t_1^* - 1\}(y^*(t) \in N).$$

The proof here is exactly the same as the proof of Theorem 3. ■

An example trajectory of (y^0, Y^1, t_1^*), i.e. the optimal growth process in $Z \subset R_+^2$ satisfying the conditions of Theorem 3 is illustrated in the Figure.

Figure Illustration to Theorem 3. The trajectory of the (y^0, Y^1, t_1^*) optimal growth process and the solution to problem (8) in the neighbourhood of turnpike $N = \{\lambda\bar{s}|\lambda > 0\} \subset \subset Z \subset R_+^2$.



Source: author's work.

6. Conclusions

In many papers devoted to the asymptotic/turnpike properties of the optimal growth processes in von Neumann-Gale-Leontief economies, production utility is assumed to be the growth criterion. The novelty of the approach proposed in this article, like in the earlier paper by Panek (2021), consists in replacing the utility of production as the standard quality criterion of economic growth processes by a minimum-time growth criterion (minimising the time needed by the economy to reach the postulated/desired target state). It was proven that changing the growth criterion does not deprive the Gale economy of its asymptotic/turnpike properties.

It would be interesting to study the turnpike properties of the solutions to the minimal-time growth problems of type (8) also in a non-stationary Gale economy with changing technology and multilane production turnpike, especially in the Gale economy with an investment mechanism (see Panek, 2022).

Probably the solution to minimal-time growth problem (8) is also characterised by a 'strong' turnpike effect (as in the case of many other optimal growth processes in the Gale economy with the maximisation of the production utility criterion).

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